Engineering Electromagnetic Fields and Waves

هندسة المجالات

هندسة المجالات
الكهرومغناطيسية والتموجات

بسم الله الرحمن الرحيم

هندسة المجالات الكهرومغناطيسية والتموجات

الجزء الثاني : □ التيارات الثابتة □ المغناطيسية الساكنة
□ طريقتي الصور والتعاكس في المسائل الحديثة

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١٩٨٦

دار الراتب الجامعية

الفصل السادس

التيار الكهربي

THE ELECTRIC CURRENT

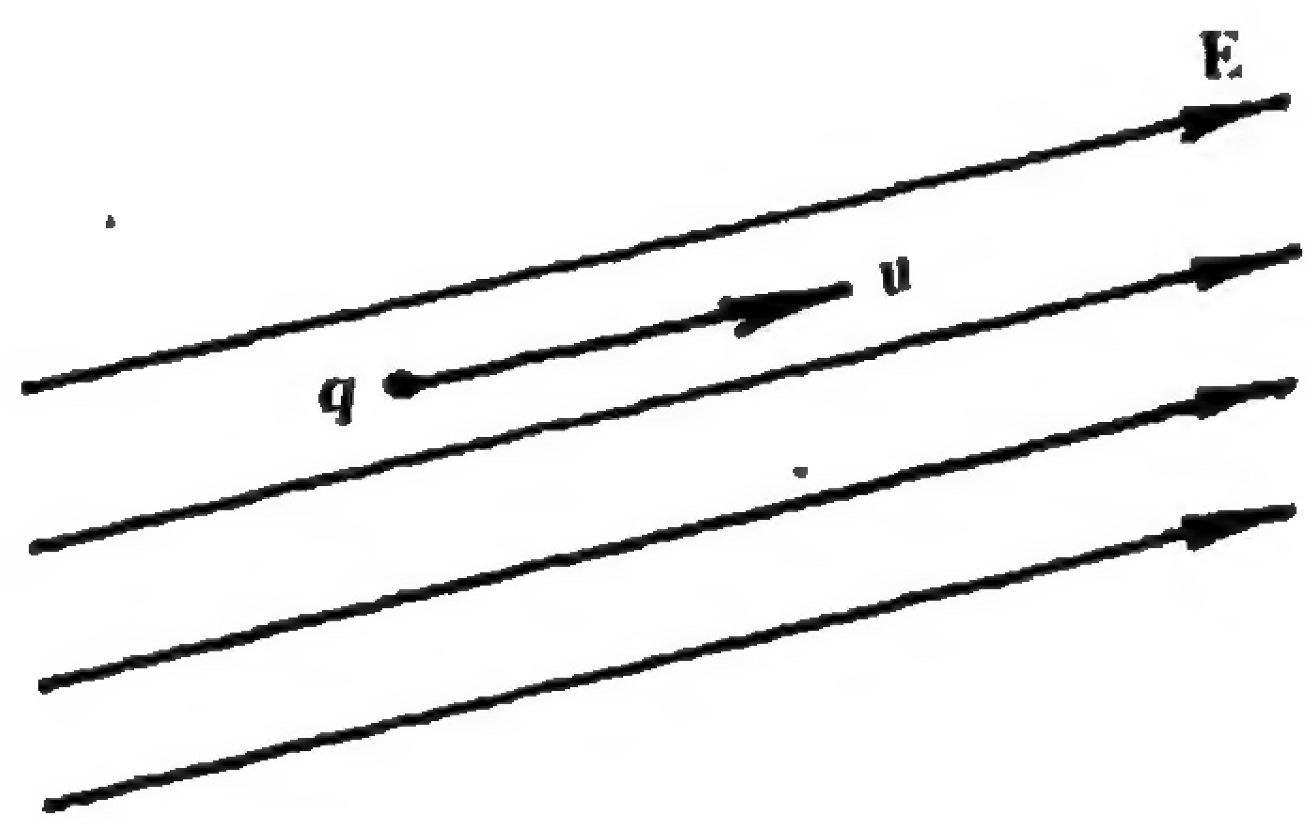
6 - 1 مقدمة :

يعرف التيار الكهربي الذي يمر بسطح ما بأنه معدل مرور الشحنات عبر هذا السطح ووحداته هي الأمبير (Amperes) . وبناء على هذا التعريف فإ أمبير واحد يساوي كولوم / ثانية . وتتحرك الشحنات الكهربية في المواد الصلبة والسائلة والغازية وكذلك في الفراغ . ويعرّف تيار الحمل (Convection Current) بأنه تيار ناتج عن حركة غاز أو سائل يحتوي على جسيمات مشحونة تحت تأثير كمية حركة مكتسبة . وتبعاً لنظرية الانتشار نجد أن الجسيمات المشحونة تنتشر في الاتجاهات التي يقل بها تركيز هذه الجسيمات مثال ذلك حركة الشحنات في المواد شبه الموصلة . وتعرف حركة الشحنات تحت تأثير قوة الانتشار بتيار الانتشار (Diffusion Current) . وعندما تتحرك الشحنات تحت تأثير مجال كهربي ينتج ما يسمى تيار التوصيل (Conduction Current) مثال ذلك حركة الالكترونات في المواد المعدنية والمحاليل الالكتروليتية . وحسب النظرية الكهرومغناطيسية فإن تيار التوصيل هو أهم أنواع التيارات الكهربية . وهناك نوع آخر من التيارات الكهربية وهو ينشأ عن معدل تغيير كثافة الفيض الكهربي مع الزمن ويسمى تيار الازاحة (Displacement Current) وهذا التيار لا يكون ناتج عن تحرك شحنات

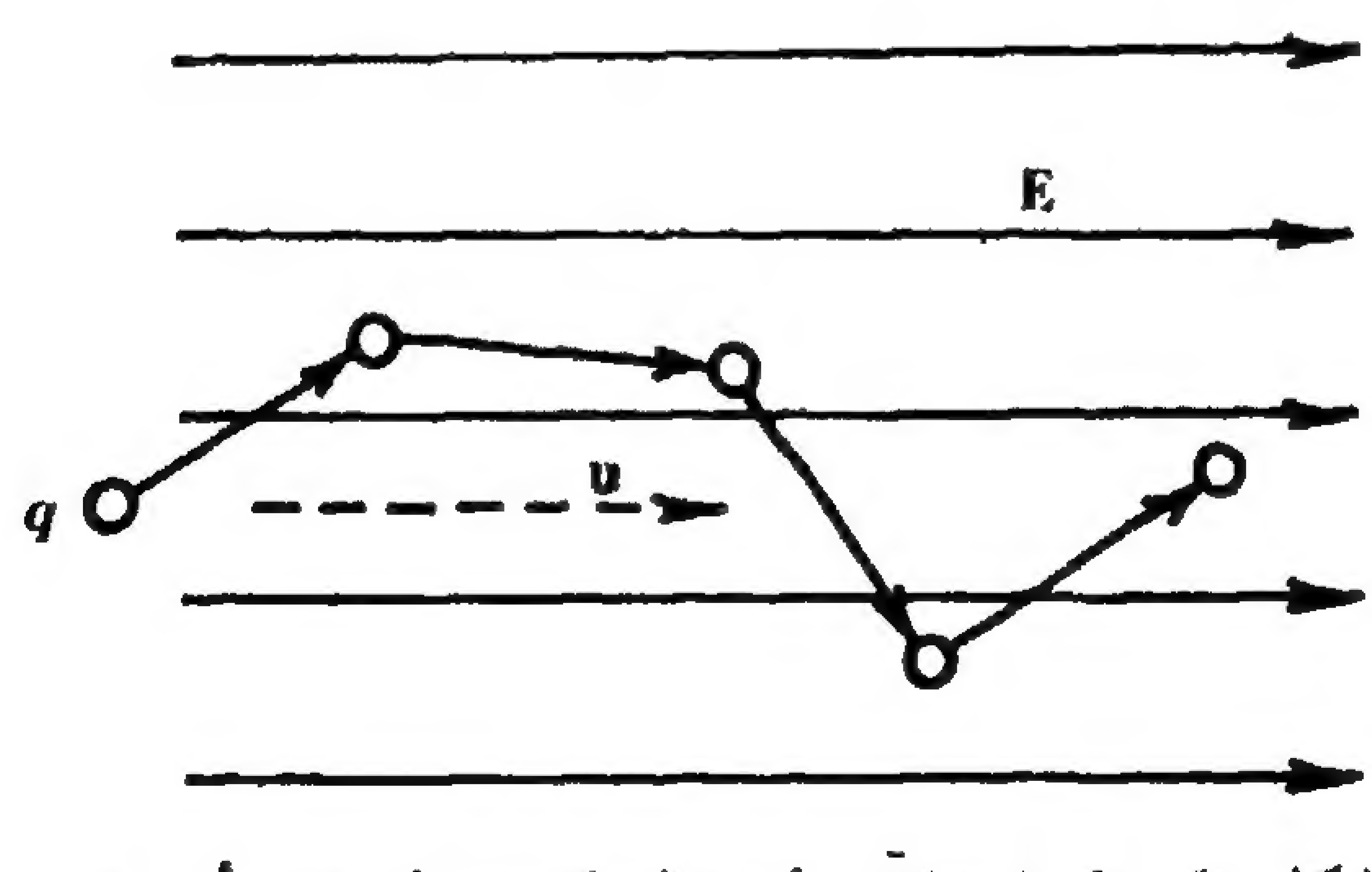
كهربية كما في الأنواع السابقة وسنتعرض لدراسة هذه التيارات كل على حده .

6 - 2 الشحنات الكهربية المتحركة :

اعتبر شحنة موجبة مقدارها q كولوم واقعة تحت تأثير مجال كهربي E في الفراغ . القوة المؤثرة على هذه الشحنة هي $q\,E$. وبناء عليه فإن هذه الشحنة تكتسب عجلة ثابتة وبذلك تتحرك في هذا المجال الكهربي بسرعة متزايدة مع الزمن طالما ظلت هذه الشحنة داخل المجال الكهربي (شكل 6 - a) . أما إذا كانت الشحنة تقع في وسط ما (صلب أو سائل أو غاز)



شكل (6 - a) شحنة في مجال كهربي في الفراغ



شكل (6 - b) شحنة q في مجال كهربي في غاز أو سائل

فإنها تصطدم اصطداماً متتالياً ينتج عنه تغيير عشوائي في سرعة الشحنة تكون

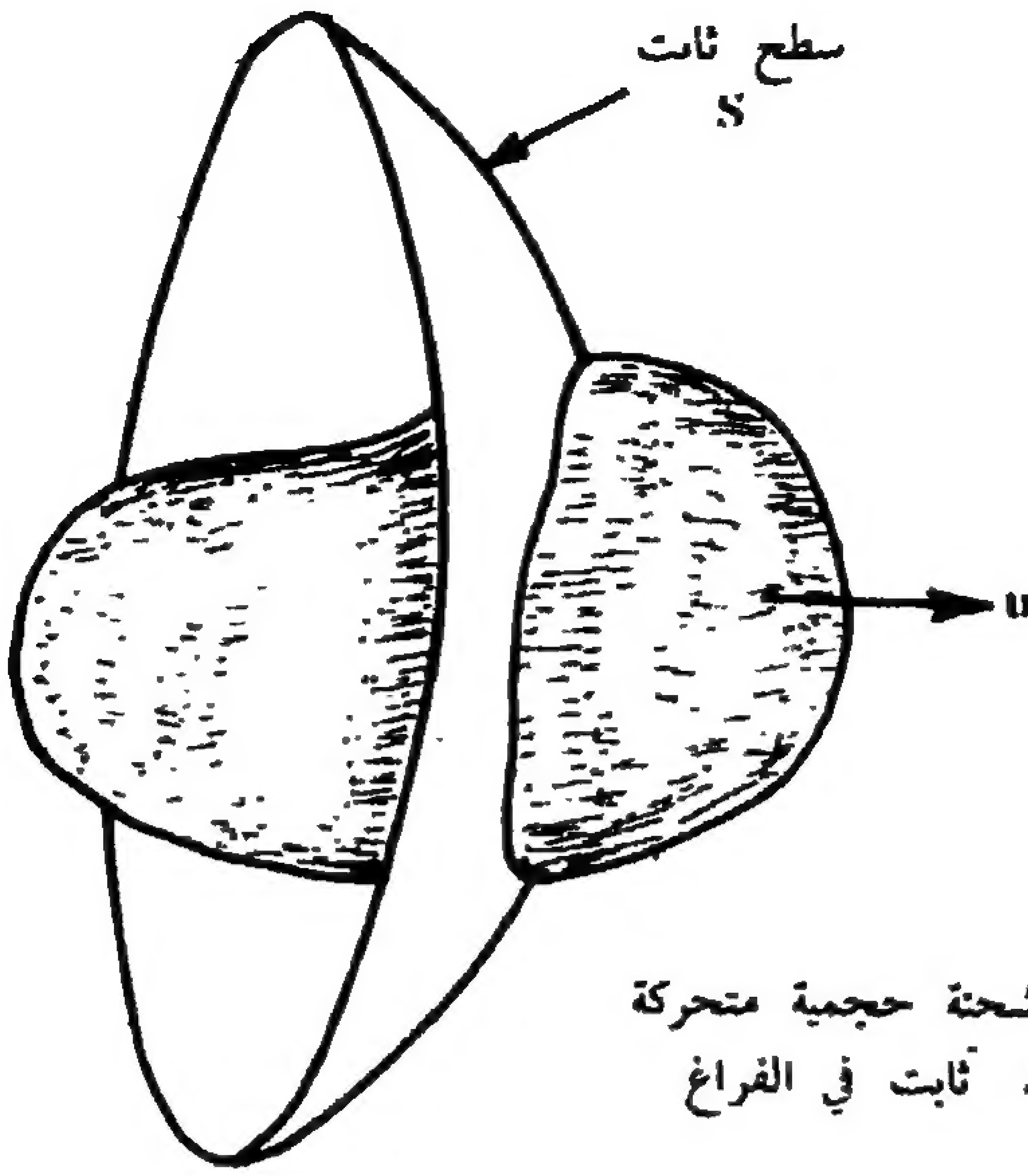
السرعة المحصلة فيه منعدمة . وإذا أثر عليها مجال كهربي E (شكل 6 - b) فإننا نلاحظ أن هناك سرعة متوسطة لهذه الشحنة في اتجاه المجال تسمى سرعة الانسياق (drift velocity) . وتتم عملية التوصيل الكهربي في المعادن بتحرك الالكترونات الموجودة بالقشور الخارجية (external shells) لذرات هذه المعادن . وحسب نظرية الغاز الالكتروني (electron gas theory) فإن هذه الالكترونات تكون لها سرعة انسياق تتناسب مباشرة مع شدة المجال الكهربي E الذي يؤثر عليها :

$$u = M\ E \qquad (1 - 6)$$

ويسمى معامل التناسب M حركية الشحنة (Mobility) ووحداتها هي $m^2 / V.\ sec$. يحتوي المتر المكعب الواحد من المعدن على حوالي 10^{28} ذرة ويكون لكل ذرة من المواد الجيدة التوصيل للكهرباء الكترون واحد أو الكترونين حرّين يساهمان في نقل التيار الكهربي في الموصل . وتتغير حركية الشحنة M مع درجة الحرارة ومع التركيب البلوري للموصل . فنلاحظ أن جزئيات الموصل لها حركة اهتزازية تزداد بزيادة درجة الحرارة فتعمل على عرقلة حركة الالكترونات الحرة وبالتالي تقل حركية الشحنة وينشأ عن ذلك انخفاض في متوسط سرعة هذه الالكترونات وبالتالي تنخفض شدة التيار الناتج من مجال كهربي معين E . ويقال في هذه الحالة أن مقاومة الموصل قد زادت بزيادة درجة الحرارة .

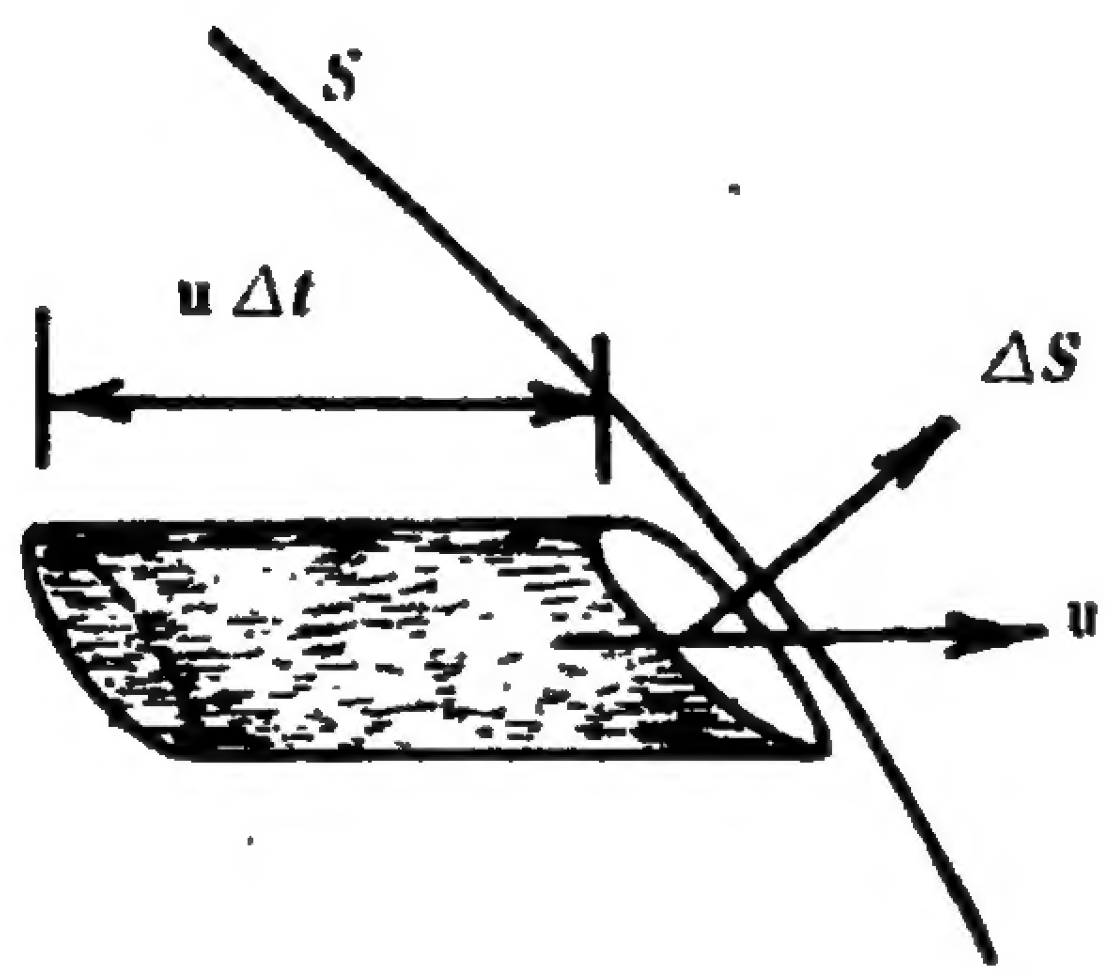
6 - 3 كثافة تيار الحمل : (Convection Current Density) :

افرض توزيع فراغي للشحنات داخل حجم معين ح ذات كثافة حجمية ؟ كولوم / م^3 وأن الحجم ح يتحرك بسرعة u كما هو مبين بشكل (6 - c) . إفرض كذلك أن كل شحنة تحتفظ بمكانها داخل الحجم ح .



شكل (c - 6) شحنة حجمية متحركة
تقطع سطح S ثابت في الفراغ

الشحنة التي تمر خلال عنصر مساحته ΔS من السطح S في زمن مقداره Δt هي ΔQ .



شكل (d - 6) الشحنة التي تمر
خلال السطح ΔS
في زمن مقداره Δt .

وبناء على ذلك يكون التيار المار خلال ΔS في الزمن Δt هو :

$$I = \Delta Q / \Delta t \qquad (2 - 6)$$

ولكن الشحنة ΔQ تقع داخل حجم مقداره $\mathrm{u}\,\Delta t.\,\Delta S$. شكل
. (d - 6)

$$\Delta Q = \rho u \, \Delta t . \Delta S$$
$$\Delta Q / \Delta t = \rho u . \Delta S$$
$$= J . \Delta S$$

حيث :

$$J = \rho u \qquad (3 - 6)$$

هو كثافة تيار الحمل أمبير / م^2 . وإذا تغير شكل الحجم v أو تغيرت كثافة الشحن ρ فإن J لا يظل ثابت مع الزمن . ويكون J منعدماً بعد مرور نهاية الحجم v من السطح S . وتيار الحمل له أهمية في دراسة النظرية الكهرومغناطيسية عندما يوجد تيار مكون من سحابة مشحونة متحركة في الوسط الكهرومغناطيسي .

6 - 4 كثافة تيار التوصيل : (Conduction Current Density) :

إذا وضع موصل معزول في مجال كهربي فإن الالكترونات الحرة داخل هذا الموصل تكتسب سرعة انسياق في اتجاه معاكس للمجال الكهربي وتعمل على وجود شحنات سطحية ، وتستمر هذه العملية حتى يتعادل المجال الكهربي الناشىء من هذه الشحنات داخل الموصل مع المجال الكهربي الأساسي وينعدم المجال الكهربي المحصل داخل الموصل . افترض أن هناك فرق جهد بين نقطتين على الموصل نتيجة لتوصيلها بطرفي بطارية مثلاً ونظراً لانحدار الجهد داخل الموصل تنتقل الالكترونات من احدى نقطتي الموصل إلى النقطة الأخرى عن طريق البطارية ولا يحدث في هذه الحالة أي تركيز للشحنات على الموصل حيث يعمل القطب الموجب للبطارية كبالوعة للالكترونات والقطب السالب كمنبع لها . ويستمر تدفق الالكترونات داخل لموصل ويكوّن هذا التدفق تيار التوصيل الذي يمر في الموصل باستمرار . وإتجاه هذا التيار هو من القطب الموجب إلى القطب السالب في الموصل أي

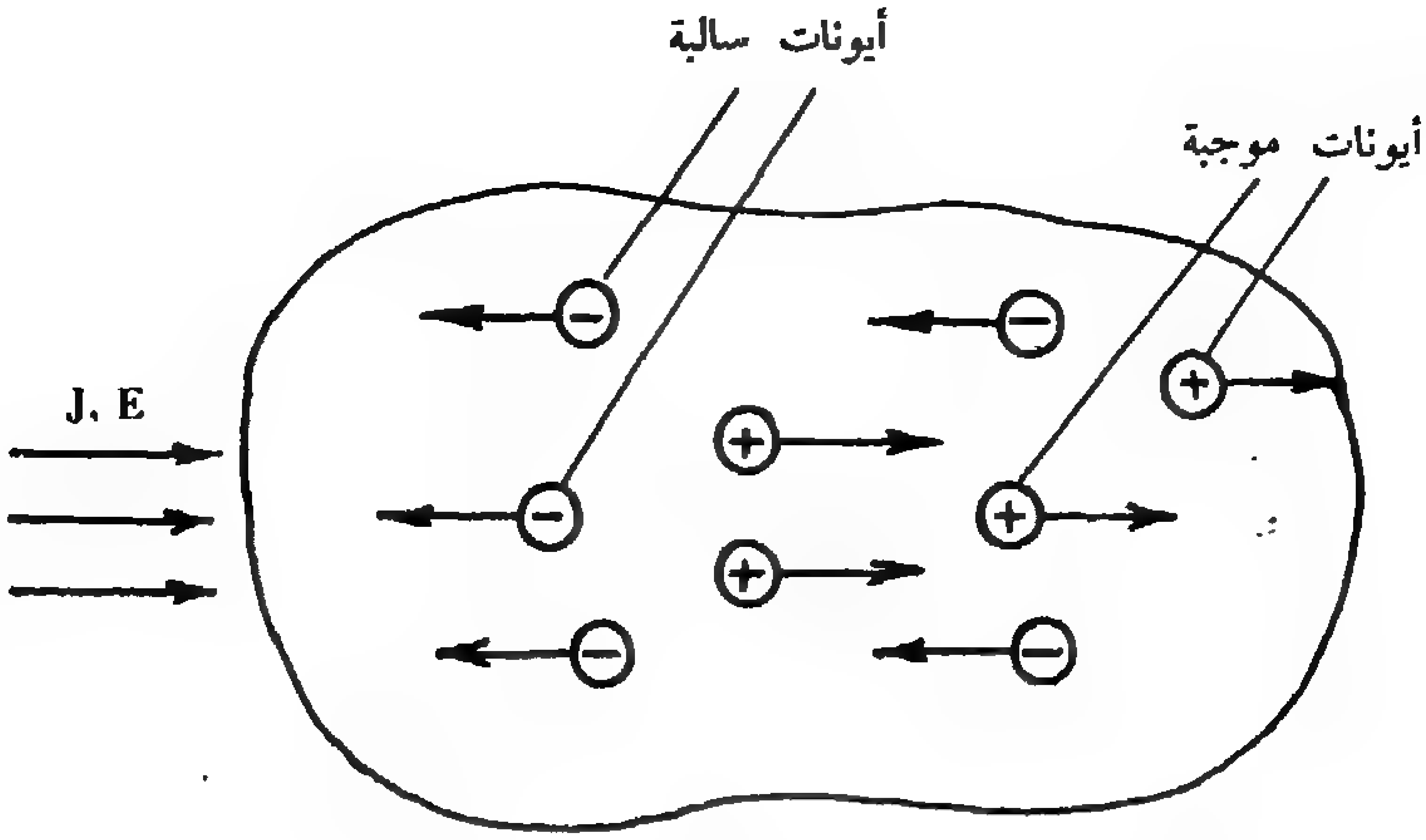
عكس اتجاه حركة الالكترونات . وعند أي لحظة نجد أن الموصل متعادل كهربياً ولكنه ليس متساو الجهد كما في الحالة الاستاتيكية التي ذكرناها سابقاً في الفصل الثالث ـ بالجزء الأول . وتكون كثافة التيار الكهربي عند أي نقطة داخل الموصل هي $J = \rho u$ وباعتبار العلاقة بين سرعة الالكترونات والمجال الكهربي $u = M E$ نجد أن :

$$J = \rho M E$$
$$= \sigma E \qquad (4 - 6)$$

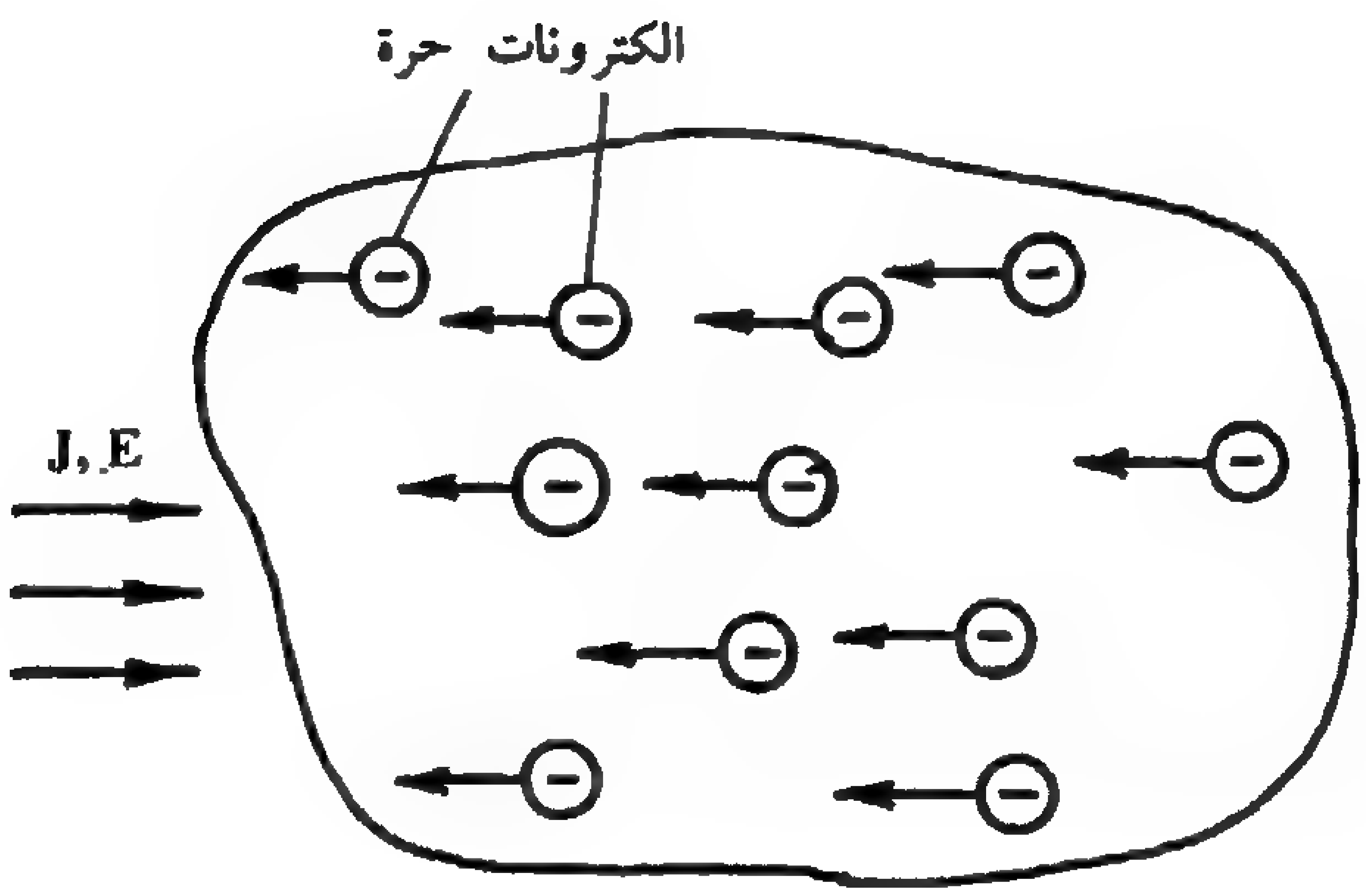
حيث σ هي الموصلية (Conductivity) ووحداتها Siemens / m . وكما ذكرنا فإن حاملات التيار في الموصلات المعدنية هي الالكترونات وهي تنساق داخل الموصل في اتجاه معاكس للمجال الكهربي E وبالتالي فإن كل من ρ ، u تكون سالبة وينتج عنهما موصلية موجبة وعلى ذلك تكون كثافة التيار الكهربي J وشدة المجال الكهربي E لهما نفس الاتجاه داخل الموصل . وتعرف العلاقة (6 - 4) بقانون أوم عند نقطة . وكما ذكرنا من قبل فإن الموصلية σ هي دالة من درجة الحرارة .

6- 5 الموصلية σ :

في الأوساط السائلة والغازية تكون حاملات التيار هي أيونات موجبة أو سالبة . بعض هذه الأيونات تحمل شحنة فردية والبعض الآخر يحمل شحنة مزدوجة أو متعددة وقد تكون لهذه الأيونات أوزان مختلفة كذلك . وبناء عليه فإن الموصلية σ لهذه الأوساط يجب أن تأخذ في الاعتبار كل هذه العوامل . فإذا كانت كل الأيونات الموجبة متماثلة وكذلك الأيونات السالبة فإن الموصلية تحتوي على حدين كما هو واضح بشكل (6 - e) . وفي حالة الموصلات المعدنية تكون الموصلية مكونة من حد واحد فقط هو حاصل ضرب كثافة الالكترونات الحرة الحجمية ρ_e والحركية M_e شكل (6 - f) . أما في المواد شبه

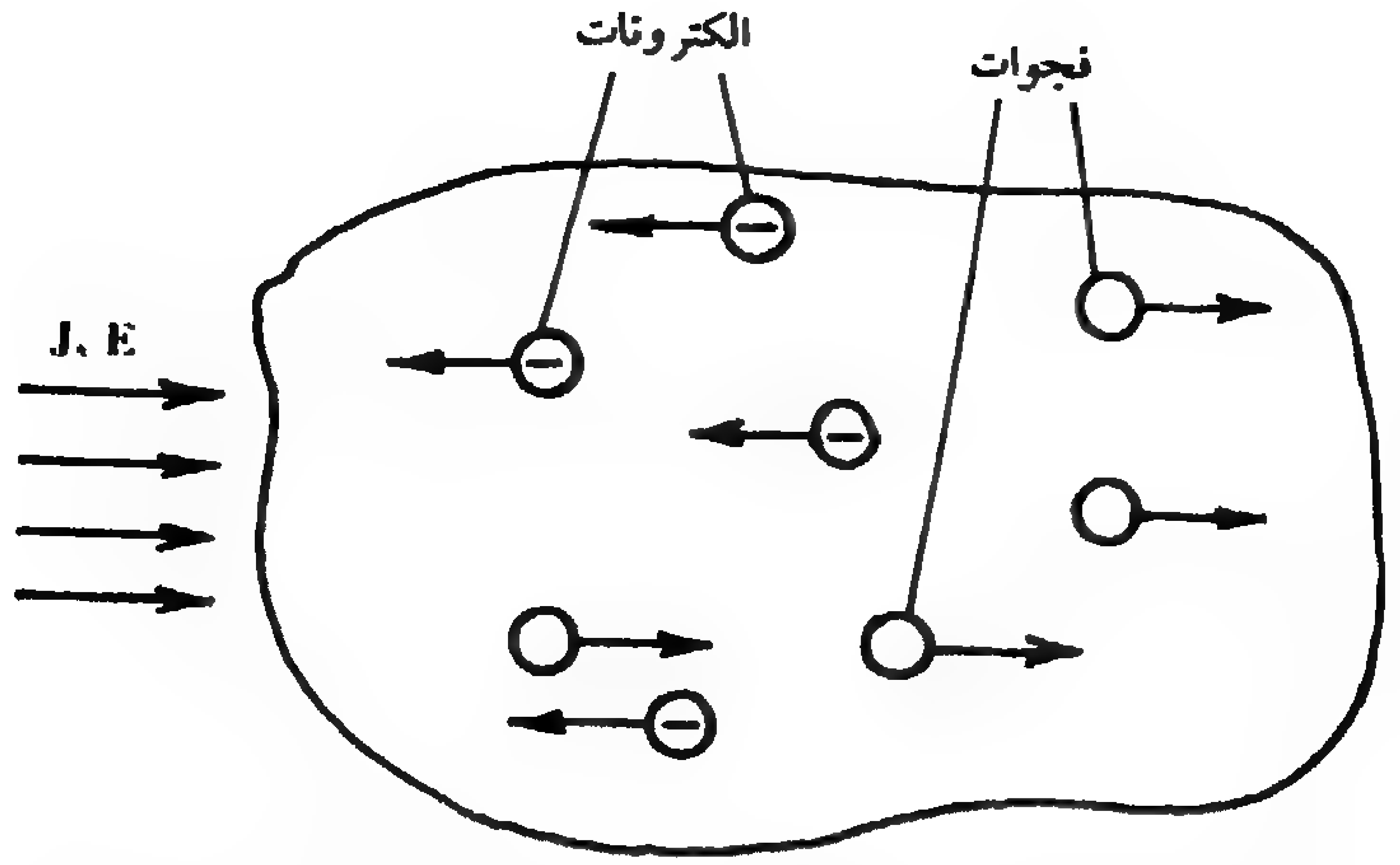


شكل (e - 6) التوصيل الكهربي في سائل أو غاز .
في هذه الحالة $\sigma = \rho_- M_- + \rho_+ M_+$



شكل (f - 6) التوصيل الكهربي في معدن . في هذه الحالة $\sigma = \rho_e M_e$

الموصلة مثل الجرمانيوم والسليكون فإن التوصيل يتم بواسطة الالكترونات والفجوات (holes) وتسمى المواد الشبه الموصلة الذاتية (Intrinsic Semiconductor) ويبين شكل (g - 6) أن الموصلية تتكون من جزئين أحدهما



شكل (6 - g) التوصيل الكهربي في شبه موصل ذاتي في هذه الحالة
$\sigma = \rho_e M_e + \rho_h M_h$

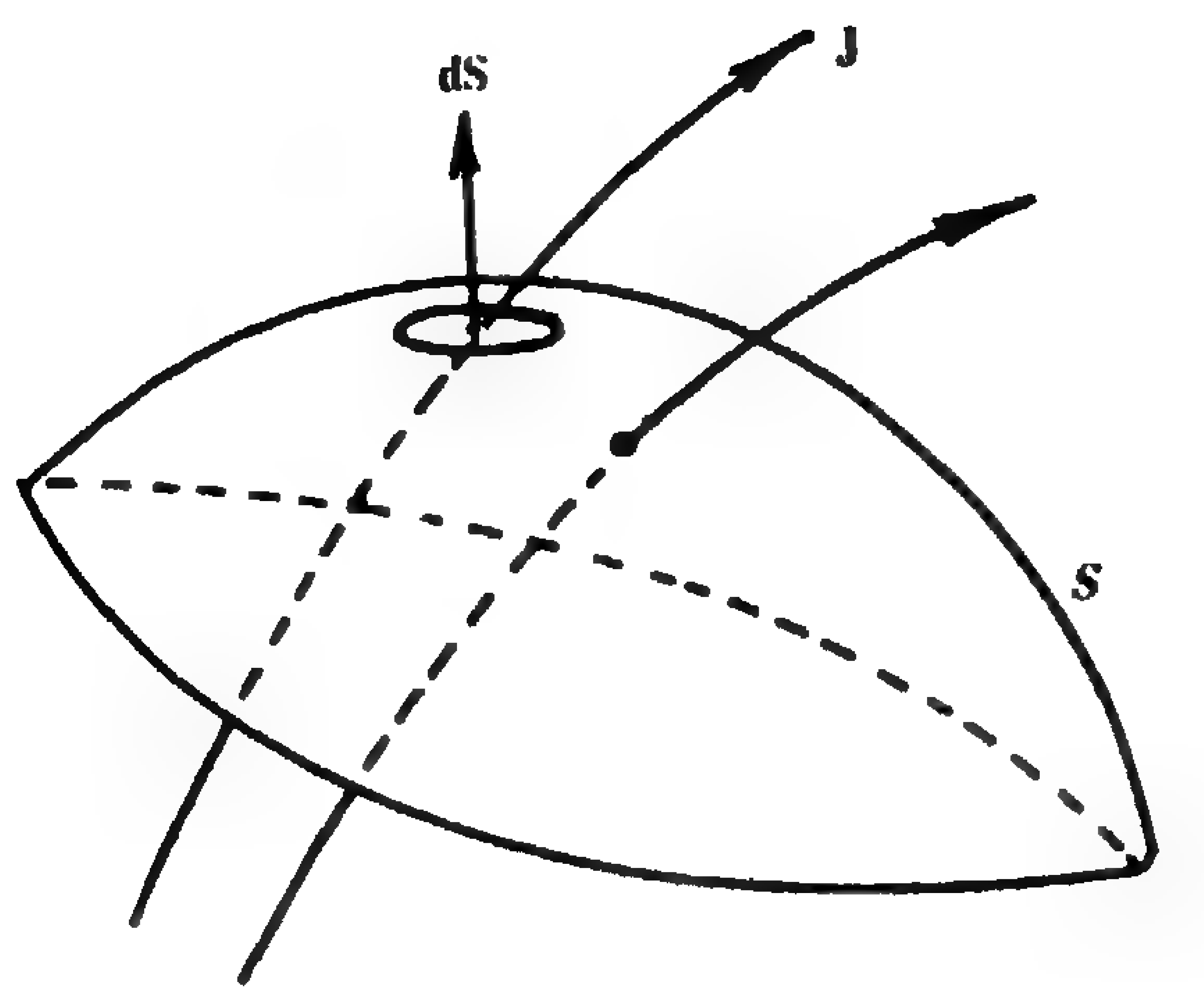
للالكترونات والآخر للفجوات . وعملياً توضع مادة شائبة مع شبه الموصل الذاتي تكون لها ثلاثة أو خمسة الكترونات تكافؤ لتكوين شبه موصل من النوع الموجب *(p - type)* أو النوع السالب *(n - type)* على التوالي . وتكون الموصلية في هذه الحالة ناتجة أساساً من الالكترونات في النوع السالب ومن الفجوات في النوع الموجب . أي أن $\rho_h << \rho_e$ في النوع السالب و $\rho_e << \rho_h$ في النوع الموجب .

6 - 6 التيار الكهربي :

التيار الكلي المار خلال سطح S . يمكن ايجاده عن طريق تكامل كثافة التيار $\mathbf{J}$ على السطح من المعادلة :

$$I = \int_S \int \mathbf{J} . d\mathbf{S} \qquad (5 - 6)$$

حيث $d\mathbf{S}$ هو متجه يمثل عنصر مساحة صغير من السطح تكون كثافة التيار المار خلالها هي $\mathbf{J}$. شكل (6 - h) . وإذا مثل السطح S مقطع موصل



شكل (h - 6) التيار خلال مساحة dS

دائري نصف قطره a فإن التيار المار بهذا الموصل هو :

$$I = \pi a^2 J \qquad Amp.$$

6 - 7 المقاومة R : (Resistance) :

إذا مر تيار كهربي في موصل طوله l ذات مقطع منتظم S فإن الجهد بين طرفيه يكون :

$$V = E\, l$$

وحيث أن التيار المار به هو :

$$I = J\, S = \gamma\, S\, E$$

وذلك يفرض أن التيار متساوي الكثافة على مقطع الموصل . وبتطبيق قانون أوم نجد أن مقاومة الموصل هي :

$$R = V / I = l / \gamma\, S \qquad ohm \qquad (6\text{-}6)$$

وإذا كان التيار غير منتظم التوزيع على مقطع الموصل فإن مقاومته تتعين من العلاقة :

$$R = V / \int\int \mathbf{J} . d\mathbf{S}$$
$$= \int \mathbf{E} . d\mathbf{l} / \int\int \mathbf{J} . d\mathbf{S} \qquad (7 - 6)$$

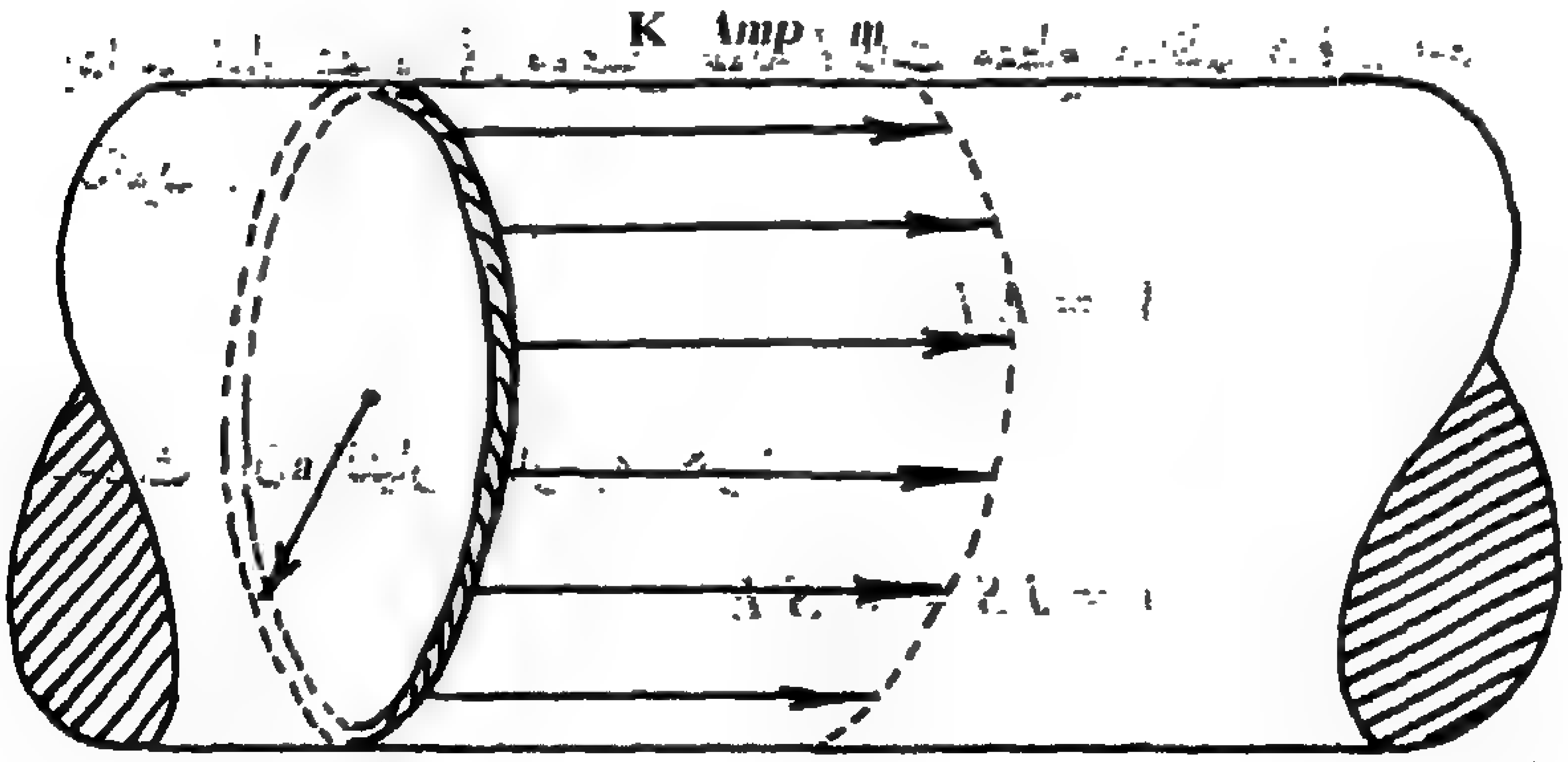
المقام في هذه المعادلة يعطي التيار الكلي المار في الموصل والبسط يعطي الجهد الكهربي بين طرفيه .

6 - 8 كثافة التيار الصفحي : (Density of Current Sheet) :

في بعض الأحيان يتركز التيار الكهربي على سطح الموصل وبناء عليه نعرف كثافة التيار الصفحي K بأنه معدل مرور الشحنات لكل وحدة طول مستعرضة . فمثلاً لموصل ذات مقطع دائري ذات نصف قطر a يحمل تيار صفحي مقداره الكلي I نجد أن :

$$I = 2 \pi a K$$

أي أن (شكل 6 - i) .



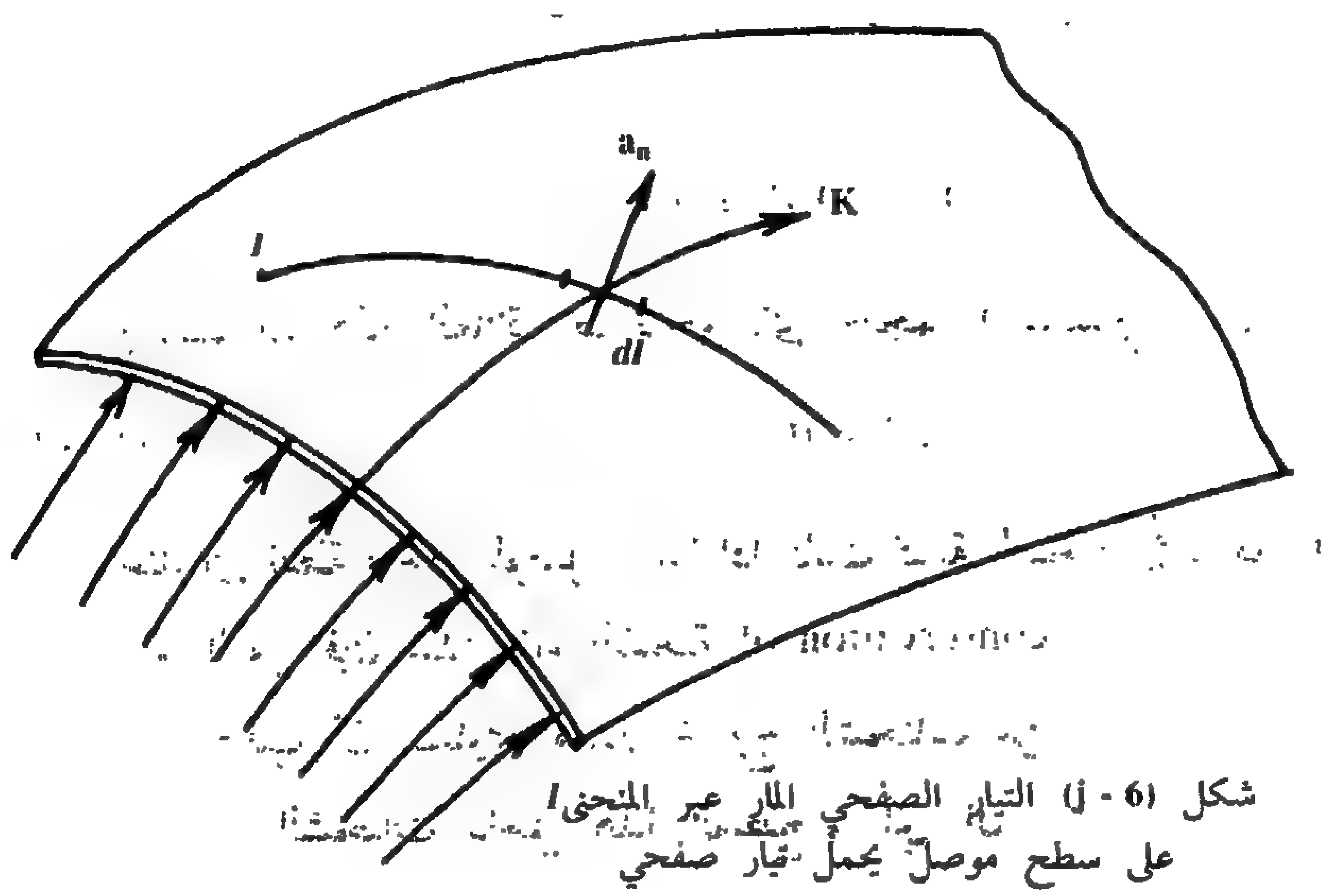
شكل (i - 6) تيار صفحي على موصل دائري المقطع

$$K = I / 2 \pi a \qquad (8 - 6)$$

وبناء عليه إذا كان هناك موصل يحمل تيار صفحي ذات كثافة K فإن التيار المار عبر خط منحنى l على سطحه هو :

$$I = \int \mathbf{K} \cdot \mathbf{a}_n \, dl \qquad (9 - 6)$$

حيث $\mathbf{a}_n$ هو متجه وحدة عمودي على dl كما هو مبين بشكل (j - 6) .



شكل (j - 6) التيار الصفحي المار عبر المنحنى l
على سطح موصل يحمل تيار صفحي

9 - 6 معادلة الاستمرارية للتيار :

(Equation of Continuity of Current)

اعتبر حجم v محدد بسطح مغلق S . إذا كانت كثافة التيار عند نقطة على السطح هي J فإن التيار المار خلال مساحة dS خارج هذا السطح هو $\mathbf{J} \cdot d\mathbf{S}$ ويكون التيار الكلي الخارج من S هو :

$$I = \oint_S \mathbf{J} \cdot d\mathbf{S} \qquad (10 - 6)$$

إذا كان معدل التغير في كمية الشحنات داخل S متلاشي (أي معدل

دخول الشحنات يساوي معدل خروجها من السطح) فإن هذا التكامل ينعدم :

$$\oint_S \mathbf{J} . d\mathbf{S} = 0 \qquad (11 - 6)$$

وهذه المعادلة ما هي إلا الصورة العامة لقانون كيرشوف للتيار . وباستخدام نظرية جاوس يمكن تحويل التكامل السطحي إلى تكامل حجمي ، أي :

$$\int_v \nabla . \mathbf{J} \, dv = 0$$

وحيث أن هذه العلاقة صحيحة لأي حجم v نستنتج أن :

$$\nabla . \mathbf{J} = 0 \qquad (12 - 6)$$

عند أي نقطة داخل الموصل . أما إذا كانت كمية الشحنة Q داخل v متغيرة مع الزمن فإن مبدأ بقاء الشحنة (principle of conservation of charge) يتطلب أن يتساوي معدل خروج الشحنات من السطح S مع معدل نقصان الشحنات داخل هذا المسطح ، أي أن :

$$\iint_S \mathbf{J} . d\mathbf{S} = - \frac{dQ}{dt}$$

$$= - \frac{\partial}{\partial t} \int_v \rho \, dv$$

ومنها يمكن أن نكتب :

$$\int_v (\nabla . \mathbf{J} + \frac{\partial \rho}{\partial t}) \, dv = 0$$

وبالتالي نجد أن :

$$\nabla . \mathbf{J} + \frac{\partial \rho}{\partial t} = 0 \qquad (13 - 6)$$

وهي تمثل معادلة الاستمرارية للتيار عند أي نقطة داخل الموصل . وتكون $\partial \rho / \partial t$ غير منعدمة في إلموصل في الحالات العابرة (transient states) . وفي الحالة المستقرة تصبح معادلة الاستمرارية للتيار على الصورة (12 - 6) .

6 - 10 تيار الازاحة : (Displacement Current) :

قانون جاوس عند أي نقطة في الوسط الكهرومغناطيسي هو :

$$\nabla . D = \rho$$

وبتفاضل الطرفين جزئياً بالنسبة للزمن نحصل على :

$$\nabla . \frac{\partial D}{\partial t} = \frac{\partial \rho}{\partial t}$$

وباستخدام معادلة الاستمرار يمكن كتابة المعادلة الأخيرة على الصورة :

$$\nabla . \left(J + \frac{\partial D}{\partial t}\right) = 0$$

من هذه المعادلة نستنتج أن وحدات $\frac{\partial D}{\partial t}$ هي وحدات كثافة التيار (Amp / m^2) وبناء عليه فهي تمثل تيار يسمى تيار الازاحة ويرمز له بالرمز J_d .

$$J_d = \frac{\partial D}{\partial t} \qquad Amp / m^2 \qquad (14 - 6)$$

ولتفهّم المعنى الطبيعي لتيار الازاحة اعتبر دائرة مكونة من مكثف ومقاومة ومنبع جهد موصلة على التوالي فإذا كان المنبع ذات تيار متردد نجد أن التيار في المقاومة هو تيار توصيل أما التيار في المكثف فهو تيار إزاحة إذ أن الالكترونات لا تمر بين لوحي المكثف وإنما يوجد مجال كهربي متغير مع الزمن بين اللوحين . ويكون تيار الازاحة بين لوحي المكثف مساوياً لتيار التوصيل في المقاومة .

6 - 11 استرخاء الشحنة : (Charge Relaxation) :

يمكن إثبات أنه في الحالة الساكنة تنعدم كمية الشحنة الكهربية داخل الموصل . وأن أي شحنة إذا وضعت داخل الموصل فإنها تطفو على سطحه بعد زمن يتراوح بين 10^{-16} إلى 10^{-19} ثانية للمواد جيدة التوصيل للكهرباء .

بالتعويض من العلاقتين $\nabla . D = \rho$ و $J = \sigma E$ في المعادلة الاستمرارية للتيار نحصل على :

$$\frac{\partial \rho}{\partial t} = -(\sigma / \epsilon_0)\rho$$

وحل هذه المعادلة هو :

$$\rho = \rho_0 \exp\left(-\frac{\sigma}{\epsilon_0} t\right) \qquad (15 - 6)$$

حيث ρ_0 هي كثافة الشحن داخل الوسط الموصل عند نقطة ما عند الزمن $t = 0$. ويسمى الزمن اللازم لتنقص كثافة الشحن عند نقطة ما من قيمتها الابتدائية ρ_0 إلى ρ_0 / e (حيث $e = 2.718$) بزمن الاسترخاء τ (relaxation time) حيث :

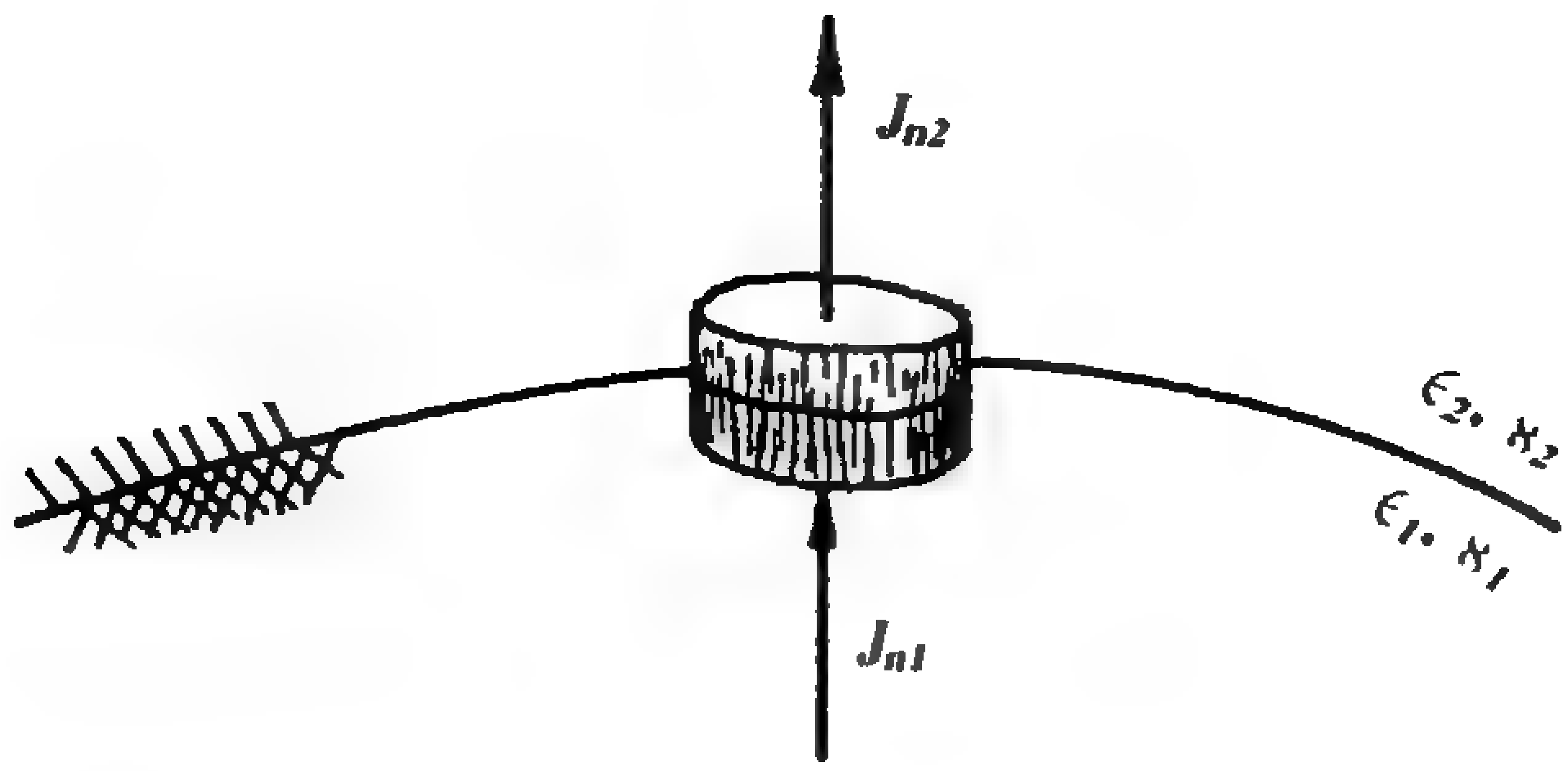
$$\tau = \epsilon_0 / \sigma \qquad sec \qquad (16 - 6)$$

وهو زمن صغير جداً بحيث يمكن أن يقال أن الشحنة داخل الموصل تنعدم تماماً في جميع الأوقات . أما بالنسبة للمواد العازلة فإن زمن الاسترخاء τ يقاس بالساعة أو باليوم وقد يكون لانهائي في الأوساط العازلة المثالية حيث تنعدم الموصلية σ .

6 - 12 الشروط الحدية للتيار الثابت بين وسطين :

(Boundary Conditions For Steady Currents Between Two Media):

اعتبر وسطين 1 ، 2 ذات سماحية وموصلية (σ_1 و ϵ_1)



شكل (k - 6) التيار العمودي بين وسطين موصلين

و ($\aleph'_2$ و ϵ_2) على التوالي كما هو مبين بشكل (k - 6) . في حالة ثبات التيار نجد أن $\nabla.J = 0$ وعليه يمكن أن نطبق هذه العلاقة على سطح أسطواني صغير مغلق وذات ارتفاع متناهي الصغر يقع سطحه العلوي في الوسط 2 سطحه السفلي في الوسط 1 .

$$\iint_S J \, . \, dS = 0$$

$$- J_{n1} \, \delta S + J_{n2} \, \delta S = 0$$

$$J_{n1} = J_{n2} \qquad (17 - 6)$$

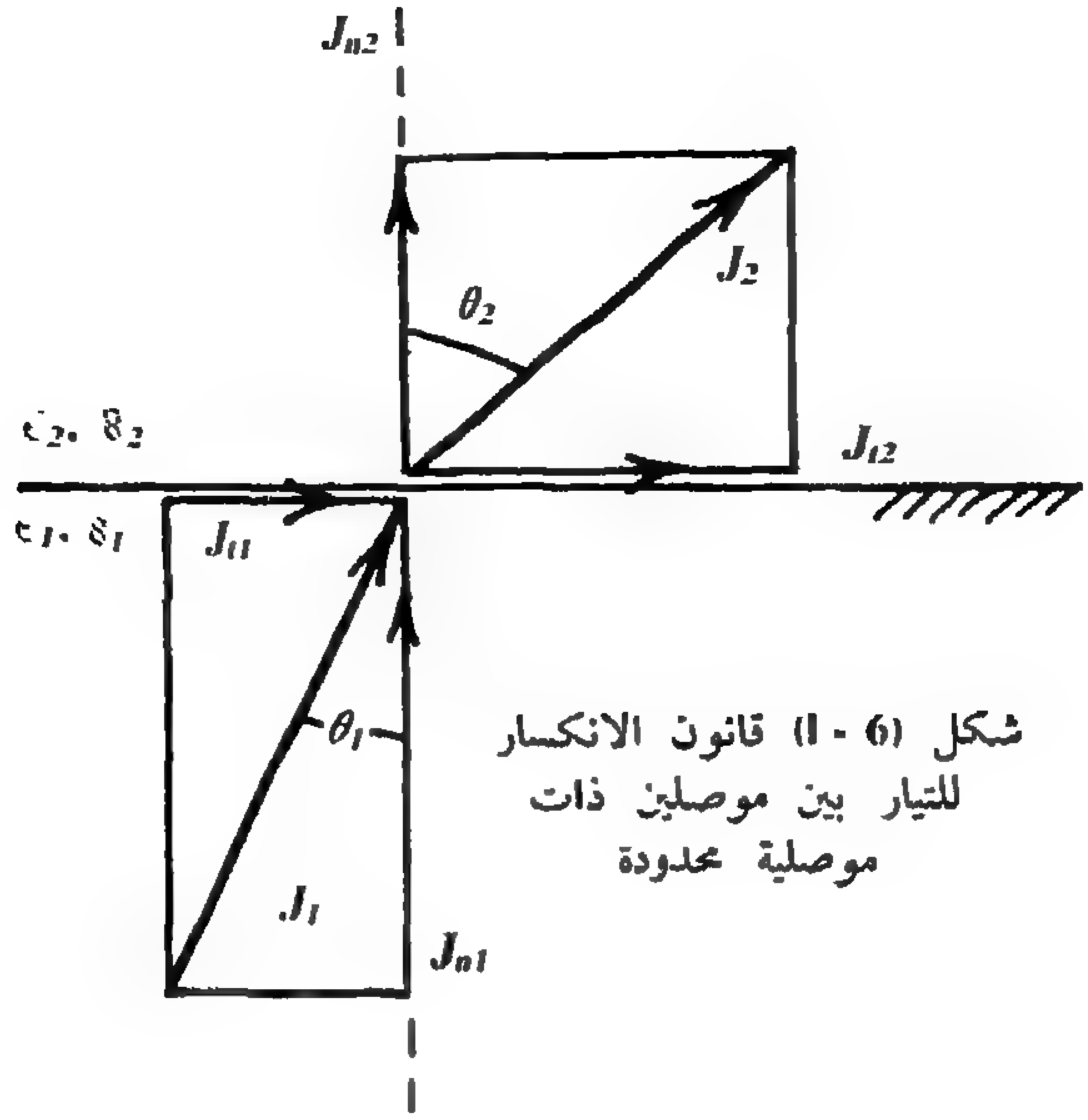
ومنها نستنتج أن مركبة التيار العمودية على السطح الفاصل بين الوسطين مستمرة . وحيث أن مركبة المجال الكهربي المماسة للسطح الفاصل مستمرة على جانبي هذا السطح .

$$E_{t1} = E_{t2} \qquad (18 - 6)$$

وباستخدام قانون أوم عند نقطة نجد أن :

$$\frac{1}{\aleph_1} J_{t1} = \frac{1}{\aleph_2} J_{t2} \qquad (19 - 6)$$

ويبين شكل (l - 6) قانون الانكسار لخطوط التيار المستمر بين الوسطين ومنها يتضح أن :



شكل (6 - l) قانون الانكسار
للتيار بين موصلين ذات
موصلية محدودة

$$\tan \theta_1 = J_{t1} / J_{n1}, \quad \tan \theta_2 = J_{t2} / J_{n2}$$

أي أن :

$$\aleph_2 \tan \theta_1 = \aleph_1 \tan \theta_2 \qquad (20 - 6)$$

فإذا كان الوسط الثاني عازل مثالي ، $\aleph_2 = 0$ ، نستنتج أن :

$$\tan \theta_1 = \infty, \quad \theta_1 = \frac{\pi}{2}$$

أي أن التيار عند السطح الفاصل يكون مماس لهذا السطح .

ويمكن تطبيق العلاقة $\nabla . D =$ ؟ على سطح جاوس الأسطواني المبين بشكل (k - 6) لاثبات أن :

$$D_{n2} - D_{n1} = \sigma \qquad (21 - 6)$$

أي أن الفرق في المركبة العمودية لكثافة الفيض الكهربي خلال السطح

الفاصل يساوي كثافة الشحنة السطحية عليه . ويمكن كتابة المعادلة الأخيرة على الصورة :

$$\frac{1}{\gamma_2} J_{n2} - \frac{1}{\gamma_1} J_{n1} = \sigma / \epsilon_0 \qquad (22 - 6)$$

ويلاحظ أنه على وجه العموم تتواجد شحنة سطحية على السطح الفاصل بين وسطين ذات موصلية محدودة وذلك عند مرور تيار كهربي ثابت بين هذين الوسطين .

6 - 13 مسائل محلولة

SOLVED PROBLEMS

1. *Figure 6.1. shows a toroidal sector of rectangular cross-section and uniform conductivity γ. Show that the resistance between the two rectangular ends is given by*

$$R = \theta / \gamma t \log (b/a)$$

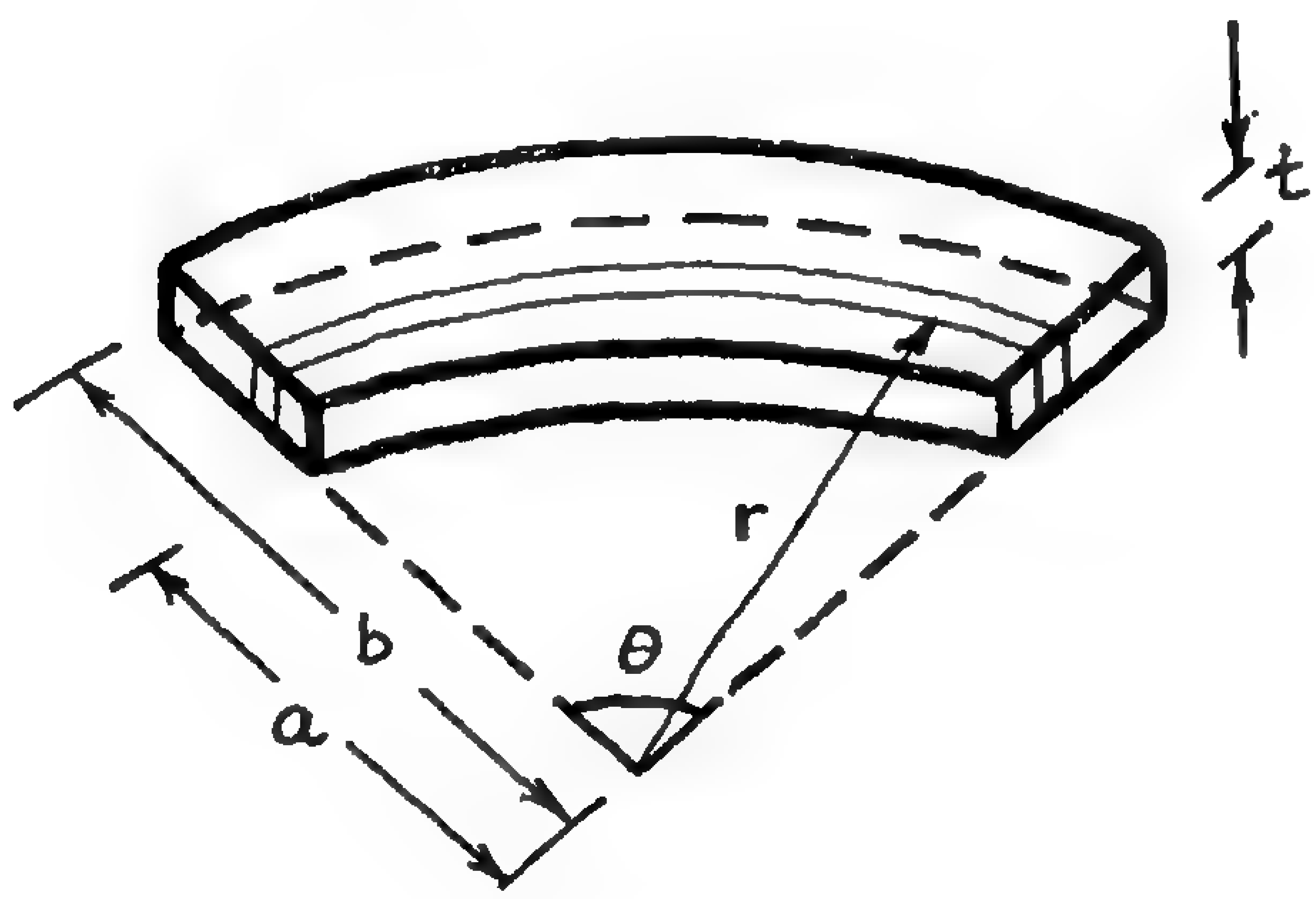


Fig. 6.1.

The resistance of an element of cross-section area $t\ dr$ and length $r\ \theta$ is

$$dR = r\ \theta \ / \ \gamma\ t\ dr$$

The total resistance between the two rectangular ends is the summation of all such resistive elements in parallel, that is

$$1/R = \int 1/dR$$

$$= (\gamma t/\theta) \int_a^b dr/r$$

$$= (\gamma t/\theta) \log (b/a)$$

and $$R = \theta/\gamma t \log (b/a)$$

2. *A metal hemisphere of negligible resistivity and radius a is embedded in the ground with its flat part level with the ground surface. If the conductivity of the ground is γ, show that the resistance beteen the hemisphere and ground is $1/2\pi\gamma a$. If a current I flows from the sphere to ground and a man is standing at a distance d from the center of the hemisphere, show that if the man takes a step of length l towards the ground electrode, then the potential difference between his feet is given by*

$$V = I\, l \,/\, 2\pi\, \gamma\, d\,(d-l)$$

Find V If $I = 1000\ A$, $l = 0.8\ m$, $d = 2m$, $\gamma = 10^{-3}\ \Omega/m$.

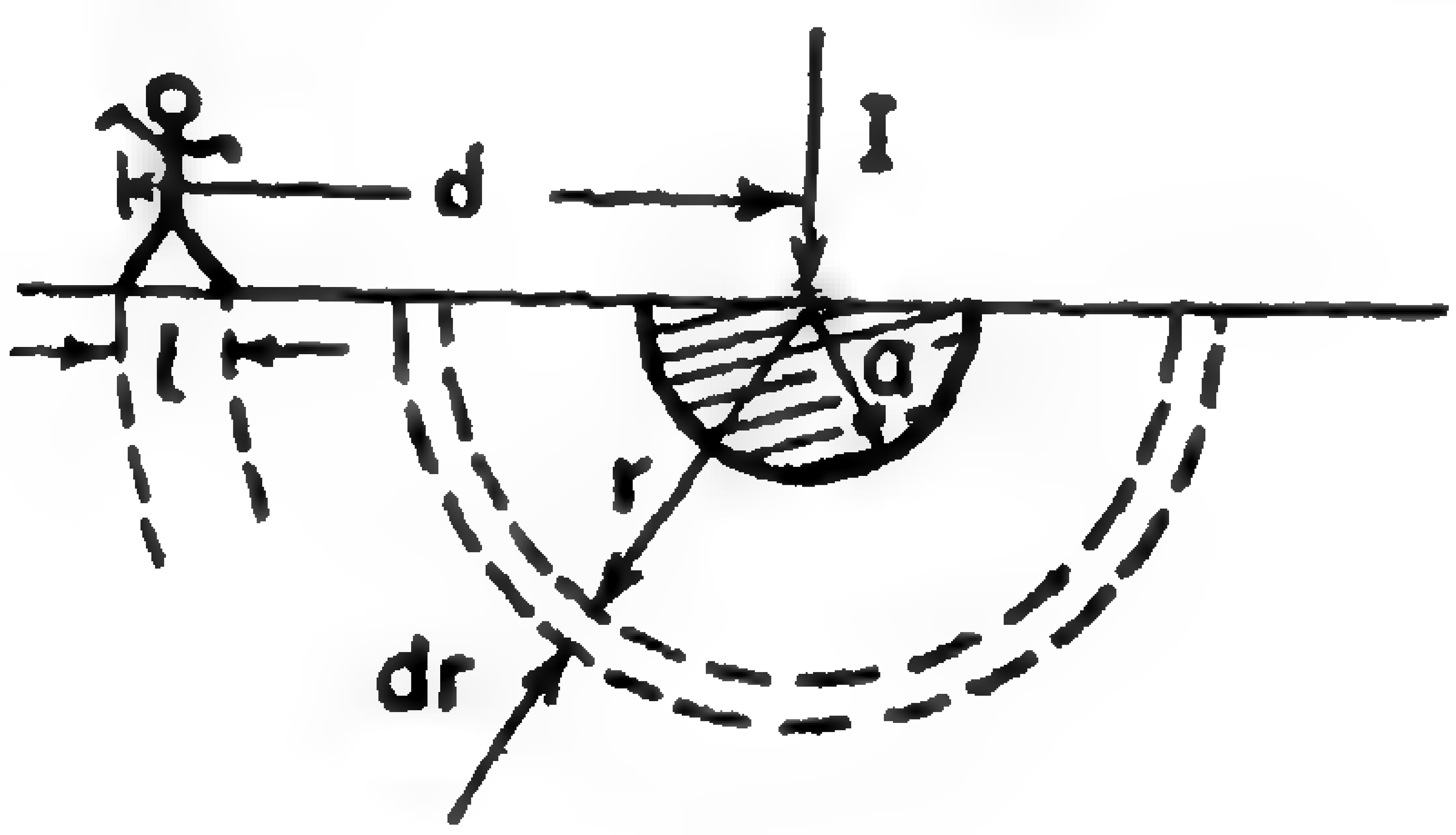


Fig. 6.2.

Since the hemispherical electrode is a perfect conductor, current will flow radially outwards from its surface. The resistance of a hemisphere of radius r and thickness dr (Fig. 6.2) is,

$$dR = dr/\gamma\, 2\, \pi\, r^2$$

Hence the resistance between the hemisphere and ground is given by

$$R = (1/2\pi\gamma) \int_a^\infty dr/r^2$$

$$= 1/2\pi\gamma a$$

The resistance between the man's feet is

$$R = (1/2\pi\gamma) \int_{d-l}^{d} dr/r^2$$

$$= l/2\pi\gamma d\,(d-l)$$

The potential difference is evidently IR. For the numerical values given in the problem we find that

$$V = 5.3\ kV$$

3. *A spherical electrode of radius a and negligible resistivity is embedded in an infinite medium of conductivity γ and permittivity ϵ. If a current I leaves the electrode, show that the total charge on the electrode surface is $\epsilon I/\gamma$.*

At the boundary between two media across which a current is flowing we have the following condition,

$$(\epsilon_2/\gamma_2 - \epsilon_1/\gamma_1)\,\mathbf{j}\cdot\mathbf{a}_n = \sigma$$

where $\mathbf{a}_n$ is the unit normal to the interface directed from medium 1 to medium 2. Applying the above condition to the present problem we have that γ_1 for the sphere is infinite so that

$$\sigma = (\epsilon/\gamma)\,j$$

The total charge over the surface of the shere is therefore

$$Q = 4\pi a^2\sigma = 4\pi a^2 j\,(\epsilon/\gamma)$$

$$= \epsilon I/\gamma$$

Alternative solution.

The resistance between the sphere and ground is evidently one half the resistance between the hemisphere and ground found in problem 2. Thus

$$R = 1/4\pi\gamma a$$

If Q is the charge on the sphere, its protential V is $Q + \pi\epsilon a$. Also

$$V = IR$$

$$Q/4\pi\epsilon a = I/4\pi\gamma a$$

$$Q = \epsilon I/\gamma$$

4. *Two spherical electrodes of radius a and negligible resistivity are embedded in a medium of infinite extent and conductivity γ; the separation d between their centers is large compared with a. If a current enters one sphere and leaves at the other, show that the resistance between the spheres is given by*

$$R = (d-a)\ /\ 2\pi\gamma da$$

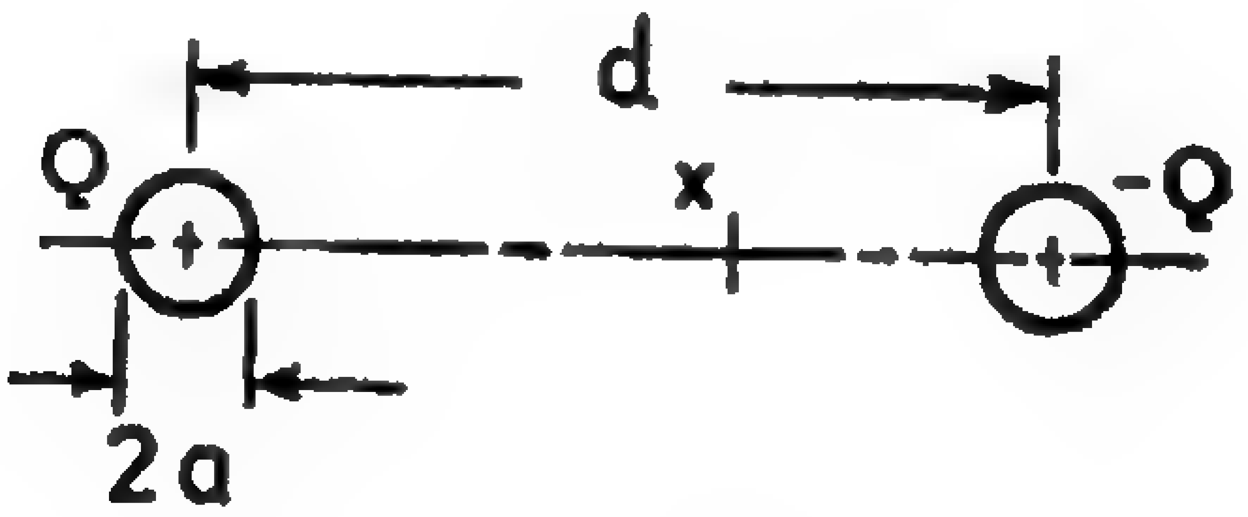


Fig. 6.3.

Since $d >> a$ we may assume that the current leaving and entering the sphere surfaces has a uniform density and that the charges on the spheres are $\pm Q = \pm I\epsilon/\gamma$ (see previous problem). The electric field intensity at a point on the line joining the sphere centers and at a distance x from the positive sphere is (Fig. 6.3),

$$E = Q/4\pi\epsilon\ [1/x^2 - 1/(d-x)^2]$$

The potential difference between the spheres is given by,

$$V = -\int_{d-a}^{a} E\,dx = Q/2\pi\epsilon\,[1/a - 1/(d-a)]$$

and this must be equal to IR where R is the resistance between the spheres. Substituting for Q and with $(d-a) \simeq d$ we obtain

$$R = (d-a) \,/\, 2\pi\gamma d a$$

5. *Two hemispherical electrodes of radius a are buried in a medium of conductivity γ as shown in Fig. 6.4. If the distance d between the electrodes is much larger than their radius, find the resistance between the electrodes.*

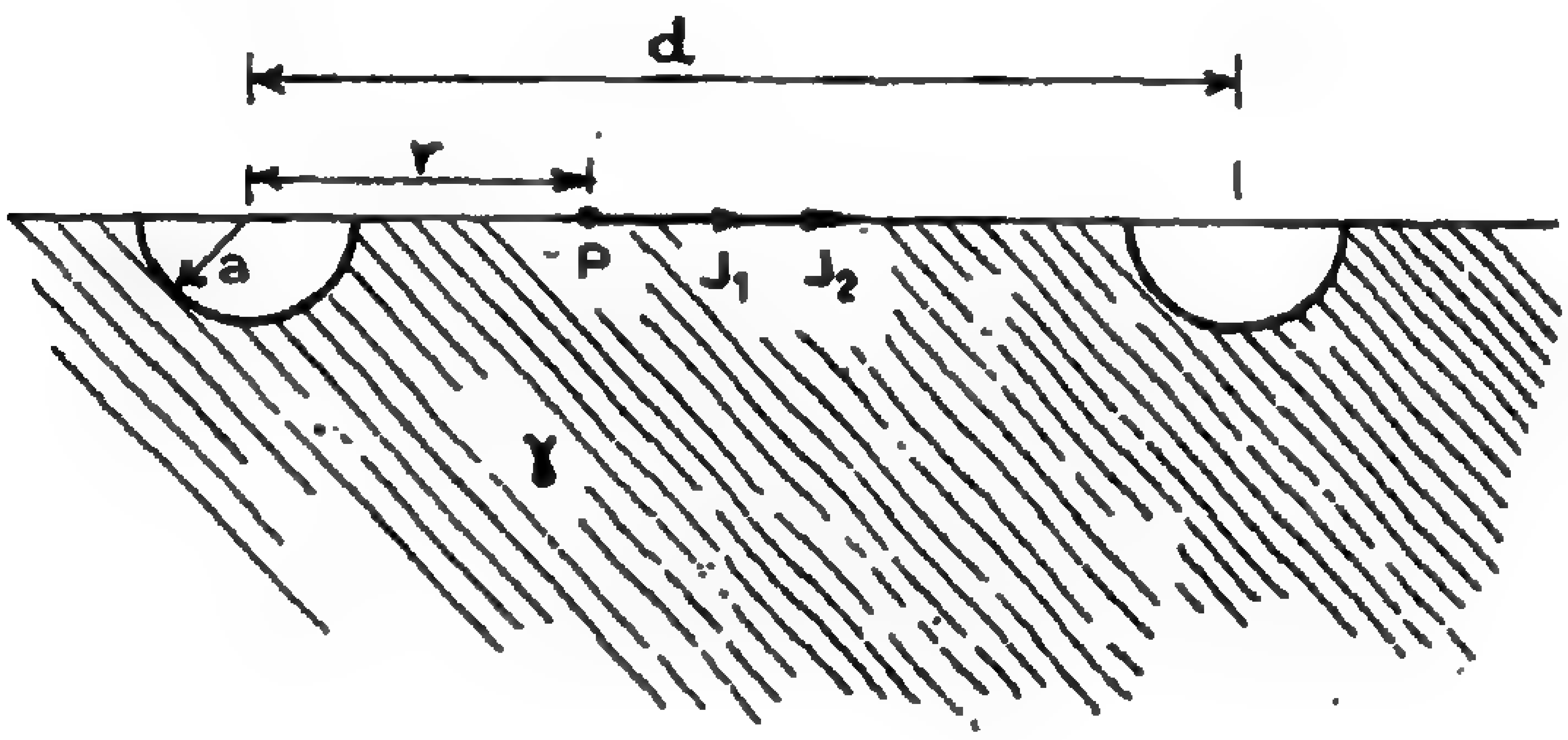


Fig. 6.4.

Suppose that a current I flows between the electrodes. At points on the surface of the earth and along the line joining the electrode centers, the current density vectors are parallel and in the same direction so that they add algebraically. The current densities at the point P are,

$$j_1 = I/2\pi r^2, \qquad j_2 = I/2\pi (d-r)^2$$

The potential difference between the electrodes is $(\bar{j} = \gamma \bar{E})$,

$$V = V_1 - V_2 = -\int_{d-a}^{a} E\, dr$$

$$= (I/2\pi\gamma) \int_{a}^{d-a} [1/r^2 + 1/(d-r)^2]\, dr$$

$$\simeq I/\pi\gamma a \qquad (d >> a)$$

The resistance between the electrodes is

$$R = V/I = 1/\pi\gamma a$$

Note. The potential difference between two spherical electrodes of radius a and charges $\pm Q$ at a distance $d >> a$ between centres and embedded in a medium of permittivity ϵ and of infinite extent is $V = Q/2\pi\epsilon a$. Hence the hemispheres may be replaced by spheres in a medium of infinite extent carrying charges $\pm 2\epsilon I/\gamma$. We shall make use of this result in the next problem.

6. *Two hemispherical grounding electrodes, each of radius a, are sunk into the surface of the earth at a distance $d >> a$ from each other and with their flat surfaces level with the ground. A second pair of electrodes, identical with the first, is sunk into the surface of the ground at a distance $h >> a$ from the first, so that the four electrodes form a rectangle of length d and width h (Fig. 6.5). If a potential difference V_{12} is applied between the first pair of electrodes, show that the potential difference which appears between the second pair is*

$$V_{34} = V_{12}\, a\, [1/h - 1/(h^2+d^2)^{1/2}]$$

As shown in the previous problem the hemispheres may be replaced by sphered carrying charges $\pm Q = \pm 2\epsilon I/\gamma$. The potentials

produced by these charges at points 3 and 4 are,

$$V_3 = (Q/4\pi\epsilon)\left[1/h - 1/(h^2+a^2)^{1/2}\right]$$

$$V_4 = (Q/4\pi\epsilon)\left[1/(h^2+a^2)^{1/2} - 1/h\right]$$

so that

$$V_{34} = V_3 - V_4 = (Q/2\pi\epsilon)\left[1/h - 1/(h^2+a^2)^{1/2}\right]$$

but since

$$V_{12} = Q/2\pi\epsilon a$$

it follows that

$$V_{34} = V_{12}\, a\left[1/h - 1/(h^2+d^2)^{1/2}\right]$$

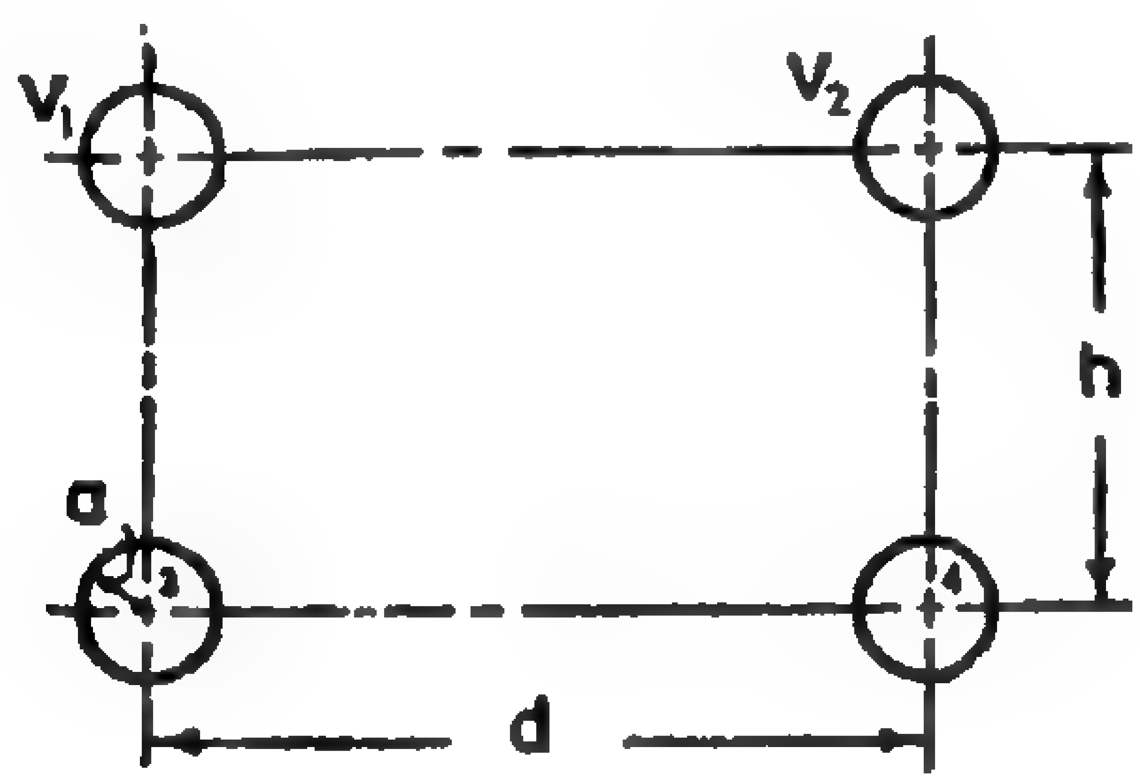


Fig. 6.5.

7. *A thin spherical shell of radius a and thickness t has a conductivity γ. Current enters and leaves the shell by two small circular electrodes of radius $b << t$ and negligible resistivity placed at diametrically opposite points of the shell. Show that the resistance of the shell between the electrodes is given by*

$$R = (1/\gamma\pi t)\log(2a/b)$$

The arrangement is shown in Fig. 6.6. Consider the resistance of an element between $\theta = \theta$ and $\theta = \theta + d\theta$. The length of the

element is $a\ d\theta$ and its cross-sectional area is $2\pi\ a \sin\theta t$. The resistance between the electrodes is therefore

$$R = (1/2\pi\ \gamma t) \int_{b/a}^{\pi - b/a} d\theta \ / \sin\theta$$

$$= (1/2\pi\ \gamma\ t) \left[\log \tan \tfrac{1}{2}\theta\right]_{b/a}^{\pi - b/a}$$

Since $b/2a$ is very small we may write,

$$\log \tan (\tfrac{1}{2}\pi - b/2a) = \log \cot b/2a$$

$$= -\log \tan b/2a \simeq -\log b/2a$$

and hence it follows that,

$$R = (1/\pi\ \gamma t) \log (2a/b)$$

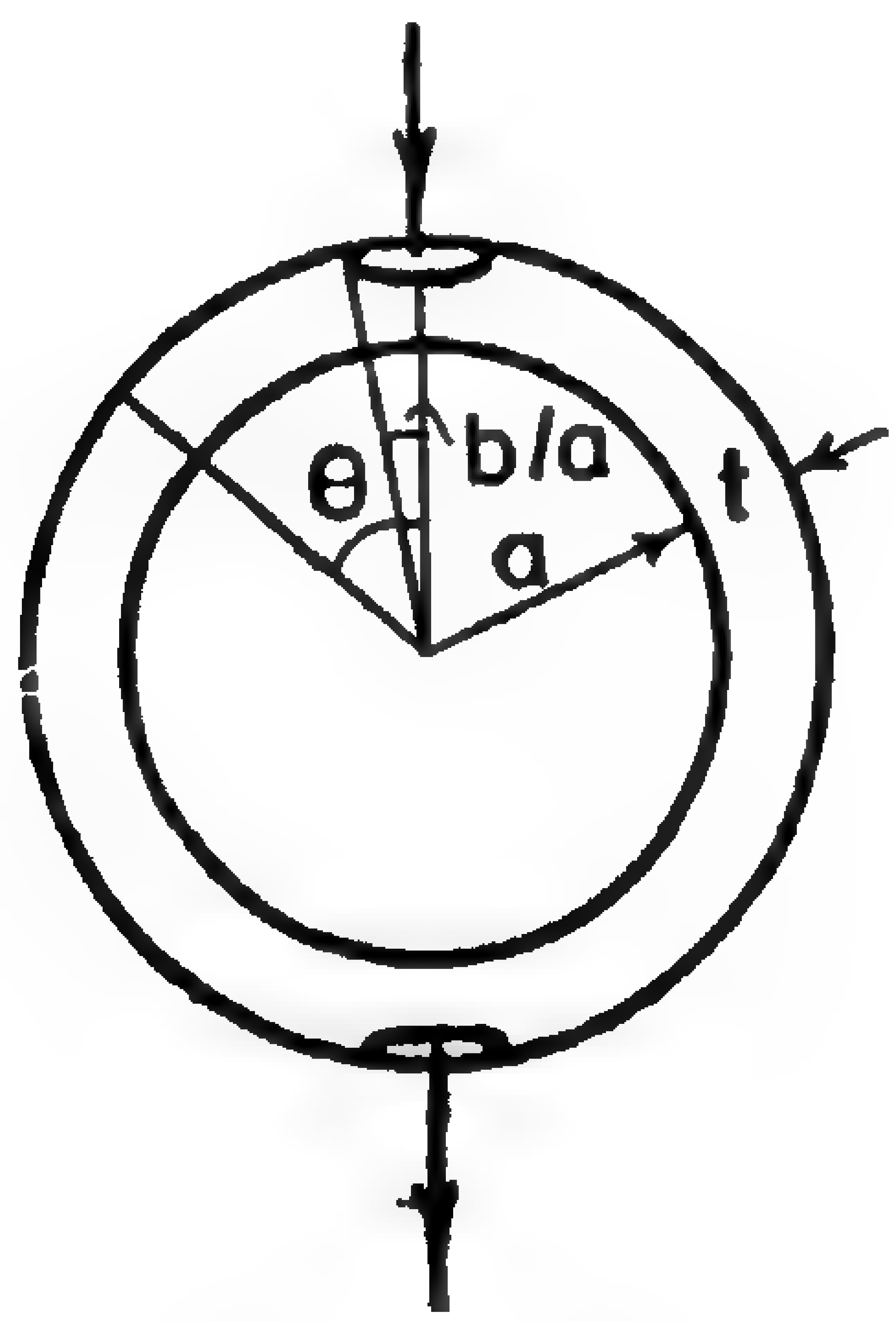


Fig. 6 6.

8. *Figure 6.7 shows the sector of a hollow conducting sphere of internal radius a and external radius b, and conductivity γ. Show that the resistance between the two spherical ends of the sector is given by*

$$R = (b-a) \ / \ ab\ 2\pi\ \gamma\ (1-\cos\theta)$$

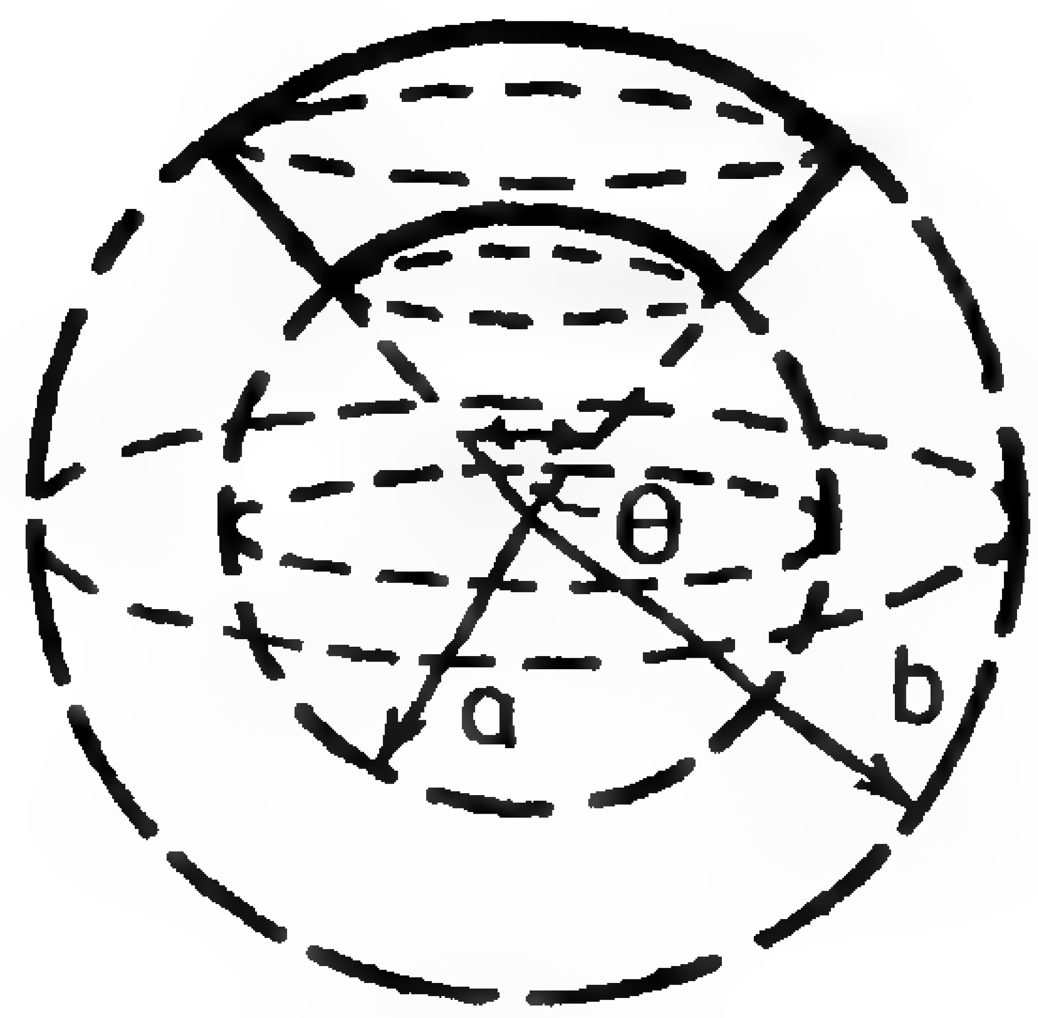


Fig. 6 7.

The area of a spherical cap of radius r and subtending an angle θ at the centre is $2\pi r^2 (1 - \cos \theta)$. The resistance between the two spherical caps is therefore

$$R = [1/2\pi \gamma (1 - \cos \theta)] \int_a^b dr/r^2$$

$$= (b-a) / ab\, 2\pi \gamma (1 - \cos \theta)$$

9. *A Faraday disc of thickness t has a radius b and a central hole of radius a (Fig. 6.8). If the conductivity of the disc material is γ, show that the resistance between the hole and the rim of the disc is given by*

$$R = (1/2\pi \gamma t) \log (b/a)$$

The resistance of an element of length dr and cross-sectional area $2\pi r t$ is,

$$dR = dr/2\pi \gamma t r$$

and the total resistance is evidently

$$R = \int_a^b dR = (1/2\pi\gamma t) \log (b/a)$$

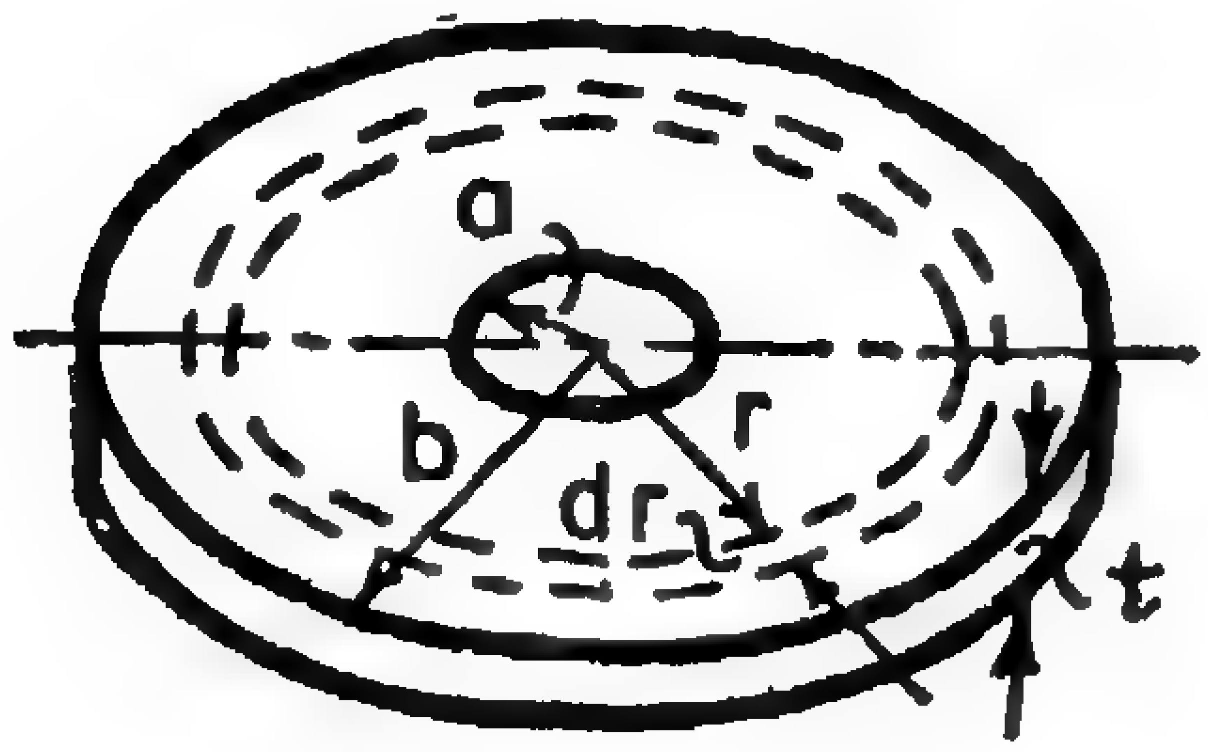


Fig. 6.8.

10. *A conductor carries a steady current of density* $\mathbf{j}$. *The x and y components of* $\mathbf{j}$ *at any point are given by* $j_x = 3\,az$, $j_y = 2\,by$. *Find the z – component of the current density.*

[For steady current flow the divergence of $\mathbf{j}$ is zero, that is

$$\partial j_x/\partial x + \partial j_y/\partial y + \partial j_z/\partial z = 0$$

Also,

$$\partial j_x/\partial x = 0$$

$$\partial j_y/\partial y = 2b$$

so that

$$\partial j_z/\partial z = -2b$$

$$j_z = -2bz + f(x, y)$$

11. *A cylindrical coaxial cable carries a current from a battery of voltage V located at z = 0 to a load resistance R and back (Fig. 6.9). Show that the potential in the space between the central wire and the sheath is*

$$V_r(z) = \{ [1 - R_a z/lR_t] \log b/r + (R_- z/lR_t) \log r/a \} V/\log (b/a)$$

where l is the length of the cable, R_a is the resistance of the central wire, R_b is the resistance of the sheath and $R_t = R_a + R_b + R$.

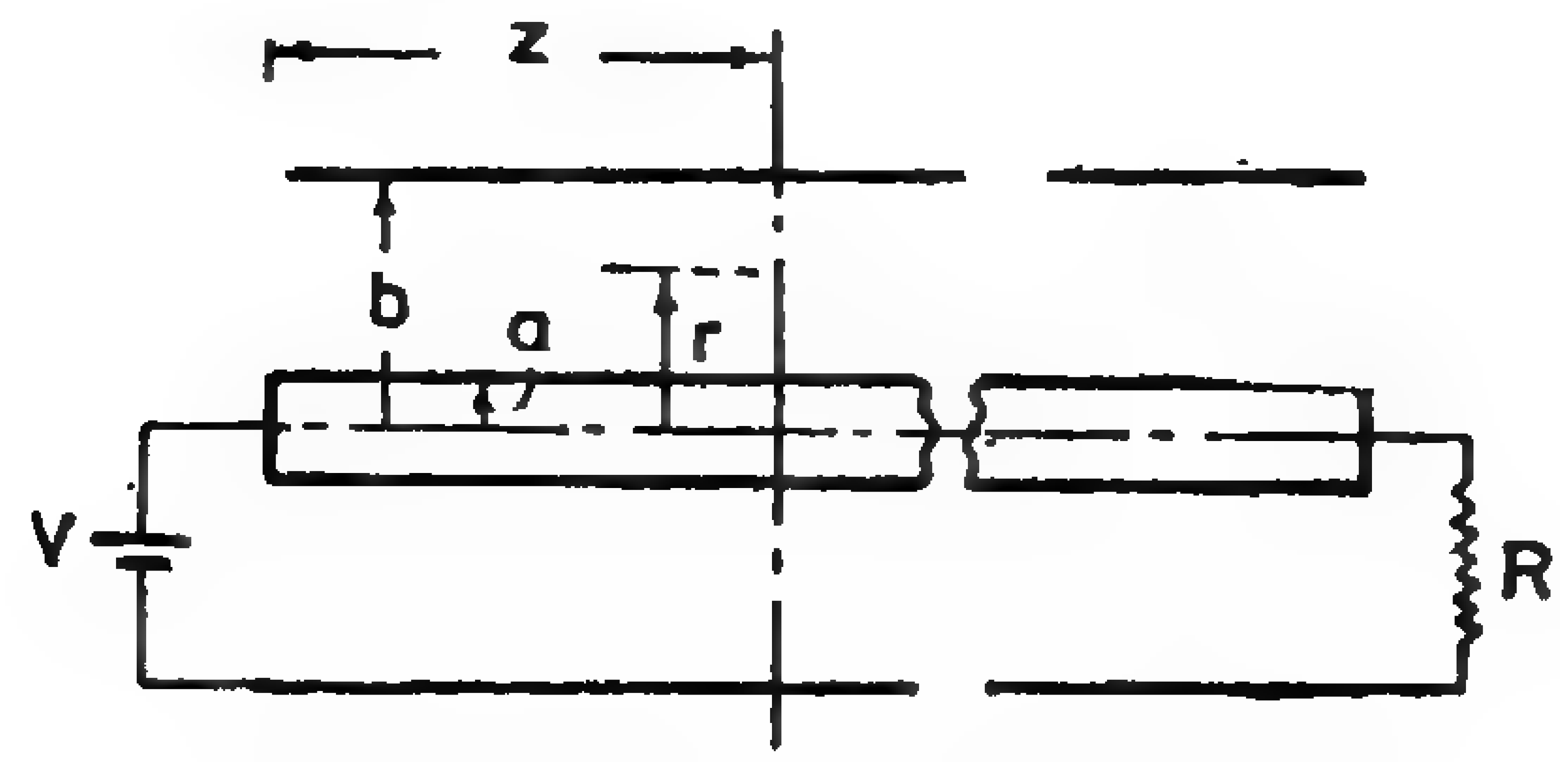


Fig. 6.9

The current is evidently $I = V/R_t$. Consider a section of the cable at a distance z from the battery. The potential of the core is

$$V_1 = V - V R_a z / l R_t \tag{1}$$

and the potential of the sheath is

$$V_2 = V R_b z / l R_t \tag{2}$$

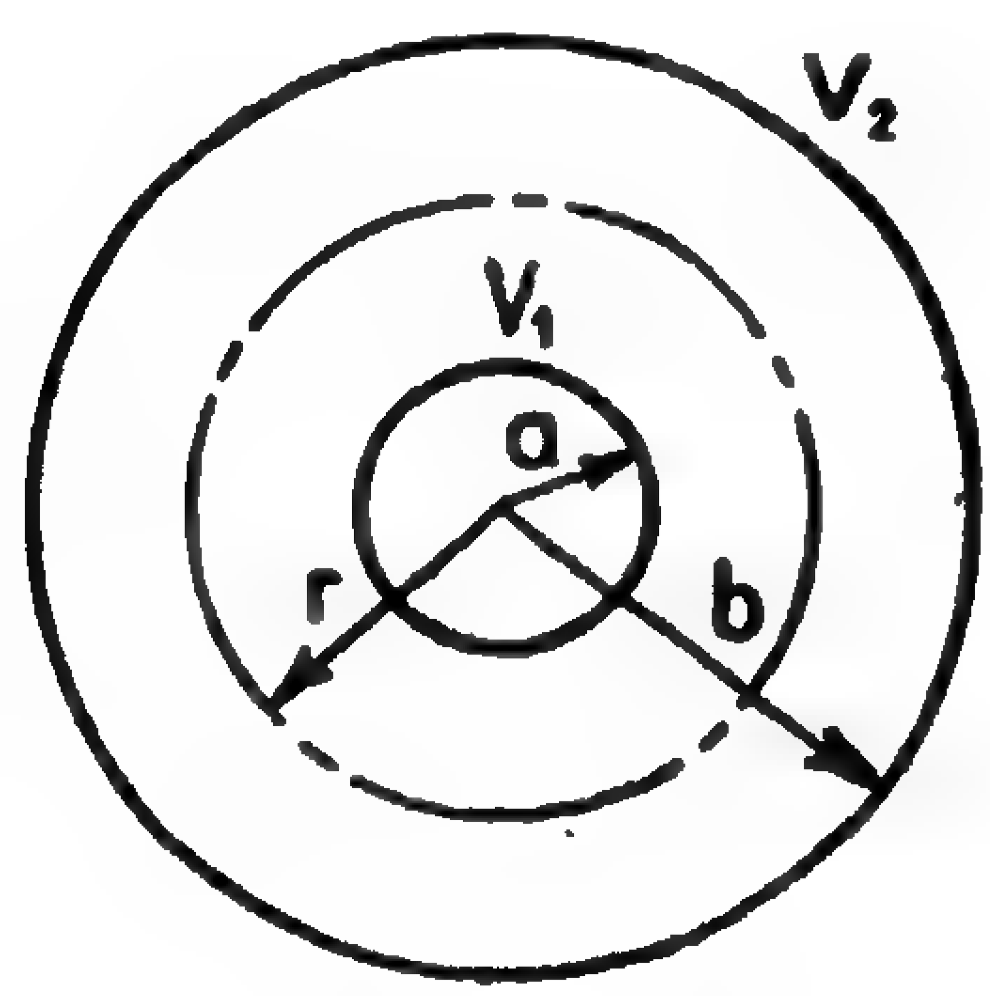


Fig. 6.10

Assuming that the cable is long and that the resistances R_a and R_b are sufficiently small so that V_1 and V_2 do not change appreciably with distance, the problem is reduced to that of finding the potential

at a distance r from the center of a concentric cable as shown in Fig. 6.10. If Q is the charge per unit length of the core we have that

$$V_r - V_1 = (Q/2\pi\epsilon) \log r/a$$

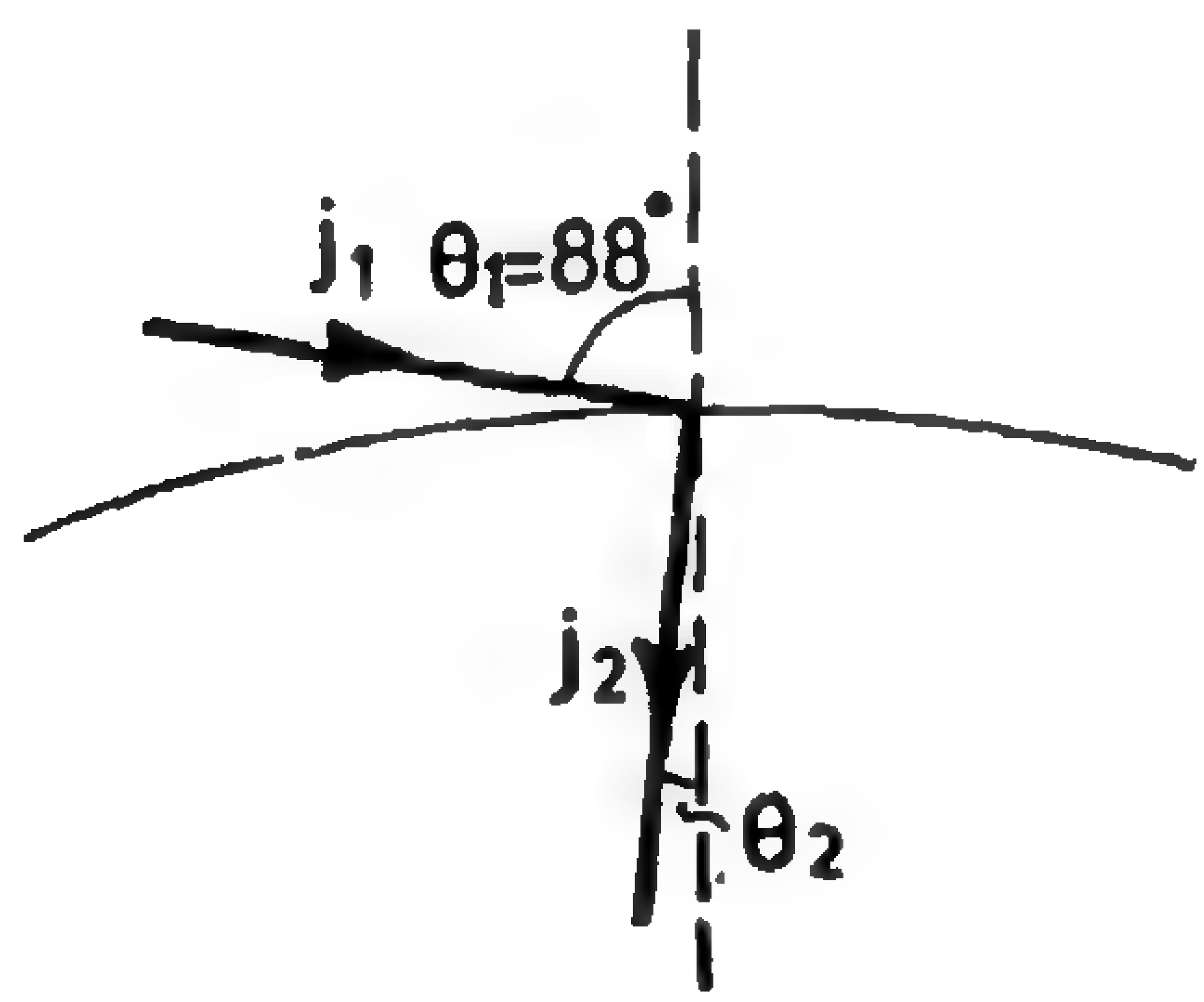
$$V_2 - V_1 = (Q/2\pi\epsilon) \log b/a$$

and hence

$$V_r = V_1 + (V_2 - V_1) \log (r/a) / \log (b/a)$$

Substituting for V_1 and V_2 from equations (1) and (2) and rearranging terms we obtain the required result.

12. *The current density at a point P lying in region 1 on the boundary between two regions 1 and 2 makes an angle $\theta_1 = 88^\circ$ with the normal at that point (Fig. 6.11). Find the angle between the current density vector and the normal at the point immediately opposite P in region 2 if $\gamma_1 = 50\,\gamma_2$ and if $\gamma_1 = 500\,\gamma_2$. What practical use can be made of this result ?*



Fi~. 6.11.

At the boundary surface between two media of conductivities γ_1 and γ_2 the current density satisfies the equation,

$$\gamma_1 \cot \theta_1 = \gamma_2 \cot \theta_2$$

where θ_1 and θ_2 are the angles between $\mathbf{j}_1$ and $\mathbf{j}_2$ and the normal to the boundary surface at the point under consideration.

If $\gamma_1 = 50\ \gamma_2$, we find that $\theta_2 = 29.8^o$ and for $\gamma_1 = 500\ \gamma_2$ we find that $\theta_2 = 3.28^o$. If region 2 is a perfect insulator, *i.e.* $\gamma_2 = 0$ then at the interface between a conductor and such an insulator there can only be a tangential flow of current in the conductor. If $\gamma_1 >> \gamma_2$ then for all values of $\theta_1 \neq 90^o$, θ_2 is extremely small. For example for $\gamma_1 = 500\ \gamma_2$ we have found that for $\theta_1 = 88^o$, $\theta_2 = 3.28^o$. Thus the current lines in medium 2 are perpendicular to the interface which is therefore an equipotential surface. A practical case is that in which medium 1 is a conductor and medium 2 is an electrolyte; the ratio γ_1/γ_2 is of the order of 10^6 so that the conductor surface is, to a very good approximation, an equipotential. Now for steady current flow we have that

$$\nabla \cdot \mathbf{j} = \nabla \cdot \gamma \mathbf{E} = 0$$

and since $E = -\nabla V$, it follows that, for a homogeneous and isotropic medium, $\nabla^2 V = 0$.

Hence in a stationary electric conduction field the potential V satisfies Laplace's equation. This means that problems involving distributions of steady currents in a homogeneous conducting medium may be solved in the same way as problems involving static field distributions in homogeneous insulating media. If the current enters and leaves the electrodes at right angles to their surfaces, these surfaces will be equipotentials. The potential distribution in the medium between the electrodes will be the same as in a capacitor whose plates are formed by the two conductors. The electric field lines, which are everywhere orthogonal to lines of constant V, are identical to lines of current flow. This fact forms the basis of operation of the *electrolytic tank* analogue which is used to obtain experimentally the potential and field distributions for problems which are too difficult or impossible to solve analytically.

13. *The dielectric of a parallel-plate capacitor of plate area S consists of two plane slabs of thicknesses d_1 and d_2, permittivities ϵ_1 and ϵ_2, and conductivities γ_1 and γ_2 (Fig. 6.12). Find the leakage resistance between the electrodes. If a voltage V is applied to the capacitor determine the electric field intensity, the displacement vector, the current density in both slabs and the densities of free and polarization charges on the three boundary surfaces.*

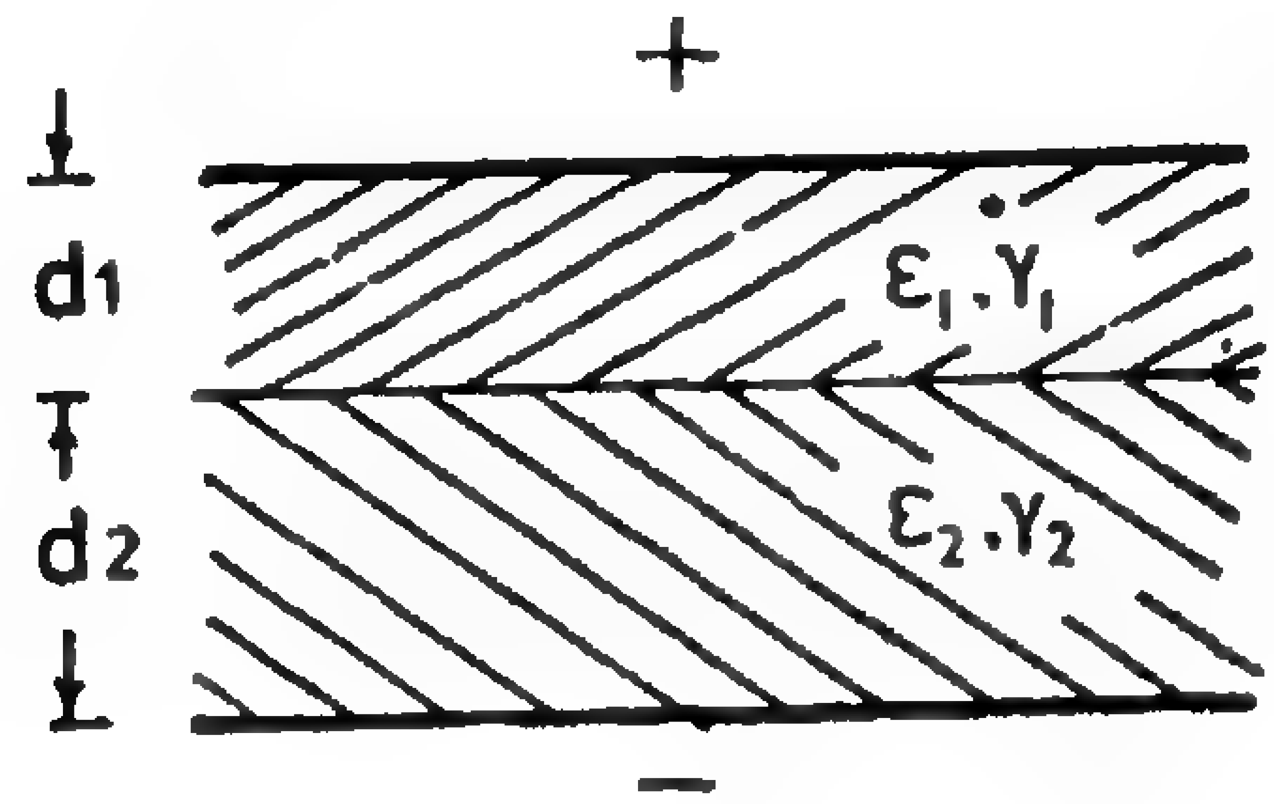


Fig. 6.12.

It is evident that the total leakage resistance is the sum of the resistance of each slab (slabs are in series), that is,

$$R = d_1/\gamma_1 S + d_2/\gamma_2 S$$

With a voltage V between the electrodes a current $I = V/R$ will flow through both slabs. The current density is,

$$j = I/S = \gamma_1\gamma_2 V / (d_1\gamma_2 + d_2\gamma_1)$$

Since this density is normal to the plates and to the boundary surface, it is the same in the two slabs (continuity of normal component of current density). From the point form of Ohm's law ($j = \gamma E$) we have that,

$$E_1 = j/\gamma_1 = \gamma_2 V / (d_1\gamma_2 + d_2\gamma_1)$$

$$E_2 = j/\gamma_2 = \gamma_1 V / (d_1\gamma_2 + d_2\gamma_1)$$

From the constitutional relationship $D = \epsilon E$ we find

$$D_1 = \epsilon_1 \gamma_2 V / (d_1 \gamma_2 + d_2 \gamma_1) \ , D_2 = \epsilon_2 \gamma_1 V / (d_1 \gamma_2 + d_2 \gamma_1)$$

It will be noted that D is not the same in the two slabs, as it would be if there were no flow of conduction current (perfect dielectrics). At the positive and negative plates the surface densities of free charges are D_1 and $-D_2$ respectively. At the interface we have that

$$\sigma = D_{n2} - D_{n1} = D_2 - D_1 = V\,(\epsilon_2 \gamma_1 - \epsilon_1 \gamma_2) / (d_1 \gamma_2 + d_2 \gamma_1)$$

The polarization charge densities at the three surfaces are ($P = D - \epsilon_0 E$)

$$\sigma_1 = -(\epsilon_1 - \epsilon_0)\, E_1$$

$$\sigma_{b2} = (\epsilon_2 - \epsilon_0)\, E_2$$

$$\sigma_{12} = (\epsilon_1 - \epsilon_0)\, E_1 - (\epsilon_2 - \epsilon_0)\, E_2$$

6 - 14 مسائل إضافية

SUPPLEMENTARY PROBLEMS

1. In the sector shown in Fig. 6.1 if $\theta = 90^o$ show that the resistance between the two horizontal surfaces is

$$R = 4\,t \,/\, \gamma\pi\,(b^2 - a^2)$$

2. For the sector of the previous problem show that the resistance between the two curved surfaces is

$$R = 2\,\log\,(b/a) \,/\, \gamma\,\pi\,t$$

3. A conducting circular cylinder of resistivity δ, inner radius a, outer radius b, and length L is cut in half along its length. The two halves are then brought together with two thin sheet electrodes inserted between the two halves. Each sheet is in contact with the entire length of the cylinder. Show that the resistance of the cylinder between the electrodes is

$$R = \pi\,\delta \,/\, 2L\,\log\,(b/a)$$

4. A coaxial cable contains an insulating material of conductivity γ_1 in its lower half and of conductivity γ_2 in its upper half. The radius of the central wire is a, that of the sheath is b, the length of the cable is l. Show that the leakage resistance of the cable is

$$R = \log\,(b/a) \,/\, \pi\,l\,(\gamma_1 + \gamma_2)$$

5. A cylindrical piece of material of conductivity γ and permittivity ϵ has cross-sectional area S and length l. Perfectly conducting electrodes of area S are attached to the flat ends and a voltage is maintained between the electrodes. Prove that if a steady current I flows from one electrode to the other, then the charges on the electrodes are equal to $\pm I\,\epsilon/\gamma$.

6. Two spherical electrodes each of radius a are sunk half-way into the surface of the ground at a distance $d >> a$ from each other. Assuming that the resistivity of the earth up to a radius b from each electrode ($b << d$) is γ_1 and that of the remaining earth is γ_2, find the resistance between the electrodes.

7. Two very long cylindrical conductors each of radius a are sunk halfway into the surface of the ground at a distance $d >> a$ from each other If the conductivity of the earth is γ show that the resistance per unit length between the conductors is

$$R = (2/\gamma \pi) \log (d/a)$$

8. A spherical electrode of radius a is placed in a medium of conductivity γ at a distance $d > a$ from a large perfectly conducting plate. Show that the resistance between the sphere and the plate is in the first approximation

$$(2d - a) / 8 \pi \gamma a d$$

9. A parallel-plate capacitor of plate separation d is filled with a laminated material so composed that its permittivity is

$$\epsilon_r = k_1 [1 + k_2 \cos (\pi x/d)]$$

and its conductivity is

$$\gamma = k_3 [1 + k_4 \sin (\pi x/d)]$$

where k_1, k_2, k_3 and k_4 are constants and x is the distance from the positive plate toward the negative one. (*a*) Find the capacitance and the leakage resistance of this capacitor. (*b*) Find the density of space charge accumulating in the laminated material when a voltage V is applied to the capacitor.

10. The capacitance between the conductors of a twin core cable is found to be 0.22 μF/km. If the relative permittivity of the insulation is 2.4 and its resistivity is 1.5×10^{9} meg ohms/cm^3, find the insulation resistance per kilometer between these conductors.

11. Show that the time needed to dissipate as much energy in the Joule heating of an ohmic conductor carrying a steady current as is stored in the electric field within this conductor is

$$t = \epsilon/2\gamma$$

12. A cubic vessel of side l has the bottom and the sides made of non-conducting material and the two ends made of perfect conductors serving as electrodes. Show that if a voltage V is applied between the electrodes, and if the vessel is filled with liquid of conductivity γ, specific heat c, and density δ, the temperature T of the liquid will rise at the rate

$$dT/dt = \gamma V^2/c\delta l^2$$

13. Find the power lost in Joule heating per unit length of the dielectric in the cable described in problem 4.

14. A spherical charge distribution, centered at the origin, contains a charge density $\delta(t)$ independent of r. Over an interval of time $\delta(t) = \delta_0/(1+at)$, where a and δ_0 are constants. Find $\mathbf{j}(\mathbf{r}, t)$ and $i_r(t)$ where the latter is the current through the sphere of radius r.

15. A wire of diameter 2 mm has a resistance of 10 ohms per kilometer. Find the field intensity in the wire when a current of 20 A is flowing through it. If there is a static surface charge on the current-carrying wire of uniform density 5×10^{-12} C/m^2, find the magnitude and direction of the field intensity just outside the surface of the wire. The medium outside the wire is air.

16. The current direction at the boundary surface between two media makes an angle of 45° with respect to the surface in medium 1; what is the angle between the current direction and the surface in medium 2. The constants for the media are as follows :

$$\gamma_1 = 100 \; mhos/m \;\; , \;\; \epsilon_{r1} \;\; 1$$

$$\gamma_2 = 1 \; mhos/m \;\; , \;\; \epsilon_{r2} = 2$$

If the total current density in medium 1 is 10 A/m^2 what is the surface charge density at the boundary.

17. Show that the ratio of the magnitudes of displacement and conduction current densities for harmonically varying fields is $\gamma/\omega\epsilon$. At what frequency is this ratio unity for ·

(a) copper, $\gamma = 5.8 \times 10^7 \; ohm^{-1} \; m^{-1}$, $\epsilon_r = 1$

(b) sea water, $\gamma = 4 \times 10^{-3} \; ohm^{-1} \; m^{-1}$, $\epsilon_r = 81$

(c) marble, $\gamma = 10^{-8} \; ohm^{-1} \; m^{-1}$, $\epsilon_r = 8$

الفصل السابع

المجال الكهربي للتيارات الثابتة في الفراغ

THE MAGNETIC FIELD OF STEADY CURRENTS IN FREE SPACE

7 - 1 مقدمة :

اكتشف هانز أورستيد (Hans Oersted) في عام 1820 أن التيار الكهربي الثابت إذا مر في سلك موصل فإنه يحرف إبرة مغناطيسية أي ينشأ عن التيار مجال مغناطيسي ساكن . وبذلك أصبح من الممكن الكشف عن التيار الكهربي بواسطة المجال المغناطيسي المصاحب له . كان اكتشاف أورستيد هو إذن الرابطة بين الكهربية والمغناطيسية وقد تلى ذلك تحديد القوة المؤثرة على سلك يحمل تياراً وموضوع في مجال مغناطيسي وذلك بتطبيق مبدأ القوة المتبادلة ، وقد استطاع أندرية أمبير (Andre Ampére) تحديد العلاقة، التي تعين المجال المغناطيسي الناتج عن سلك يحمل تيار . وكذلك استطاع تعيين القوة بين سلكين يحملان تياراً وتحديد التيار الحلقي الذي يكافىء مغناطيس دائم . وقد استحق أمبير الوصف الذي أطلقه عليه جيمس ماكسويل (James Maxwell) بأنه نيوتن علم الكهربية . وقد تلي ذلك أعمال كل من بيوت (Biot) وسافرت (Savart) لتحديد العلاقة التي تعطي المجال المغناطيسي نتيجة لعنصر تيار كهربي (electric current element) .

7 - 2 عنصر التيار :

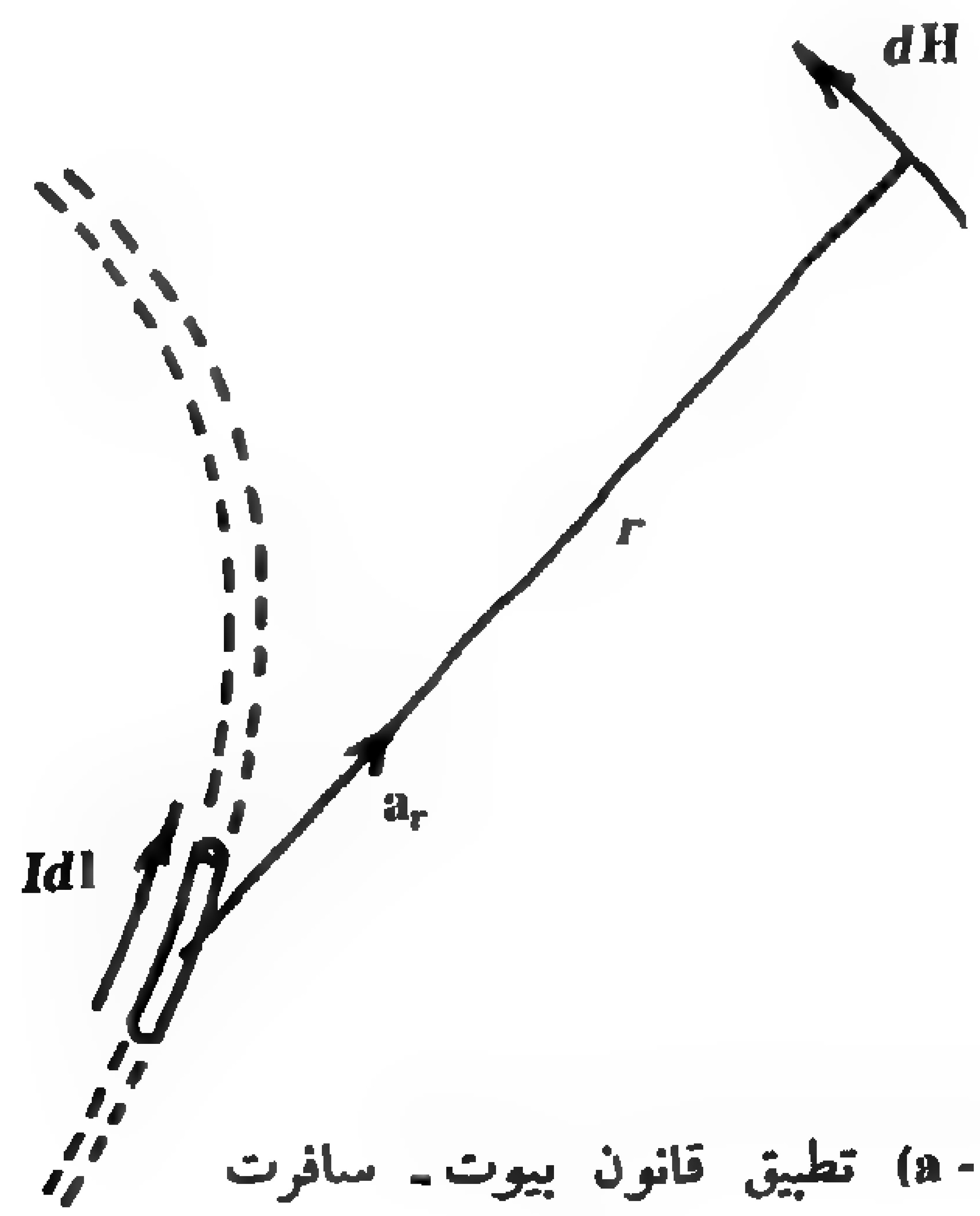
إذا مر تيار كهربي I في سلك طوله l فإننا نعرّف عنصر التيار بأنه متجه مقداره Idl ويكون في اتجاه التيار I . الطول dl هو جزء لانهائي القصر من السلك . ويؤخذ في اتجاه التيار وبناء على ذلك يمكن كتابة عنصر التيار على الصورة $I\,dl$. وكذلك يمكن كتابة عنصري التيار السطحي والحجمي بالعلاقتين $K\,dS$ و $J\,dv$ على التوالي .

7 - 3 قانون بيوت ـ سافرت : (Biot - Savart Law) :

يعطي هذا القانون شدة المجال المغناطيسي dH الناشىء عن عنصر تيار Idl .

$$d\mathrm{H} = \frac{Id\mathrm{l} \times \mathrm{a}_r}{4\,\pi\,r^2} \qquad (1 - 7)$$

حيث a_r هو متجه وحدة يشير من النقطة التي يحتلها عنصر التيار إلى النقطة المراد تعيين $d\mathrm{H}$ عندها (شكل 7 - a) و r هي المسافة بين نقطة P



شكل (a - 7) تطبيق قانون بيوت ـ سافرت

وعنصر التيار . وحيث أن عنصر التيار هو جزء من دائرة متصلة C ولذلك يكون المُجال المغناطيسي الكلي عند نقطة P هو :

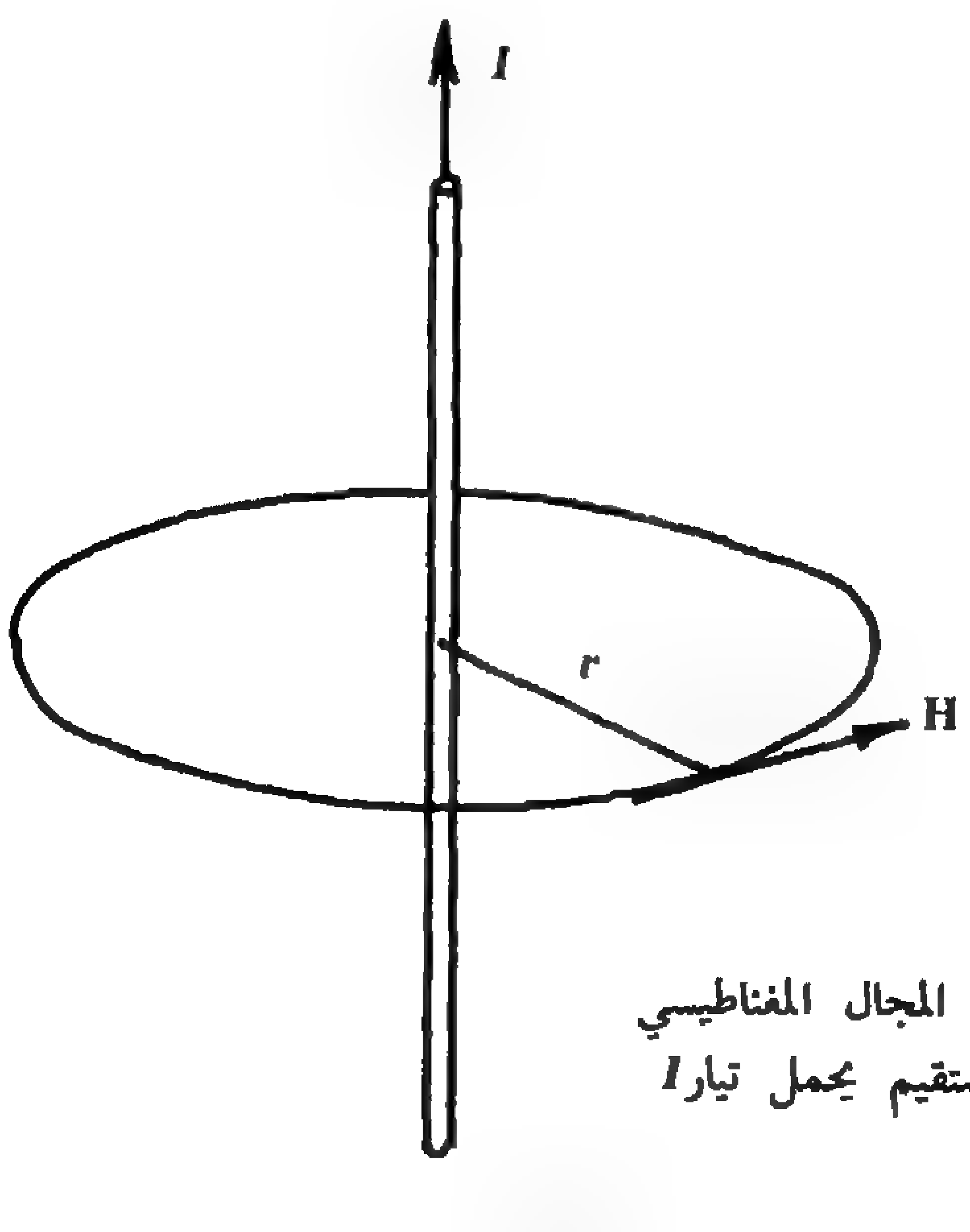
$$H = \oint_C \frac{I d\mathbf{l} \times \mathbf{a}_r}{4 \pi r^2} \tag{2 - 7}$$

4 - 7 قانون أمبير : (Ampere's Law) :

ينص قانون أمبير على أن التكامل الخطي للمجال المغناطيسي H حول منحنى مقفل C يساوي التيار المحاط بهذا المنحنى :

$$\oint_C \mathbf{H} . d\mathbf{l} = I \tag{3 - 7}$$

ويلاحظ عند تطبيق هذا القانون أن التيار I يكون معطي والمطلوب تعيين المجال المغناطيسي H . وهذا القانون يناظر قانون جاوس في النظرية الكهروستاتيكية حيث يعطي توزيع معين من الشحنات ويراد تعيين كثافة الفيض الكهربي D عند أي نقطة . وأبسط تطبيق لقانون أمبير هو عند إيجاد المجال المغناطيسي الناشيء من سلك مستقيم لانهائي الطول ويحمل تياراً I شكل (7 - b) .



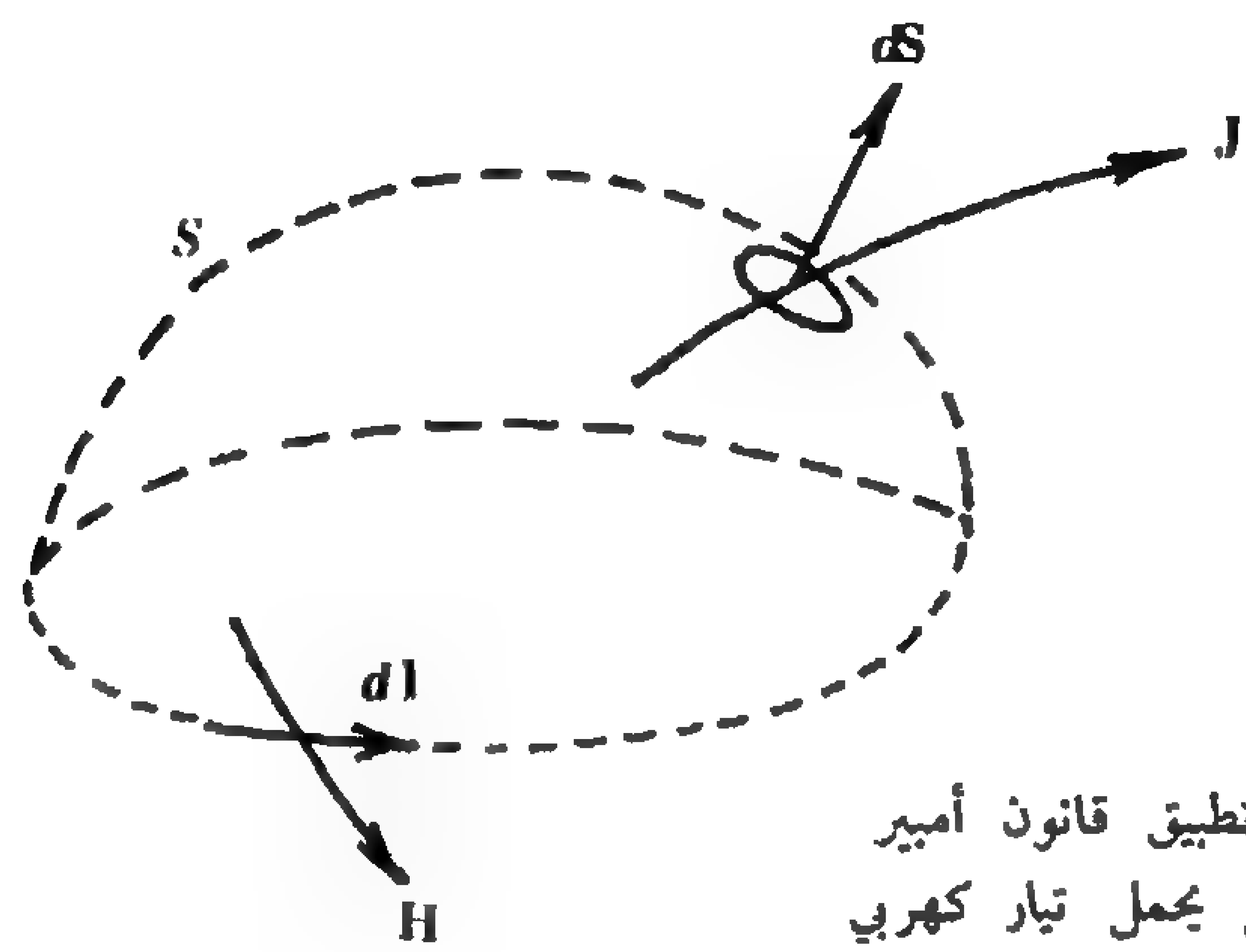
شكل (7 - b) المجال المغناطيسي حول سلك مستقيم يحمل تياراً I

$$\oint \mathbf{H} . d\mathbf{l} = 2 \pi r H = I$$

$$\mathbf{H} = \frac{I}{2 \pi r} \mathbf{a}_{\varnothing} \qquad (4 - 7)$$

a 4 - 7 قانون أمبير في وسط موصل :

افرض منحنى مغلق C على هيئة حلقة صغيرة في وسط موصل شكل



شكل (7 - c) تطبيق قانون أمبير في وسط موصل يحمل تيار كهربي

(7 - c) بتطبيق قانون أمبير على هذه الحلقة :

$$\oint \mathbf{H} . d\mathbf{l} = \int \int \mathbf{J} . d\mathbf{S} = I \qquad (5 - 7)$$

حيث I هو التيار الكلي المار بالحلقة و S أي سطح محاط بالمنحنى C وبتطبيق نظرية ستوك يمكن تحويل التكامل الخطي في هذه المعادلة إلى تكامل سطحي :

$$\int \int_S \nabla \times \mathbf{H} . d\mathbf{S} = \int \int_S \mathbf{J} . d\mathbf{S}$$

وهذه المعادلة يمكن كتابتها على الصورة :

$$\int \int_S (\nabla \times \mathbf{H} - \mathbf{J}) . d\mathbf{S} = 0$$

وحيث أن هذه العلاقة تنطبق على أي منحنى مغلق داخل الوسط الموصل .

نجد أن العلاقة الآتية تتحقق .

$$\nabla \times H = J \qquad (6 - 7)$$

وتعرف هذه العلاقة بقانون أمبير عند نقطة (Point Form of Ampére's Law) أو بمعادلة ماكسويل الأولى في حالة التيار الثابت .

7 - 5 الفيض المغناطيسي : (The Magnetic Flux) :

يعرف كثافة الفيض المغناطيسي B من العلاقة المساعدة (auxiliary relation) .

$$B = \mu H \qquad (7 - 7)$$

حيث M هي انفاذية الوسط (Permeability) وهي معطاة بالعلاقة :

$$\mu = \mu_0 \mu_r \qquad (8 - 7)$$

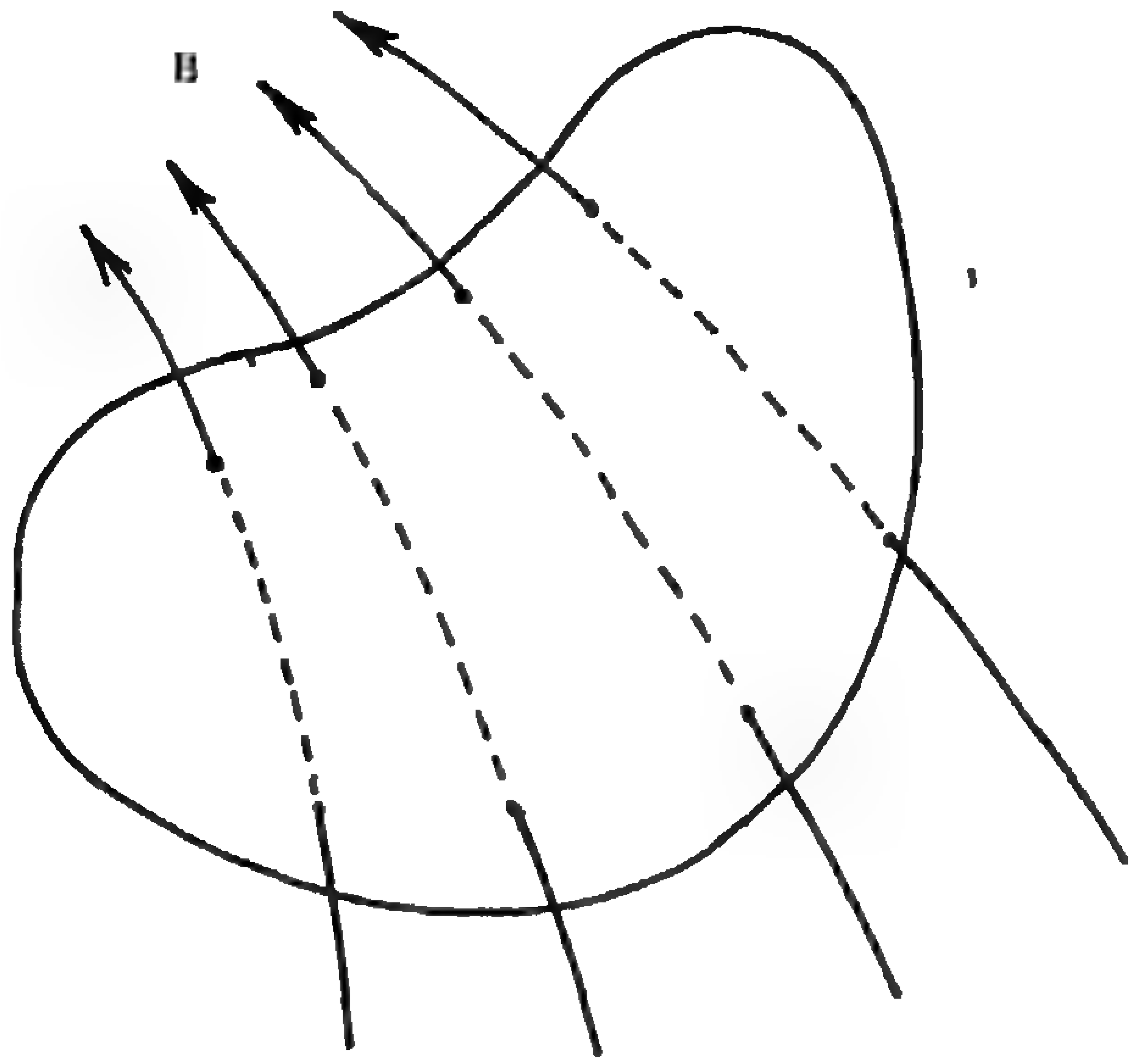
وتسمى μ_r الانفاذية النسبية (relative permeability) و μ_0 انفاذية الفراغ (free space permeability) ومقدارها $4 \pi \times 10^{-7}$ *henry* .

ويمكن تعيين كمية الفيض المغناطيسي المار بسطح S بالعلاقة :

$$\emptyset = \int \int_S B . dS \quad weber \qquad (9 - 7)$$

وبذلك يمكن أن نستنتج أن وحدات كثافة الفيض المغناطيسي B هي $weber / m^2$. ويلاحظ أن خطوط القوى المغناطيسية مقفلة بعكس خطوط القوى الكهربية التي تبدأ عند الشحنات الموجبة وتنتهي عند الشحنات السالبة . وبناء على ذلك فكل الخطوط المغناطيسية التي تدخل حجم معين v

شكل $(7 - d)$ تخرج منه مرة أخرى أي أن المجال B ليس له منابع أو مصبات



شكل (7 - d) استنتاج أن المجال B يكون دائما حلزونيا

وبناء على ذلك نجد أن تكامل المجال B على سطح مغلق ينعدم :

$$\oint\int_S \mathbf{B} \cdot d\mathbf{S} = 0 \qquad (10 -7)$$

وبتطبيق نظرية جاوس نستنتج العلاقة الآتية :

$$\nabla \cdot \mathbf{B} = 0 \qquad (11 - 7)$$

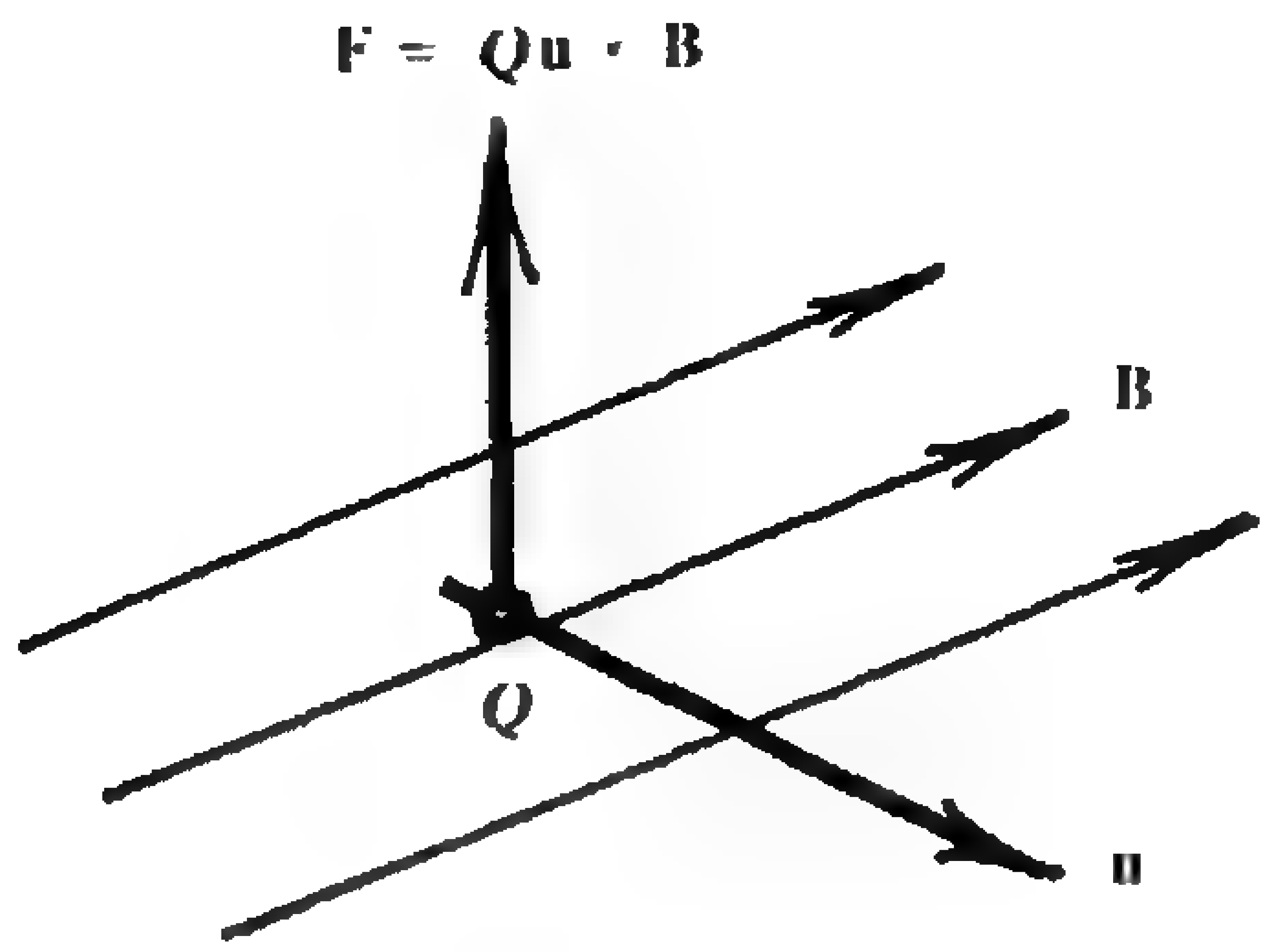
ومنها نجد أن المجال B حلزوني (Solenoidal) وهذه العلاقة عامة سواء كان B ثابتاً أو متغيراً مع الزمن .

7 - 6 القوة المغناطيسية على شحنة متحركة :

(Magnetic Force on a Moving Point Charge):

إذا تحرك جسيم يحمل شحنة Q بسرعة u في مجال مغناطيسي ساكن له

كثافة فيض مغناطيسي B فإنه يعاني قوة F تكون عمودية على كل من B، u وتحدد بالعلاقة (أنظر شكل 7 - e) .



شكل (7 - e) شحنةQ متحركة بسرعة u في مجال مغناطيسي

$$F = Q\,u \times B \qquad (12 - 7)$$

من هذه العلاقة نستنتج أن المجال المغناطيسي B يعمل على انحراف مسار الشحنة Q إذا كانت سرعتها u لا توازي هذا المجال . ويلاحظ أن القوة F تكون دائماً عمودية على اتجاه مسار الشحنة وبالتالي فهي لا تبذل شغل في اتجاه هذا المسار وتظل طاقة حركة الجسيم ثابتة . وإذا فرضنا أن المجال B منتظم وكانت u عمودية عليه فإن مسار الجسيم المشحون يكون دائري نصف قطره r يمكن تحديده بمساواة قوة الطرد المركزي على الجسيم بالقوة المغناطيسية المؤثرة عليه :

$$Q\,u\,B = m\,u^2 / r$$

ومنها نستنتج أن :

$$r = mu / QB$$

أي أن نصف قطر المسار يتناسب طردياً مع كمية حركة الجسيم mu .

وإذا كان الجسم يتحرك بسرعة معطاة بالعلاقة :

$$u = u_1 \, a_x + u_2 \, a_y + u_3 \, a_z$$

وكثافة الفيض المغناطيسي المؤثرة عليه هي :

$$B = B_1 \, a_x + B_2 \, a_y + B_3 \, a_z$$

فإن القوة المؤثرة على الجسيم المشحون تصبح على الصورة العامة :

$$F = (u_2 B_3 - u_3 B_2) \, a_x + (u_3 B_1 - u_1 B_3) \, a_y + (u_1 B_2 - u_2 B_1) \, a_z$$

ويلاحظ أنه إذا وضع جسيم مشحون بشحنة Q في مجال كهربي شدته E فإن القوة المؤثرة عليه QE تعمل على تعجيله في اتجاه المجال الكهربي (إذا كانت Q موجبة) وبذلك يبذل المجال شغل ويغير من طاقة حركة الجسيم .

7 - 7 . حركة جسيم مشحون في مجال كهرومغناطيسي :

(Motion of a Charged Particle in an Electromagnetic Field):

إذا تحرك جسيم كتلته m مشحون بشحنة Q في مجال كهربي شدته E ومجال مغناطيسي كثافة الفيض المغناطيسي له B فإن القوة المؤثرة عليه هي :

$$\begin{aligned} F &= Q E + Q \, u \times B \\ &= Q \, (E + u \times B) \end{aligned} \qquad (13 - 7)$$

وهي تعرف بقوة لورانس (Lorentz Force) ويمكن تحديد مسار الجسيم وذلك بمساواة هذه القوة بحاصل ضرب كتلته في عجلته وتعويض الحالة الابتدائية لهذا الجسيم .

7 - 8 القوة المغناطيسية المؤثرة على عنصر تيار :

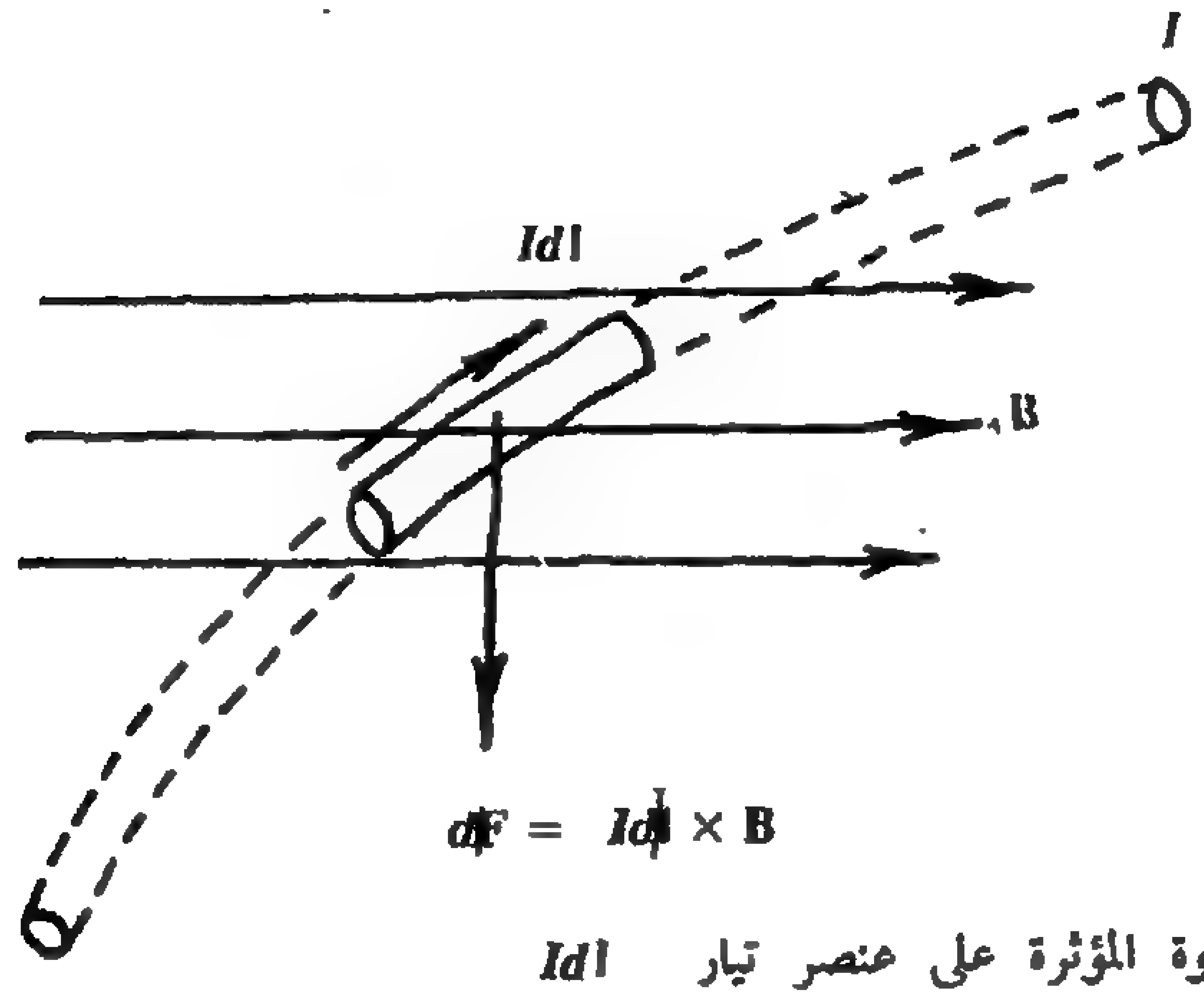
(Magnetic Force on a Current Element):

لتعيين القوة المؤثرة على عنصر تيار Idl موضوع في مجال مغناطيسي

نعبر عن التيار I على الصورة :

$$I = \frac{dQ}{dt}$$

وبالتالي يمكن تحديد هذه القوة كما يأتي . (شكل (7 - f))



شكل (7 - f) القوة المؤثرة على عنصر تيار $I d\mathbf{l}$

$$\begin{aligned} d\mathrm{F} &= dQ\, \mathrm{u} \times \mathrm{B} \\ &= I\,(dt\,\mathrm{u}) \times \mathrm{B} \\ &= I\, d\mathrm{l} \times \mathrm{B} \qquad (14 - 7) \end{aligned}$$

حيث $d\mathrm{l} = \mathrm{u}dt$ هو متجه يمثل طول عنصر التيار في اتجاه التيار I .

ويلاحظ أن القوة $d\mathrm{F}$ تؤثر أساساً على الالكترونات الحرة داخل الموصل الذي يكوّن عنصر التيار . وحيث أن هذه الالكترونات مجبرة على الحركة داخل هذا الموصل فتنتقل هذه القوة إلى التركيب البلوري للموصل .

7 - 9 المجال المغناطيسي الناشىء عن شحنة نقطية متحركة :

(The Magnetic Field of a Moving Point Charge):

بالتعويض عن الشحنة Q المتحركة بدلالة عنصر التيار المكافىء نحصل على :

$$I\,dl = J'\,dv = \rho\, u\, dv$$
$$= \rho\, dv\, u = Q\, u$$

حيث $\rho\, dv$ هي كمية الشحنة في حجم متناهي الصغر ويمكن اعتبارها شحنة نقطية Q ، ويعطي قانون بيوت يسافرت (7 - 1) .

$$H = \frac{Q}{4\pi r^2}\, u \times a_r \qquad (15 - 7)$$

حيث a_r هو متجه وحدة من الشحنة إلى النقطة التي يراد عندها تعيين H . ويلاحظ أن العلاقة (7 - 15) تظل صحيحة فقط إذا كانت سرعة الشحنة u صغيرة بالمقارنة بسرعة الضوء .

7 - 10 الشغل والقدرة في المجال المغناطيسي :

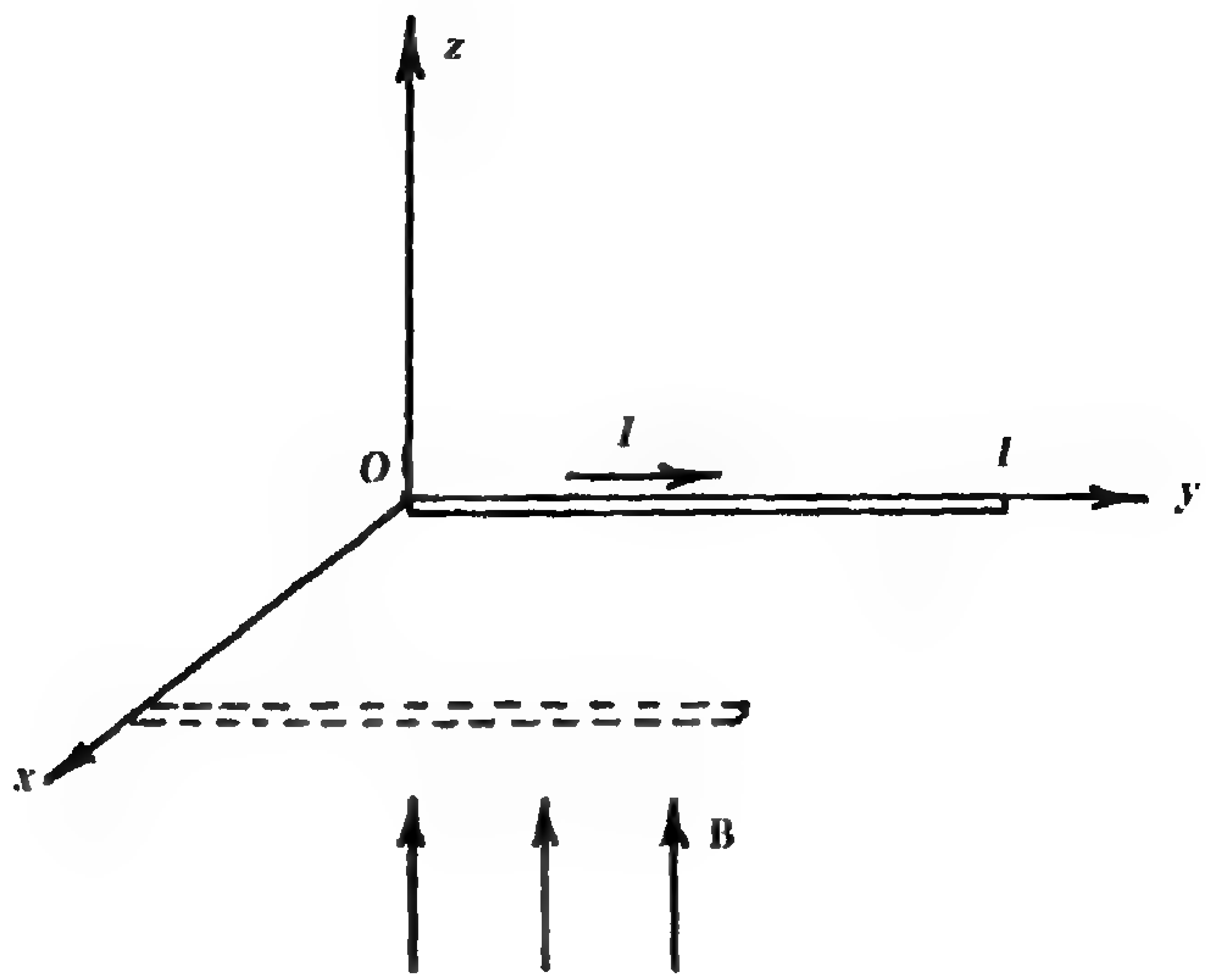
(Work and Power in a Static Magnetic Field):

إذا كانت F هي القوة المؤثرة على سلك يحمل تيار ويقع في مجال مغناطيسي ساكن فإن الشغل المبذول ليتحرك هذا السلك مسافة dl هو $-F.\,dl$ ونحصل على الشغل المبذول ليتحرك السلك من وضع ما 1 إلى وضع آخر 2 بالتكامل :

$$W = -\int_1^2 F.\, dl \qquad (18 - 7)$$

إذا كانت W موجبة فإن هناك شغل خارجي مبذول اما إذا كانت W سالبة فإن المجال المغناطيسي نفسه هو الذي يبذل الشغل .

وحيث أن القوة المغناطيسية F هي على وجه العموم غير محافظة (Nonconservative) فيجب وصف المسار من 1 إلى 2 بدقة وذلك لأن W تعتمد عليه . فمثلاً الشغل المبذول لتحريك سلك مستقيم طوله l يحمل تياراً I ومنطبق على محور y (شكل 7 - g) لمسافة مقدارها h متر هو :



شكل (7 - g) سلك طوله l يحمل تياراً I ويتحرك في مجال مغناطيسي

$$W = - \int_0^h \mathbf{F} \cdot dx\, \mathbf{a}_x$$

حيث :

$$\mathbf{F} = I\mathbf{l} \times \mathbf{B} = I\,l\,B\ \mathbf{a}_x$$

ومنه نستنتج أن $W = - IlBh$ جول . الاشارة السالبة تبين أن هذا الشغل قد بذله المجال المغناطيسي نفسه لتحريك السلك في الاتجاه المبين .

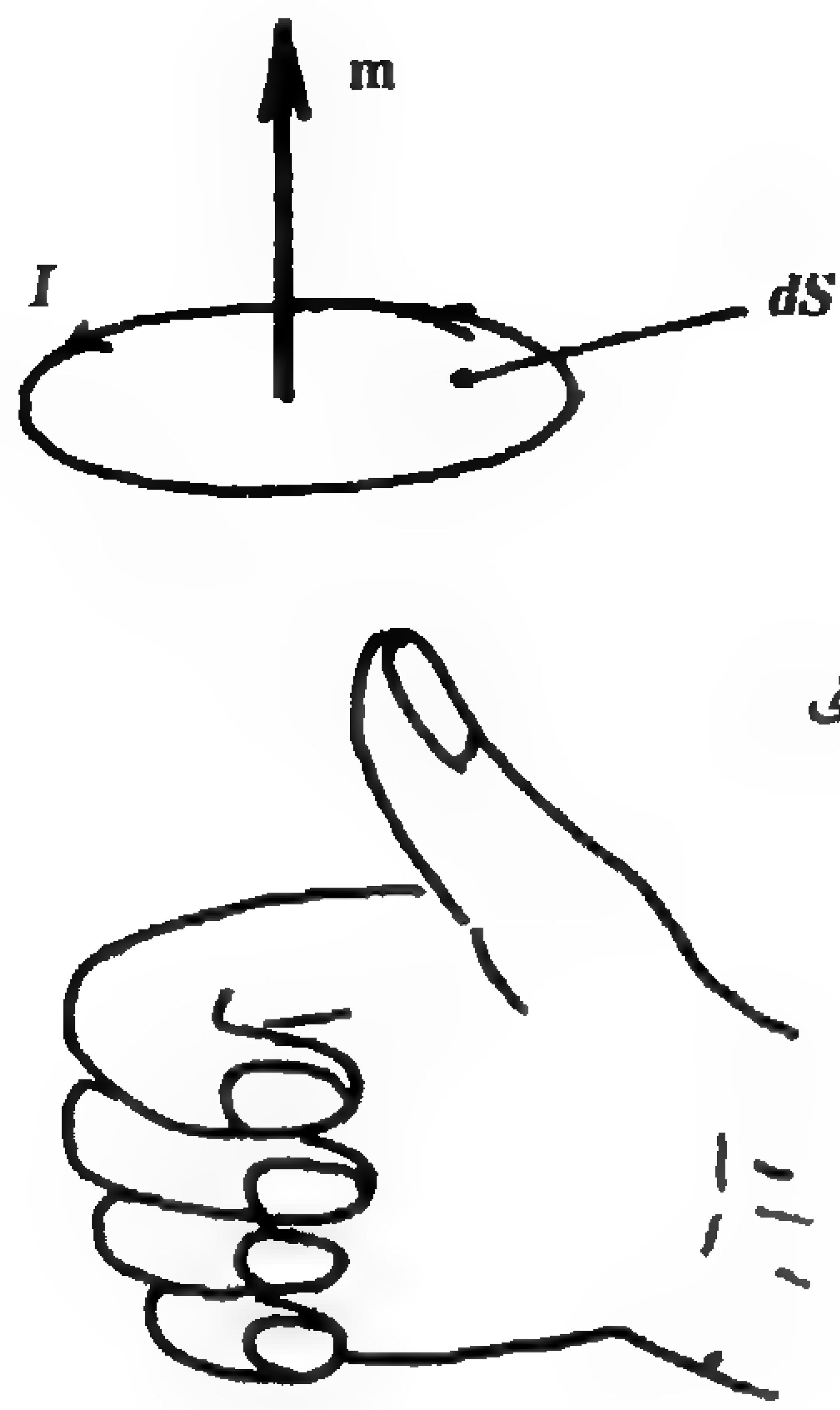
7 - 11 العزم المغناطيسي لملف مستو :

Magnetic Moment of a Planar Coil:

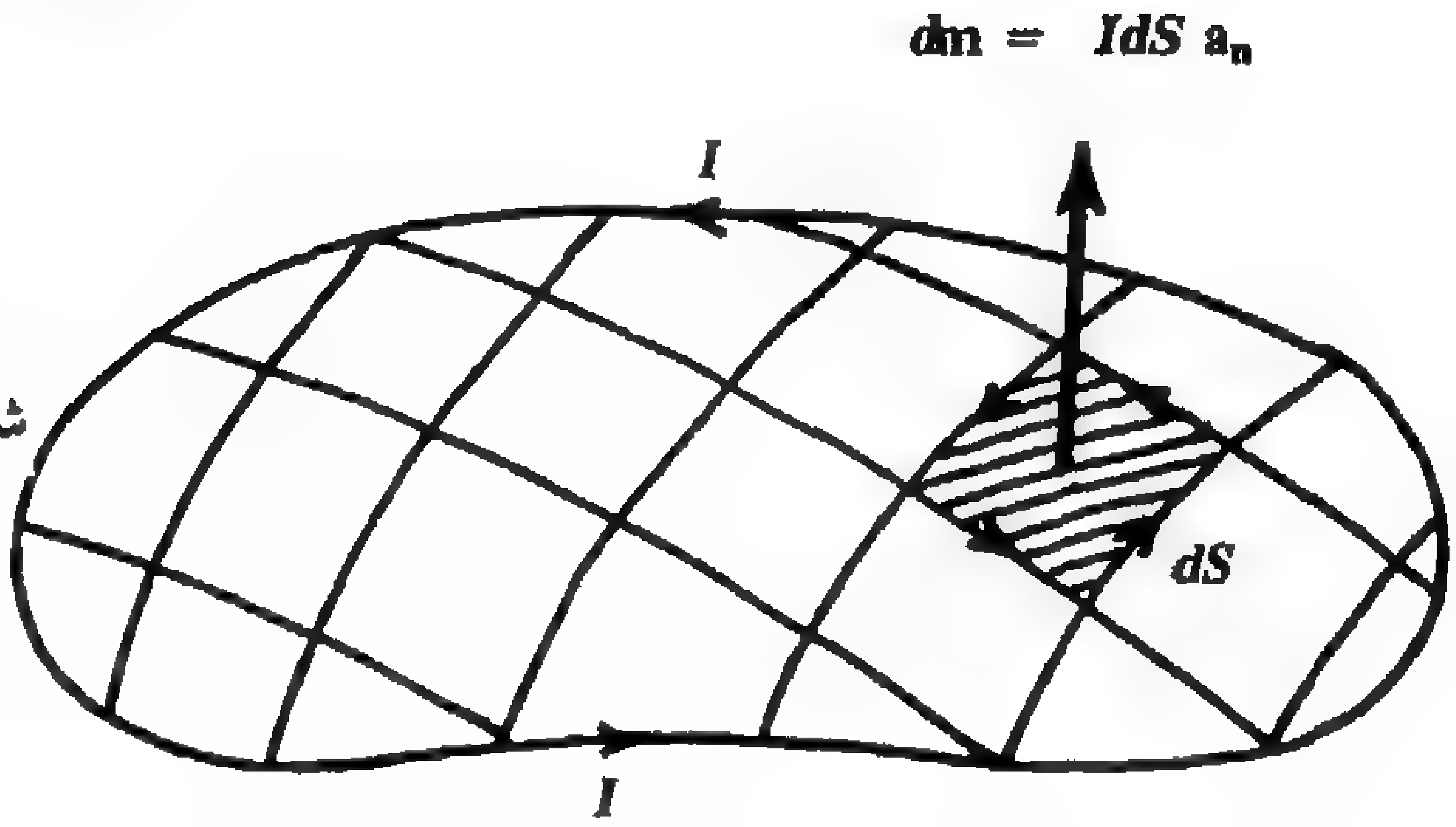
يعرف العزم المغناطيسي m لملف صغير مستوى مساحته dS ويحمل تياراً I ومكون من لفة واحدة بالعلاقة :

$$\mathbf{m} = I dS\, \mathbf{a}_n \qquad (19 - 7)$$

حيث a_n هو متجه عمودي على مستوى هذا الملف ويعين بقاعدة اليد اليمنى . شكل (7 - h) . وإذا كان الملف كبيراً فيمكن تقسيمه إلى عدد



شكل (7 - h) تطبيق قاعدة اليد اليمنى لتعيين اتجاه العزم لحلقة تيار

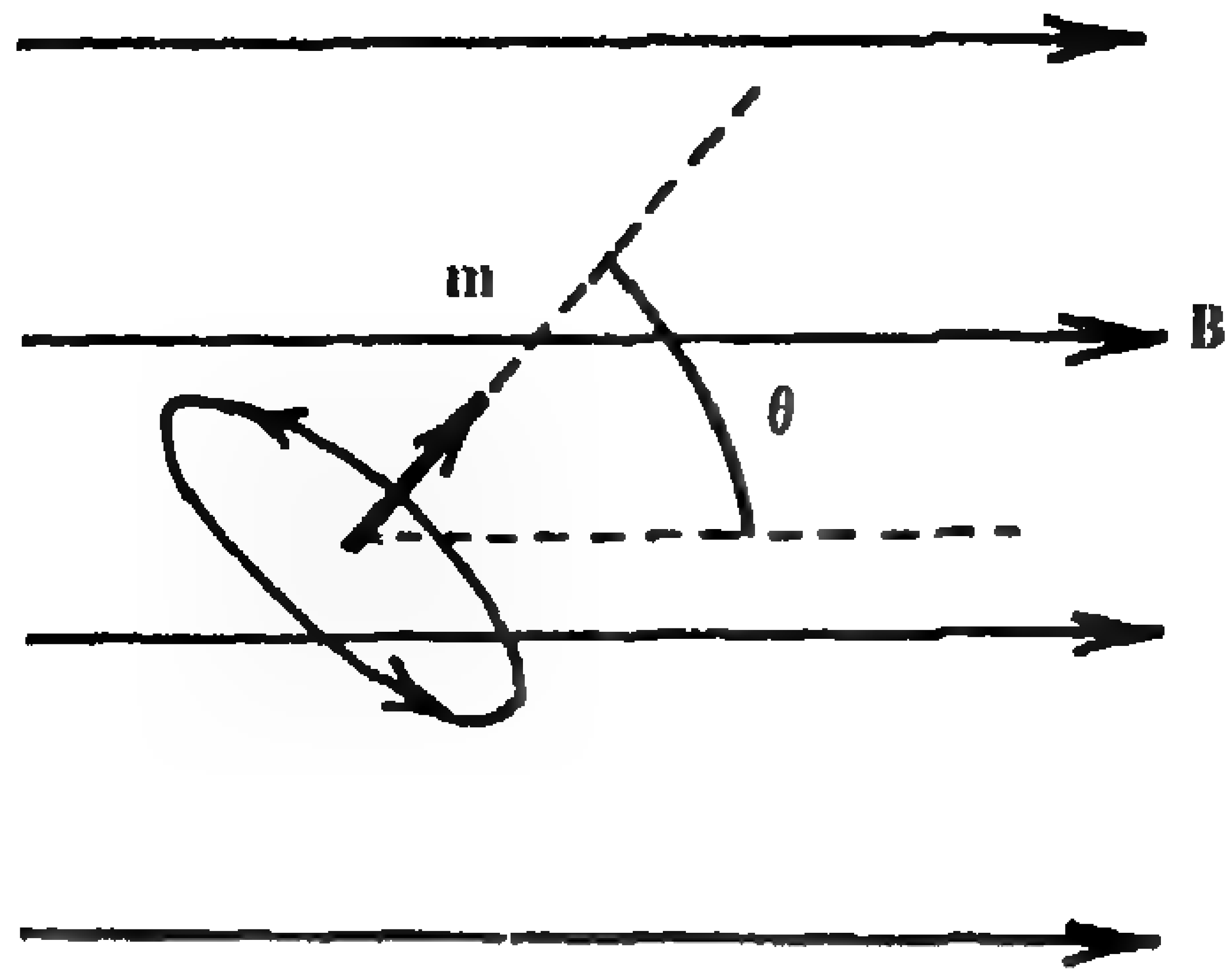


شكل (7 - i) تعيين العزم المغناطيسي المحصل لملف كبير يحمل تيارI

لانهائي من الملفات الصغيرة (شكل 7 - i) العزم المغناطيسي المحصل لها هو :

$$m = \int_S I\, dS \qquad (20 - 7)$$

إذا وضع الملف في مجال مغناطيسي منتظم شكل (7 - j) فإن العزم المؤثر عليه يمكن إيجاده بالتناظر مع ثنائي القطب الكهروستاتيكي :



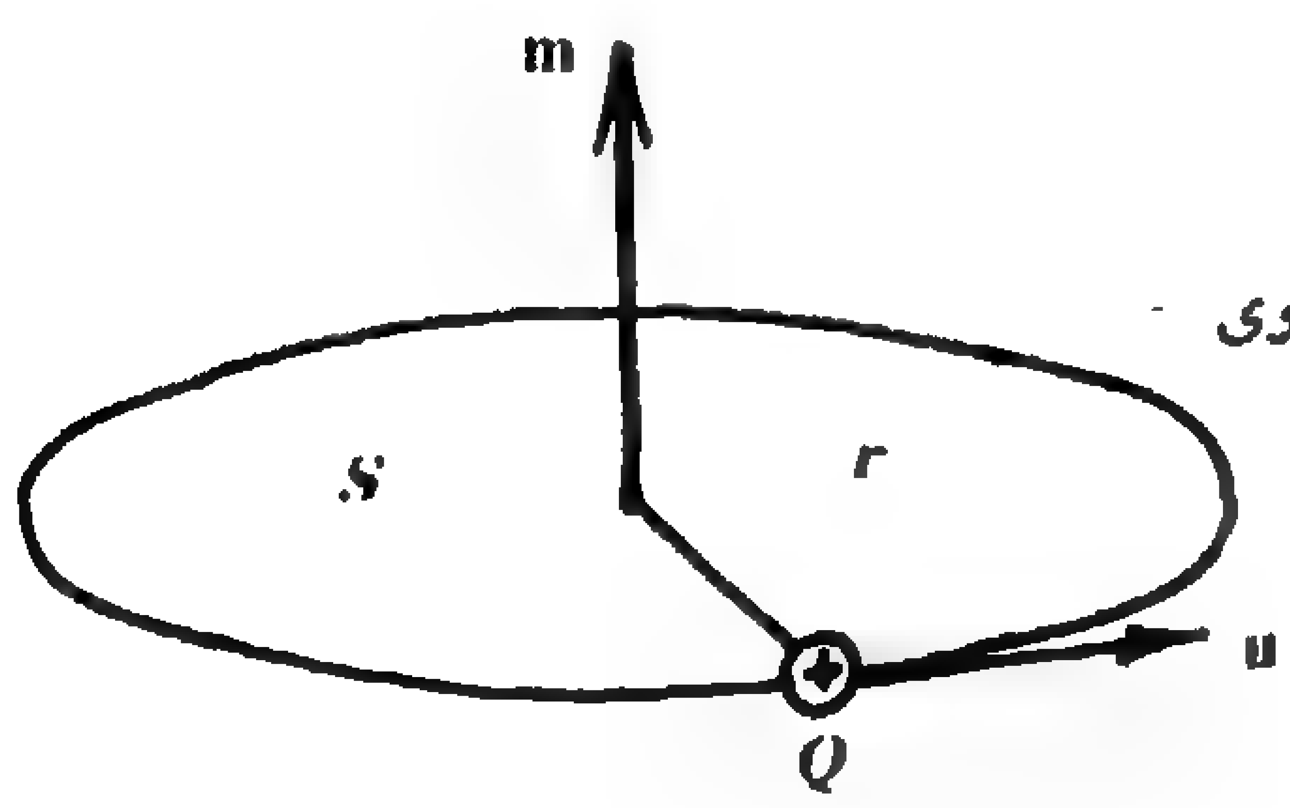
شكل (7 - j) ملف يحمل
تيار وموضوع في مجال مغناطيسي

$$\begin{aligned} \mathrm{T} &= \mathrm{m} \times \mathrm{B} \\ &= m\, B \sin\theta\, \mathrm{a}_n \end{aligned} \tag{21 - 7}$$

حيث a_n هو متجه وحدة عمودي على كل من B ، m والزاوية θ تقع بين B و m .

ويمكن استخدام مفهوم العزم المغناطيسي لفهم طبيعة الأجسام المشحونة التي تدور في مدارات دائرية . افترض شحنة نقطية Q تدور في مدار دائري مستوى نصف قطره r بسرعة زاوية ω .

شكل (7 - k) التيار المكافىء لهذه الشحنة هو :



شكل (7 - k) العزم المغناطيسي
لشحنه تدور في مدار دائرة مستوى .

$$I = \frac{\omega}{2\pi} Q \tag{22 - 7}$$

والعزم المغناطيسي الناشىء من دوران هذه الشحنة هو :

$$m = \frac{\omega}{2\pi} Q S a_n$$

7 - 12 جهد الكمية المغناطيسية المتجهة :

(Magnetic Vector Potential)

أشرنا من قبل أنه لا يمكن فصل الأقطاب المغناطيسية عن بعضها وبناء على ذلك استنتجنا أن B هو مجال حلزوني . وتنص نظريات التحليل الاتجاهي أنه يمكن تمثيل B على صورة دوامية متجه آخر A .

$$B = \nabla \times A \qquad (23 - 7)$$

ويسمى A جهد الكمية المغناطيسية الموجهة ووحداته W/m . وإذا أدخلنا شرط لورانس في الحالة الاستاتيكية على A أي $\nabla . A = 0$ فإنه يمكن تعيين A نتيجة لأي توزيع للتيار الكهربي الثابت .

(i) تيار خطي التوزيع (7 - 24)
$$A = \frac{\mu}{4\pi} \oint \frac{I dl}{r}$$

(ii) تيار صفحي التوزيع (7 - 25)
$$A = \frac{\mu}{4\pi} \int \frac{K dS}{r}$$

(iii) تيار حجمي التوزيع (7 - 26)
$$A = \frac{\mu}{4\pi} \int \frac{J dv}{r}$$

حيث r هي المسافة بين عنصر التيار والنقطة التي يراد عندها تعيين A . ويلاحظ أنه في الحالات الثلاثة (i) و (ii) ، (iii) يتلاشى A عندما تصبح المسافة r لانهائية .

7 - 13 دالة الجهد المغناطيسي القياسية :

(The Scalar Magnetic Potential Function)

قانون أمبير على صورته التكاملية في الوسط المغناطيسي هو :

$$\oint_C \mathbf{H} . d\mathbf{l} = I \qquad (27 - 7)$$

وإذا انعدم التيار المار في المحاط C فإن الطرف الأيمن يتلاشى ويمكن أن نستنتج العلاقة الآتية :

$$\nabla \times \mathbf{B} = 0 \qquad (28 - 7)$$

وهذه العلاقة صحيحة عند كل النقط التي ينعدم فيها التيار وبناء على ذلك يمكن كتابة (7 - 28) على الصورة :

$$\nabla \times (- \nabla V_m) = 0$$

$$\mathbf{B} = - \nabla V_m . \qquad (29 - 7)$$

وتسمى V_m بدالة الجهد المغناطيسي ووحداتها W/m . وبالرجوع للمعادلة (7 - 27) نجد أن التكامل حول أي منحنى مغلق يساوي صفر وبالتالي التكامل حول منحنى مفتوح AB يعتمد فقط على نقط البداية A والنهاية B .

$$V_{mAB} = - \int_B^A \mathbf{B} . d\mathbf{l} = V_{mA} - V_{mB} \qquad (30 - 7)$$

ويلاحظ أن الدالة V_m ليست لها نفس الأهمية الطبيعية التي تعطي لدالة الجهد الكهربي V والسبب في ذلك يرجع إلى استحالة عزل القطب المغناطيسي وبالتالي لا يمكن القول أن وحدات V_m هي طاقة الوضع لوحدة الأقطاب المغناطيسية الموجبة .

ويمكن تعريف السطح المتساوي الجهد المغناطيسي بالعلاقة :

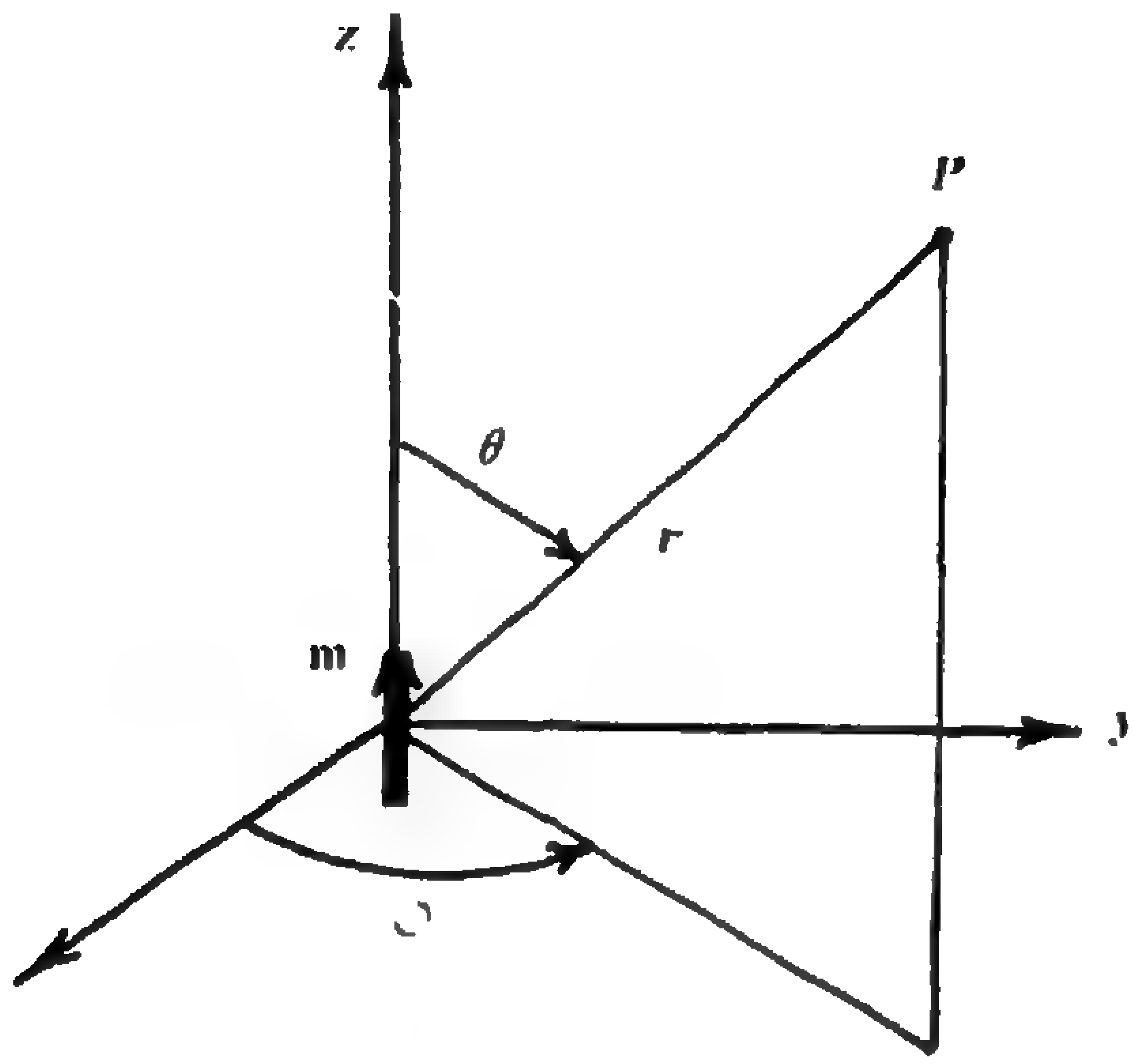
$$V_m = Constant$$

وتكون خطوط القوى المغناطيسية عمودية عند أي نقطة على هذا السطح .

وحيث أن $\nabla . \mathrm{B} = 0$ فإن دالة الجهد المغناطيسي V_m تحقق معادلة لابلاس :

$$\nabla^2 V_m = 0, \ J = 0$$

ويمكن كتابة الجهد المغناطيسي الناشىء عند نقطة P شكل $(l - 7)$ عن



شكل (7 - l) تعيين الجهد المغناطيسي لمغناطيس صغير

مغناطيس قصير عزمة m وذلك بالتناظر مع ثنائر القطب الكهربي :

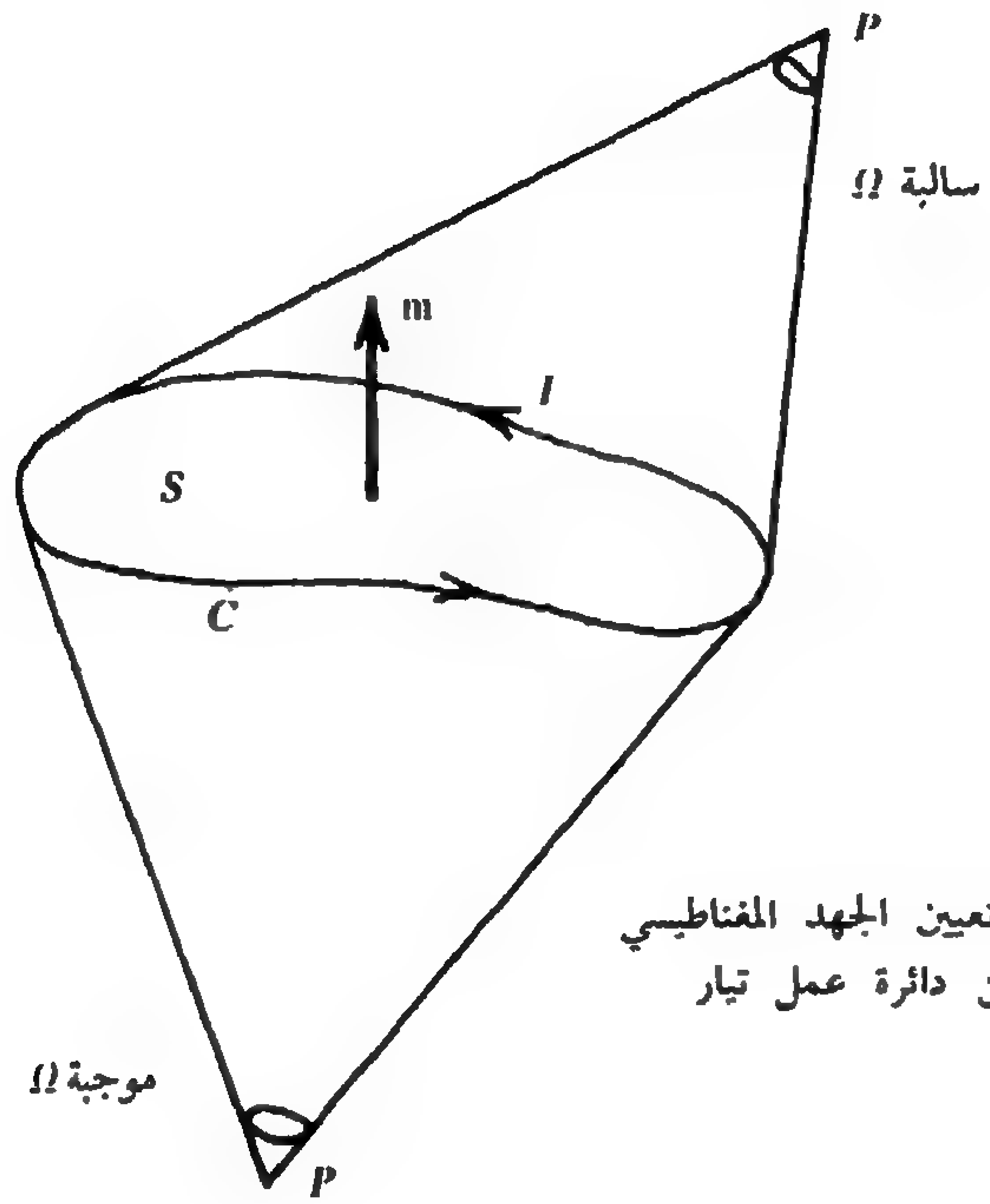
$$V_m = - \frac{\mu_0}{4 \pi} \mathrm{m} . \nabla \left(\frac{l}{r} \right) \qquad (31 - 7)$$

ويمكن استخدام هذه العلاقة لايجاد الجهد المغناطيسي عند نقطة P الناشىء عن تيار يمر في دائرة مغلقة كما هو مبين بشكل $(m - 7)$.

افرض أن S هو أي سطح يكون C محاط له . بتقسيم هذه المساحة إلى عناصر مساحية متناهية الصغر مساحة كل منها dS يمكن اعتبار العنصر dS

بأنه حلقة يمر بها تيار I في نفس اتجاه التيار في الدائرة C وبالتكامل نجد أن كل التيارات الداخلية تتلاشى وتصبح V_m .

$$V_m = \frac{\mu_0}{4\pi} \int_S I dS\, \dot{a}_n \nabla \left(\frac{I}{r}\right)$$



شكل (m - 7) تعيين الجهد المغناطيسي الناشيء عن دائرة عمل تيار

$$V_m = \frac{\mu_0 I}{4\pi} \int_S \frac{dS \cos\theta}{r^2} = \frac{\mu_0 I}{4\pi} \int d\Omega$$

$$= -\frac{\mu_0 I}{4\pi} \Omega \qquad (32 - 7)$$

حيث Ω هي الزاوية المجسمة المحددة بالسطح S . ويلاحظ أن Ω قد تكون موجبة أو سالبة (شكل m - 7) حسب اتجاه التيار I وموقع النقطة P

7 - 14 المحاثة الذاتية والتبادلية :

(Self - and Mutual Inductance):

تتناسب كثافة الفيض المغناطيسي B تناسباً مباشراً مع التيار I وذلك حسب قانون بيوت ـ سافرت . وبناء عليه يكون الفيض المغناطيسي $\varnothing$ المحاط بأي دائرة C متناسب مع التيار I المار في هذه الدائرة :

$$\varnothing = L\, I \qquad (33 - 7)$$

ويسمى معامل التناسب L بالمحاثة الذاتية للدائرة ووحداتها وبر / أمبير = هنري . ويمكن تعريف محاثة الدائرة الذاتية بأنها كمية الفيض المغناطيسي الذي تحيطه هذه الدائرة إذا مر بها تيار مقداره أمبير واحد .

وإذا مر تيار I_1 في دائرة معينة C_1 ونتج عن ذلك فيض مغناطيسي $\varnothing_{21}$ في دائرة مغلقة C_2 تقع على مقربة من الدائرة الأولى تكون العلاقة بين I_1 و $\varnothing_{21}$ على الصورة :

$$\varnothing_{21} = M_{21}\, I_1$$

وتسمى M_{21} بالمحاثة التبادلية بين الدائرتين C_1 و C_2 ووحداتها هنري .

7 - 15 المحاثة الذاتية لبعض الأنظمة البسيطة :

(Self Inductance for Some Simple Systems):

(i) المحاثة لوحدة الأطوال من كابل محوري (Coaxial Cable) .

كثافة الفيض المغناطيسي بين موصلي الكابل على بعد r من المحور هي :

$$\mathbf{B} = \mu_0\ I / 2\pi r)\, \mathbf{a}_{\varnothing}$$

ويكون الفيض المغناطيسي بين موصلي الكابل لكل متر واحد طولي هو :

$$\emptyset = \int^{b}_{a} \int^{l}_{0} \left(\frac{\mu_0 I}{2 \pi r} \right) dz \ dr = \frac{\mu_0 I}{2 \pi} \ ln \frac{b}{a}$$

وعلى ذلك فإن المحاثة لوحدة الأطوال هي :

$$L = \frac{\mu_0}{2 \pi} ln \frac{b}{a} \qquad henry$$

(ii) محاثة ملف حلقي (Toroidal Coil) نصف قطره r ومساحة مقطعة S متر مربع وعدد لفاته N هي :

$$L = \frac{\mu_0 N^2 S}{2 \pi r} \qquad henry$$

(iii) المحاثة لوحدة الأطوال لخط نقل مكون من سلكان متوازيان (Parallel Wire Transmission Line) نصف قطر كل منهما a والمسافة بينهما d هي :

$$L = \frac{\mu_0}{\pi} cosh^{-1} \frac{d}{2a} \qquad henry$$

(iv) المحاثة لوحدة الأطوال لنظام مكون من سلك نصف قطره a ويبعد مسافة $\frac{d}{2}$ من سطح موصل مستوى مؤرض لانهائي المساحة هي :

$$L = \frac{\mu_0}{2 \pi} \ cosh^{-1} \left(\frac{d}{2a} \right)$$

(v) محاثة ملف مكون من طبقة واحدة نصف قطره a وطوله l ويحتوي على N لفة هي :

$$L = \frac{39.5\ N^2 a^2}{9a + 10\,l} \qquad micro\ \ henry$$

(vi) محاثة ملف مكون من عدة طبقات نصف قطره الداخلي a والخارجي b وطوله l وعدد لفاته N هي :

$$L = \frac{31.6\ N^2\ a^2}{6a + 9\,l + 10\,(b - a)} \quad micro\ henry$$

7 - 16 مسائل محلولة

SOLVED PROBLEMS

1. *An infinitely long copper conductor is placed in a uniform horizontal magnetic field of strength 10^{-3} tesla; the field is perpendicular to the conductor axis. What must the current density in the conductor be in order to float the conductor in the earth's gravitational field. Density of copper 8.9 g. cm^{-3}*

   The magnetic force per unit length of conductor,

   $$BI = BjS$$

   where j is the current density and S the cross-sectional area of the conductor. The weight per unit length of conductor is Sd where d is the density.

   At equilibrium,

   $$jBS = Sd$$

   $$j = d/B = 8.9\times10^{6} \quad A/m^2$$

   $$= 8.9 \quad A/mm^2$$

2. *A square loop of side l carries a current I. Find the force acting on any one side of the loop and derive an expression for the flux density B at points on the axis perpendicular to the plane of the loop and passing through its center. Hence show that the flux density at the center of the loop is $\mu_0\, 2\sqrt{2}\, I/\pi l$.*

The magnetic flux density at a distance r from a current segment is

$$B \quad (\mu_o I / 4\pi)\ (\cos\ \theta_1 - \cos\ \theta_2) / r$$

Where θ_1 and θ_2 are the angles between the directed segment and the radius vectors from the end of the segment to the point.

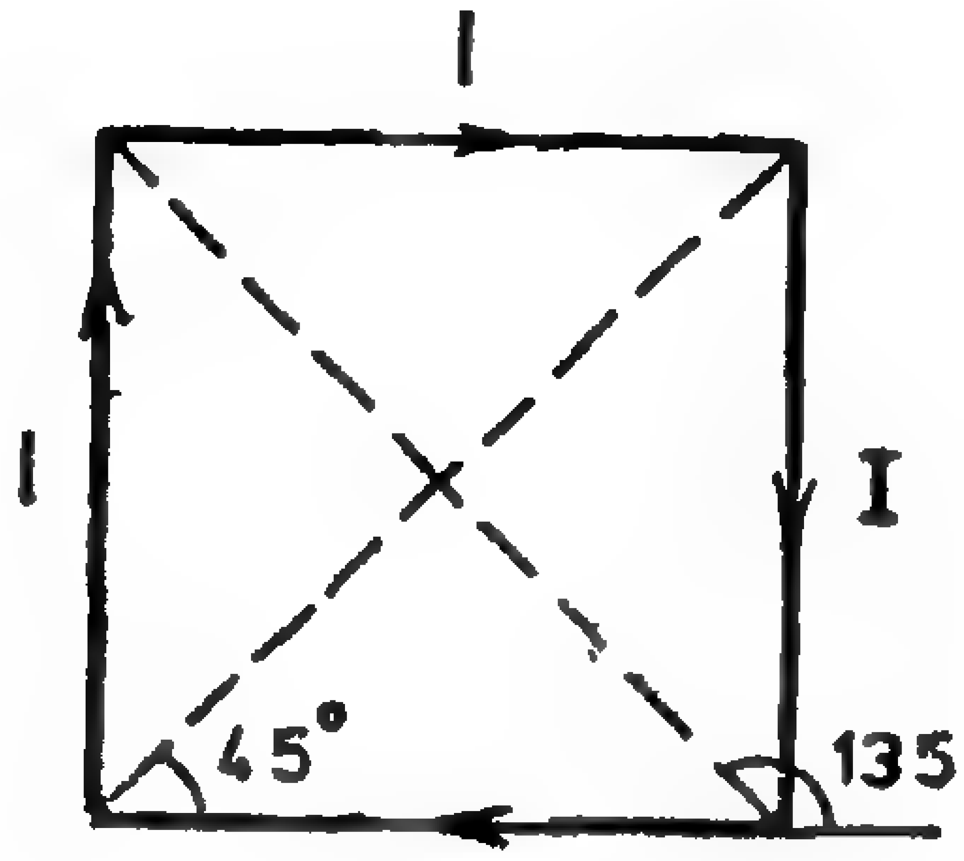


Fig. 7.1.

At the center of the square (Fig. 7.1), and for one side of the loop,

$$r = l/2 \ , \ \theta_1 = 45^o \ , \qquad \theta_2 = 135^o$$

so that

$$B_l = (\mu_o I/4\pi)\ 2\ (1/\sqrt{2} + 1/\sqrt{2}) / l$$

$$= \mu_o I/\sqrt{2}\ \pi\ l$$

The total flux density is therefore,

$$B = 4\ B_l = \mu_o\ 2\sqrt{2}\ I/\pi\ l$$

3. *In the previous problem find* **B** *at the center of the loop by using the equation* **B** $= - \nabla V_m$.

The magnetic potential at the point P on the *x-axis* (Fig. 7.2) is given by

$$V_m = -(\mu_o I/4\pi)\quad \Omega$$

where Ω is the solid angle subtended by the loop at the point P. Making use of the result of problem 1.28 we have

$$\Omega = 4 \arctan (l')^2 / z (z^2+2 l'^2)^{1/2}$$

Hence

$$\mathbf{B} = - \nabla V_m = (\mu_o I/\pi) \, \partial\Omega/\partial z$$

Differentiating Ω with respect to z and writing $z = 0$ and $l' = l/2$, we obtain finally

$$\mathbf{B} = \mu_o 2 \sqrt{2} I / \pi l$$

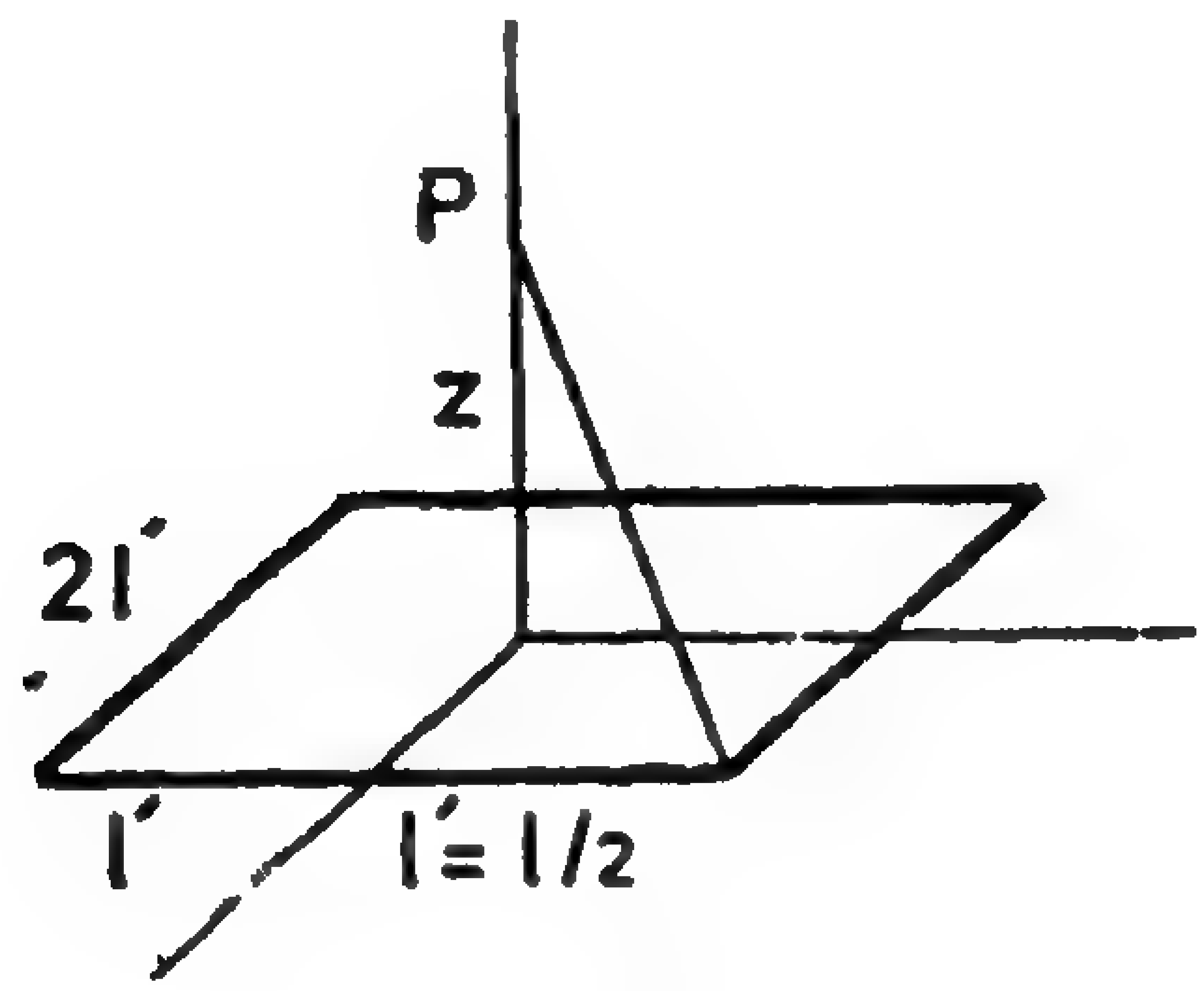


Fig. 7.2.

4. *A current loop is in the form a plane regular polygon of n sides inscribed in a cricle of radius R. If the loop carries a current I show that the flux density at the center of the loop is given by*

$$\mathbf{B} = (\mu_o I n/2 \pi R) \tan (\pi/n)$$

From this expression derive **B** *at the center of a circular current loop of radius R.*

From Fig. 7.3 it is evident that for each section

$$\theta_1 = \tfrac{1}{2}\pi - \pi/n$$

$$\theta_2 = \tfrac{1}{2}\pi + \pi/n$$

$$r = R \cos (\pi/n)$$

The flux density at the center due to one current segment is

$$B_1 = (\mu_0 I/4\pi r)\ [\cos(\tfrac{1}{2}\pi - \pi/n) - \cos(\tfrac{1}{2}\pi + \pi/n)]$$

the total flux density due to the n segments is

$$B = n\, B_1$$

Substituting for r , and B_1 we get

$$B = (\mu_0 In/2\pi R)\tan(\pi/n)$$

If the number of segments n increases, the regular polygon tends to a circle of radius R. In this case $\tan \pi/n$ tends to π/n, so that

$$B = \mu_0 I/2R$$

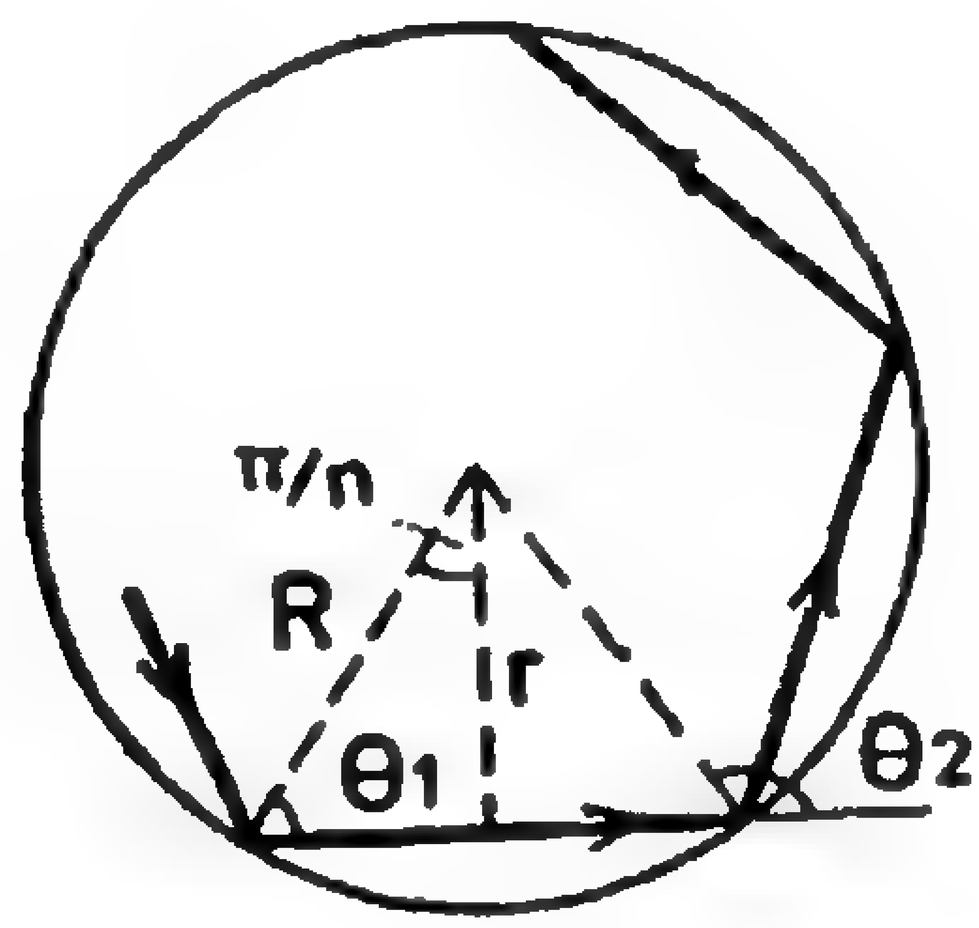


Fig. 7.3.

5. *Show that the integral from* $-\infty$ *to* ∞ *of the flux density on the axis of a circular current loop carrying a current* I *is exactly* $\mu_0 I$.

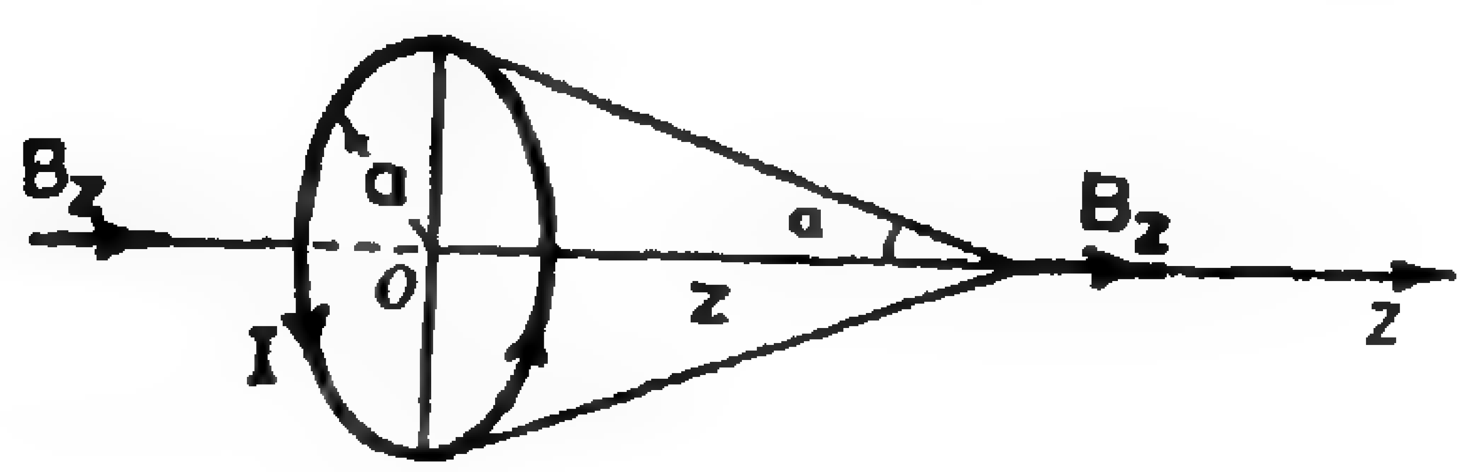


Fig. 7.4.

The flux density at any point on the axis of a plane circular loop (Fig. 7.4) is given by,

$$B_z = (\mu_0 I/2a) \sin^3 \alpha$$

From Fig. 7.4, $z = a \cot \alpha$

so that $dz = -a \operatorname{cosec}^2 \alpha \, d\alpha$

Thus

$$\int_{-\infty}^{\infty} B_z \, d_z = -\tfrac{1}{2} \mu_0 I \int_{\pi}^{0} \sin \alpha \, d\alpha$$

$$= \mu_0 I$$

6. *A three-phase transmission line consists of three very long parallel conductors lying in a horizontal plane with the outer conductors at distance D from the central one. At a given instant the current in the central conductor is I and that on each of the outer conductors is* $-\sqrt{3}\, I/2$. *Find the force on each of the conductors.*

The force between two wires carrying currents I_1 and I_2 and at a distance r apart is

$$f = \pm (\mu_0 I_1 I_2 / 2\pi r) \, \mathbf{a}_r \qquad N/m$$

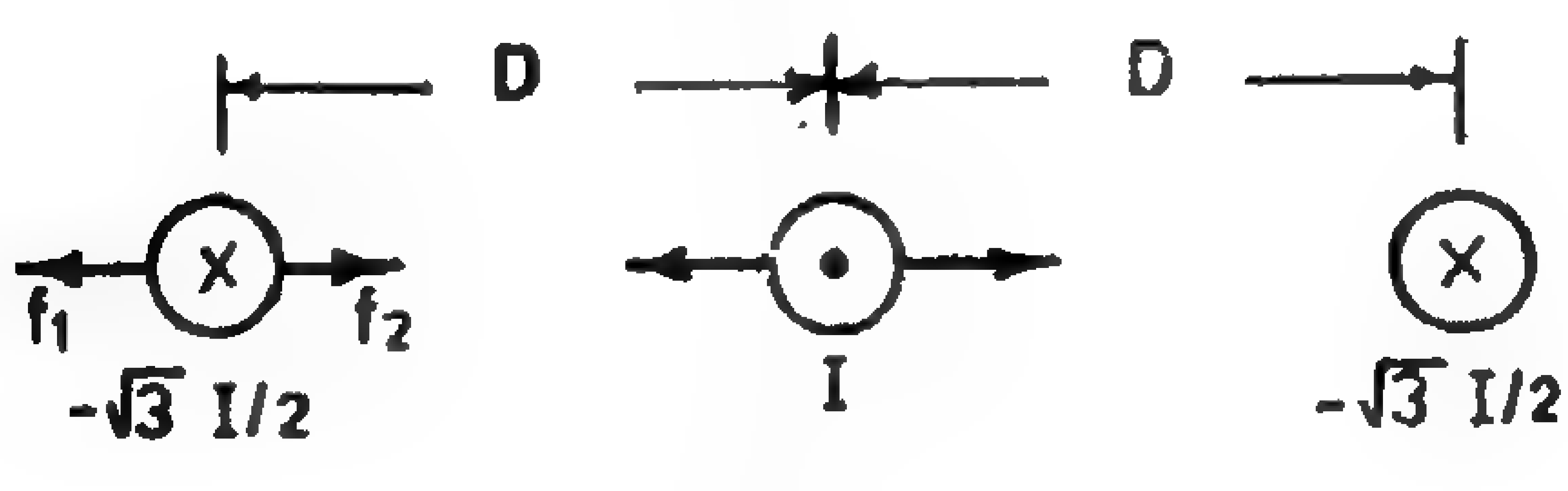


Fig. 7.5.

The force is attractive if the currents are flowing in the same direction and repulsive if in opposite directions. Thus the force on the central

wire is zero as shown in Fig. 7.5. The force between the outer wire and the central wire

$$f_2 = (\mu_0/2\pi D)\ I\ (\sqrt{3}\ I/2) = \sqrt{3}\ \mu_0 I^2/4\pi D\ ,\ N/m$$

And the force be the two outer wires is,

$$f_1 = [\mu_0/2\ \pi\ (2D)]\ (\sqrt{3}\ I/2)\ (\sqrt{3}\ I/2)$$
$$= 3\mu_0\ I^2/16\pi\ D \quad N/m$$

The force on each of the outer conductors is

$$f_2 - f_1 = (\mu_0 I^2\ /\ 4\ \pi D)\ (4\sqrt{3} - 3) \quad N/m$$

7. *Show that the flux inside a toroid of rectangular cross-section is given by*

$$\phi = (\mu_0 N\ Ib\ /\ 2\pi)\ \log\ [(R+\tfrac{1}{2}d)\ /\ (R-\tfrac{1}{2}d)]$$

where b is the breadth of the coil, R the radius of the coil, d is the radial depth of the coil, and N is the number of turns of the coil.

Applying Ampere's law to a circular path of radius r (Fig. 7.6)

$$NI = H_r\ 2\pi\ r$$
$$H_r = NI\ /\ 2\pi\ r$$

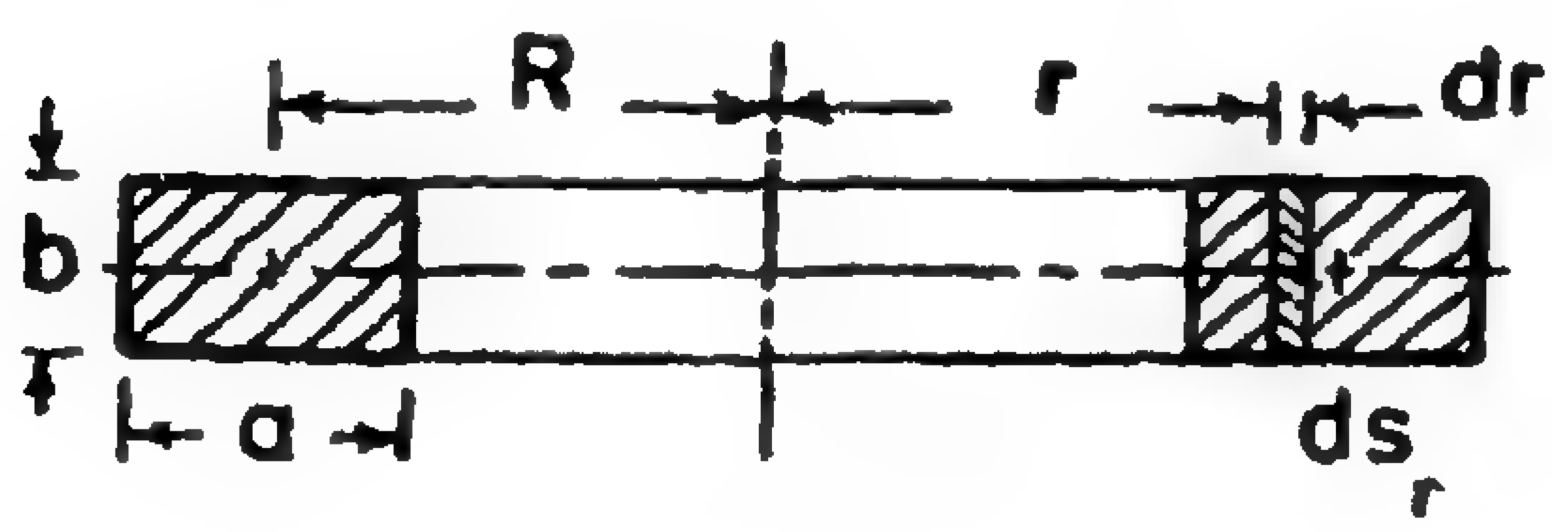


Fig. 7.6.

The flux through the element of area $dS_r = b\,dr$ is

$$d\phi_r = B_r\ b\ dr = \mu_0 H_r\ b\ dr$$

Integrating we get,

$$\phi = (\mu_o NIb / 2\pi) \int_{R-\frac{1}{2}d}^{R+\frac{1}{2}d} (1/r)\, dr$$

$$= (\mu_o N I b/2\pi) \log [(R+\tfrac{1}{2} d) / (R-\tfrac{1}{2} d)]$$

8. *Show that the flux inside a toroid of circular cross-section is given by*

$\mu_o N I \; [R - \sqrt{R^2 - a^2}]$

Applying Ampere's law to a circular path of radius $R + r \cos \theta$ (Fig. 7.7),

$$2\pi (R+r \cos \theta) H_r = N I$$

Hence $H_r = N I / [2\pi (R+r \cos \theta)]$

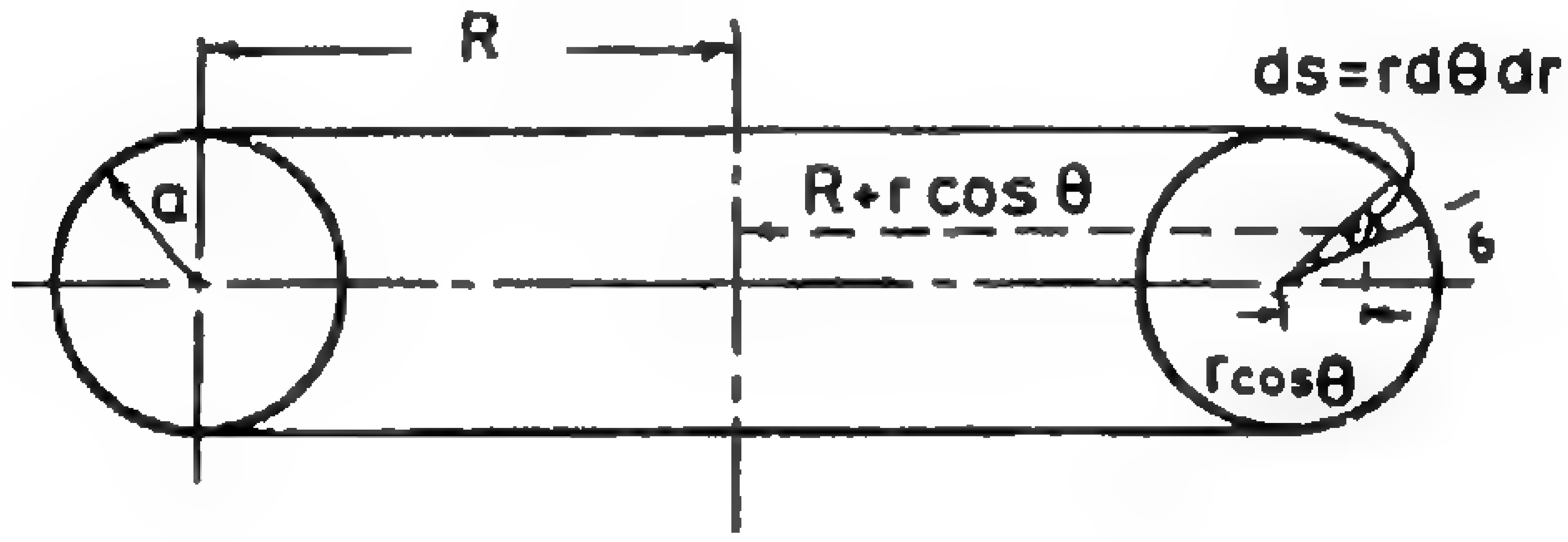


Fig. 7.7

The flux through the element of area $dS = r\, d\theta\, dr$ is

$$d\phi_r = B_r\, r\, d\theta\, dr = \mu_o H_r\, r\, d\theta\, dr$$

$$\phi = (\mu_o N I/2\pi) \int_{\theta=0}^{2\pi} \int_{r=0}^{a} (R+r \cos \theta)^{-1}\, r\, dr\, d\theta$$

Now $$\int_0^{2\pi} \frac{d\theta}{R+r \cos \theta} = 2 / (R^2 - r^2)^{1/2} \left[\tan^{-1} \frac{(R-r) \tan (\theta/2)}{(R^2-r^2)^{1/2}} \right]_0^{2\pi}$$

$$= 2\pi / (R^2-r^2)^{1/2}$$

Thus, $$\phi = \mu_o N I \int_0^a [r/ (R^2-r^2)^{1/2}]\, dr$$

$$= \mu_o N I \; [R - (R^2 - a^2)^{1/2}]$$

9. *A wire carrying a current I is in the shape of a plane curve whose equation in plane polar coordinates is* $r = f(\theta)$. *Show that the magnetic flux density at the pole is given by*

$$B = (\mu_0 I / 4\pi) \int d\theta / r$$

the integration being extended over the length of the wire.

Using the derived result show that the magnetic flux density at some point P in the plane of a circular loop of radius a is given by

$$B = [\mu_0 I a \,/\, 4\pi (a^2 - h^2)]\; E\,(2\pi,\ h/a)$$

where I is the current in the loop, h is the distance of the point P from the center of the loop, and $E(2\pi, h/a)$ *is the elliptic integral of the second kind.*

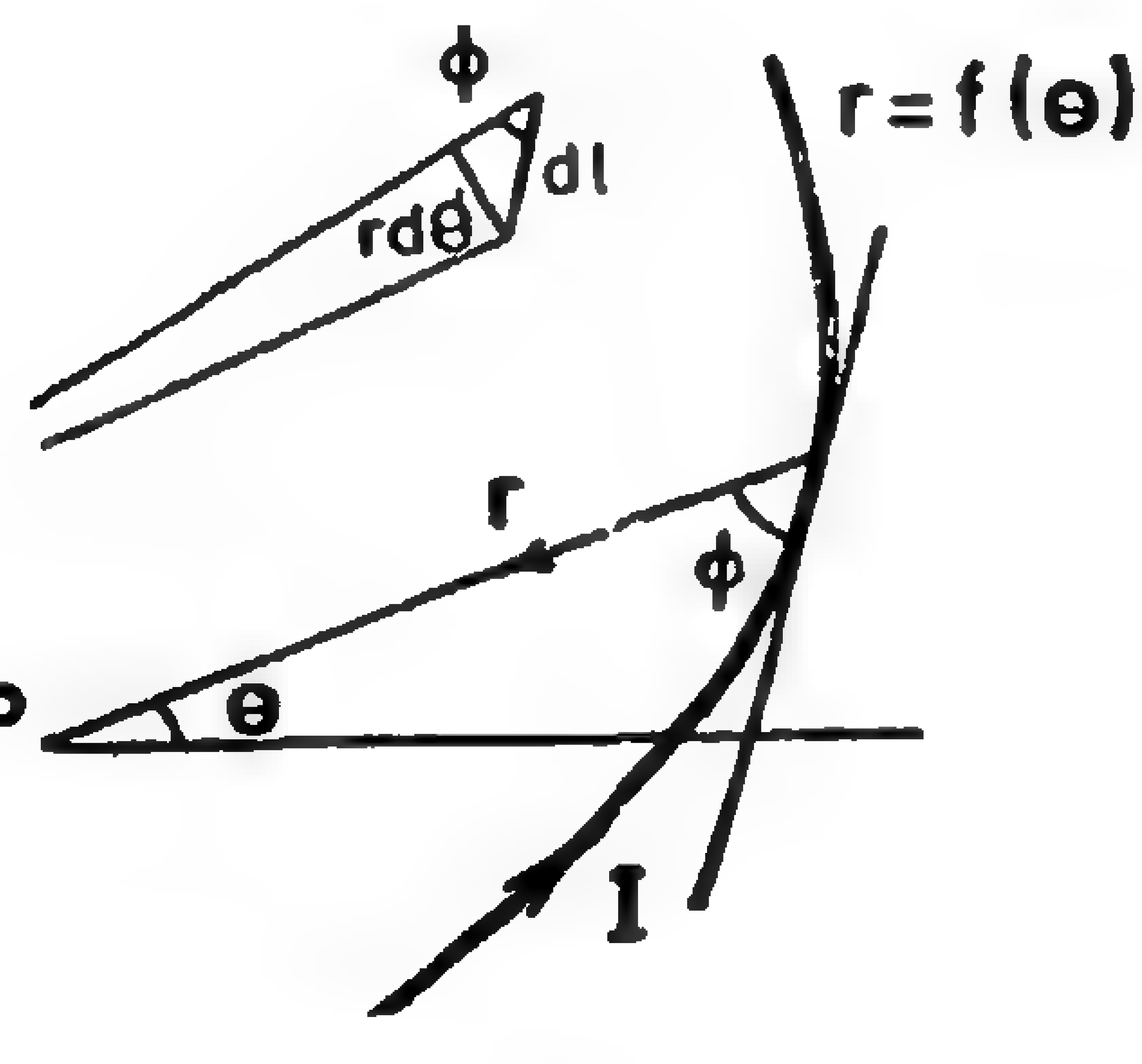


Fig. 7.8

The contribution to B at the pole P of an element of length **dl** (Fig. 7.8) is, by the Biot-Savart law,

$$dB = (\mu_0 I / 4\pi r^2)\ \mathbf{dl} \times \mathbf{a}_r$$

$$= \mu_0 I\ dl\ \sin\phi\ /\ 4\pi\ r^2$$

in a direction perpendicular to the plane of the curve. Since $dl \sin\phi = r\, d\theta$, it follows that,

$$B = (\mu_o I \,/\, 4\pi) \int d\theta \,/\, r$$

Now consider a plane circular loop as shown in Fig. 7.9; it is clear that

$$r = h \cos\theta + (a^2 - h^2 \sin^2\theta)^{1/2}$$

so that,

$$1/r = 1 \,/\, [h \cos\theta - (a^2 - h^2 \sin^2\theta)^{1/2}]$$

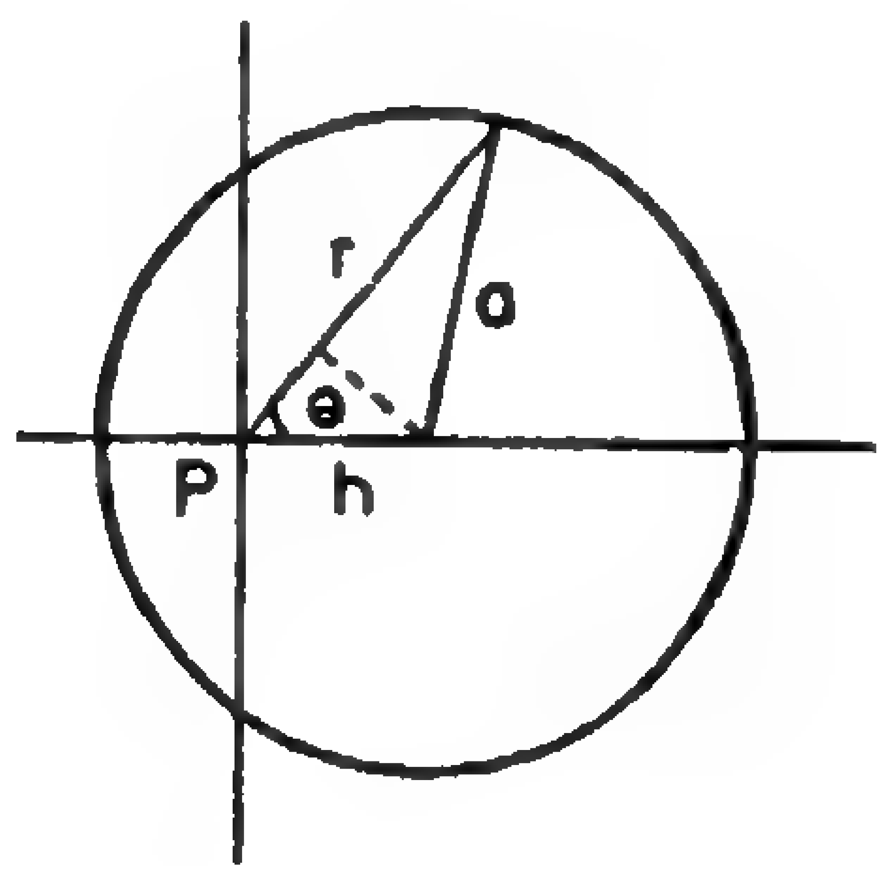


Fig. 7.9.

Multiplying the numerator and denominator by the conjugate, we obtain

$$1/r = [(a^2 - h^2 \sin^2\theta)^{1/2} - h \cos\theta] \,/\, (a^2 - h^2)$$

Hence the flux density is given by

$$B = [\mu_o I a \,/\, 4\pi\, (a^2 - h^2)] \int_0^{2\pi} [1 - (h^2/a^2) \sin^2\theta]^{1/2}\, d\theta$$

since the integral of $h \cos\theta$ between zero and 2π is zero. Also,

$$\int_0^{2\pi} [1 - k^2 \sin^2\theta]^{1/2}\, d\theta = E\,(2\pi, k)$$

is an elliptic integral of the second kind.

Hence $B = [\mu_o I a / 4\pi (a^2 - h^2)] E(2\pi, h/a)$

10. *By making use of the scalar magnetic potential find the flux density at a point on the axis of a solenoid of length L and number of turns N.*

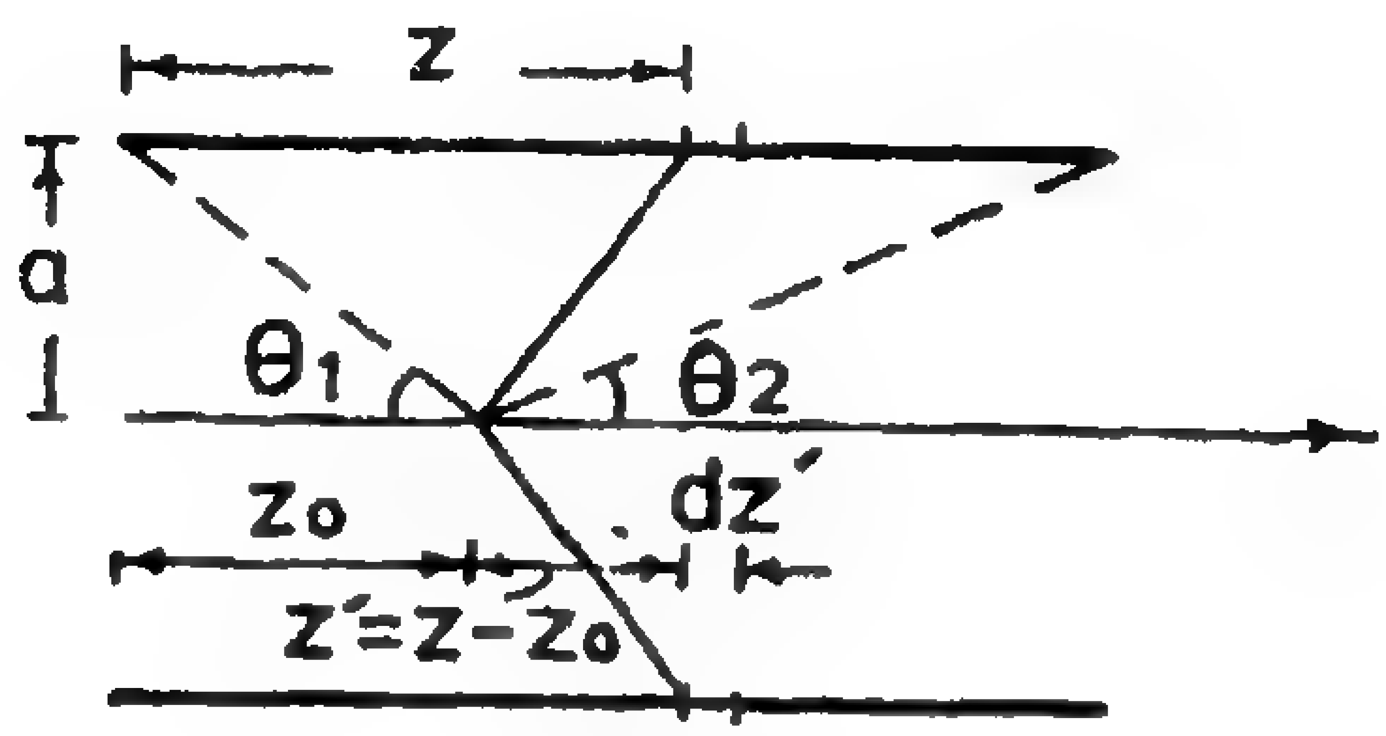


Fig. 7.10

Consider the solenoid to be located with its axis coincident with the z-axis as shown in Fig. 7.10. If the spacing between the turns is small the N turns are considered to be distributed with linear density $n = N/L$. Now the magnetic potential at point P $(z = z_0)$ on the axis of the solenoid, due to one turn only of the solenoid is given by

$$V'_m = -(\mu_o I / 4\pi) \ \Omega$$

where Ω is the solid angle at P subtended by the single current loop.

A section of length dz, is equivalent to a circular loop of current $n\, I\, dz$. The magnetic potential at P due to this loop is

$$d V_m = -(\mu_o n I dz / 4\pi) \ \Omega \tag{1}$$

thus $\quad dB_z = -\nabla_p (dV_m)$

$$= (\mu_o n I / 4\pi) \, dz \, \partial\Omega / \partial z_o$$

Since Ω is a function of $(z-z_0)$ it follows that

$$\partial \Omega / \partial z_0 = - \partial \Omega / dz$$

thus $$\int dB_z = -(\mu_0 n I / 4\pi) \int_{\Omega_1}^{\Omega_2} d\Omega$$

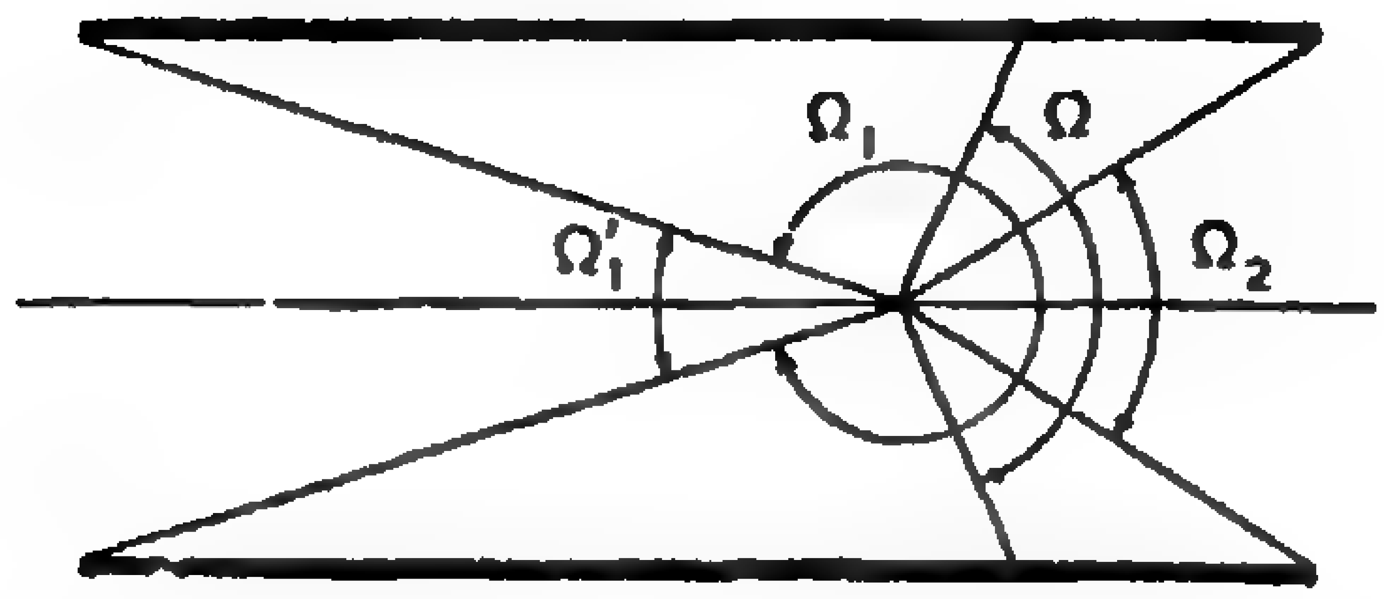


Fig. 7.11.

where, Ω_1 and Ω_2 are shown in Fig. 7.11. It is clear that

$$\Omega_1 = 4\pi - \Omega'_1$$
$$\Omega_2 = 2\pi (1 - \cos\theta_2)$$
$$\Omega'_1 = 2\pi (1 - \cos\theta_1)$$

Hence

$$\Omega_1 = 4\pi - 2\pi (1 - \cos\theta_1) = 2\pi (1 + \cos\theta_1)$$

Thus we have

$$B_z = (\mu_0 n I / 4\pi)(\Omega_1 - \Omega_2)$$
$$= \tfrac{1}{2} \mu_0 n I \; (\cos\theta_1 + \cos\theta_2)$$

For a long solenoid

$$\Omega_1 = 4\pi , \qquad \Omega_2 = 0 , \qquad \text{and}$$

$$B_z = \mu_0 n I$$

Alternative solution for the previous problem is available. Substituting for Ω in (1) gives

$$d V_m = -(\mu_0 n I dz' / 4\pi)\ \Omega$$

$$\Omega = 2\pi (1 - \cos\ \theta)$$

with $\cos\ \theta \quad z' / (a^2 + z'^2)^{1/2}$

Hence

$$d V_m = -\tfrac{1}{2}\ \mu_0 n I\ [1 - z'^2 / (a^2 + z'^2)^{1/2}]\ dz'$$

Integrating between the limits $z' = -z_0$ and $z' = L - z_0$ gives

$$V_m = -\tfrac{1}{2}\ \mu_0 n I \{ L - [(L-z_0)^2 + a^2]^{1/2} + (z_0^2 + a^2)^{1/2} \}$$

Now the magnetic flux density is given by

$$\mathbf{B} = -\nabla_p V_m$$

From the expression of V_m it is clear that we have only B_z

$$B_z = -\partial V_m / \partial z_0 = -\tfrac{1}{2}\ \mu_0 n I \{ (L - z_0) / [(L-z_0)^2 + a^2]^{1/2} + z_0 / (z_0^2 + a^2)^{1/2} \}$$

$$= \tfrac{1}{2}\ \mu_0 n I\ (\cos\ \theta_2 + \cos\ \theta_1)$$

which is the same result obtained before.

11. *An infinitely long line carrying a current I lies in the xy-plane and is parallel to the x-axis at $y = b$. Find the magnetic scalar potential at the point $(0, 0, z)$.*

The magnetic potential at the point $(0, 0, z)$ is

$$V_m = -(\mu_0 I / 4\pi)\ \Omega$$

where Ω is the solid angle subtended at P by the infinite line. This solid angle is equal to the solid angle at P subtended by the part of the xy-plane bounded by $x = \pm\infty$ and $y = -\infty, b$ so that

$$\Omega = \int (\cos\ \delta / r^2)\ dS$$

where θ is the angle between the z-axis and the line joining P (0, 0, z) to the point (x, y) in the $x\,y$-plane, and $dS = dx\,dy$

$$\Omega = -\int_{y=-\infty}^{b} \int_{-\infty}^{\infty} z\, dx\, dy \,/\, (x^2+y^2+z^2)^{3/2}$$

$$= -z \int_{-\infty}^{b} [dy\,/(y^2+z^2)]\; [x\,/\,(x^2+y+z^2)^{1/2}]_{-\infty}^{\infty}$$

$$= -2z \int_{-\infty}^{b} dy \,/\, (y^2+z^2)$$

$$= -2\,[\tan^{-1}(b/z) + \pi/2] = -(2\theta+\pi)$$

$$= -2\,(\pi/2-\phi) - \pi = -2\,(\pi-\phi)$$

Hence, $\Omega = -2\,(\pi-\phi)$

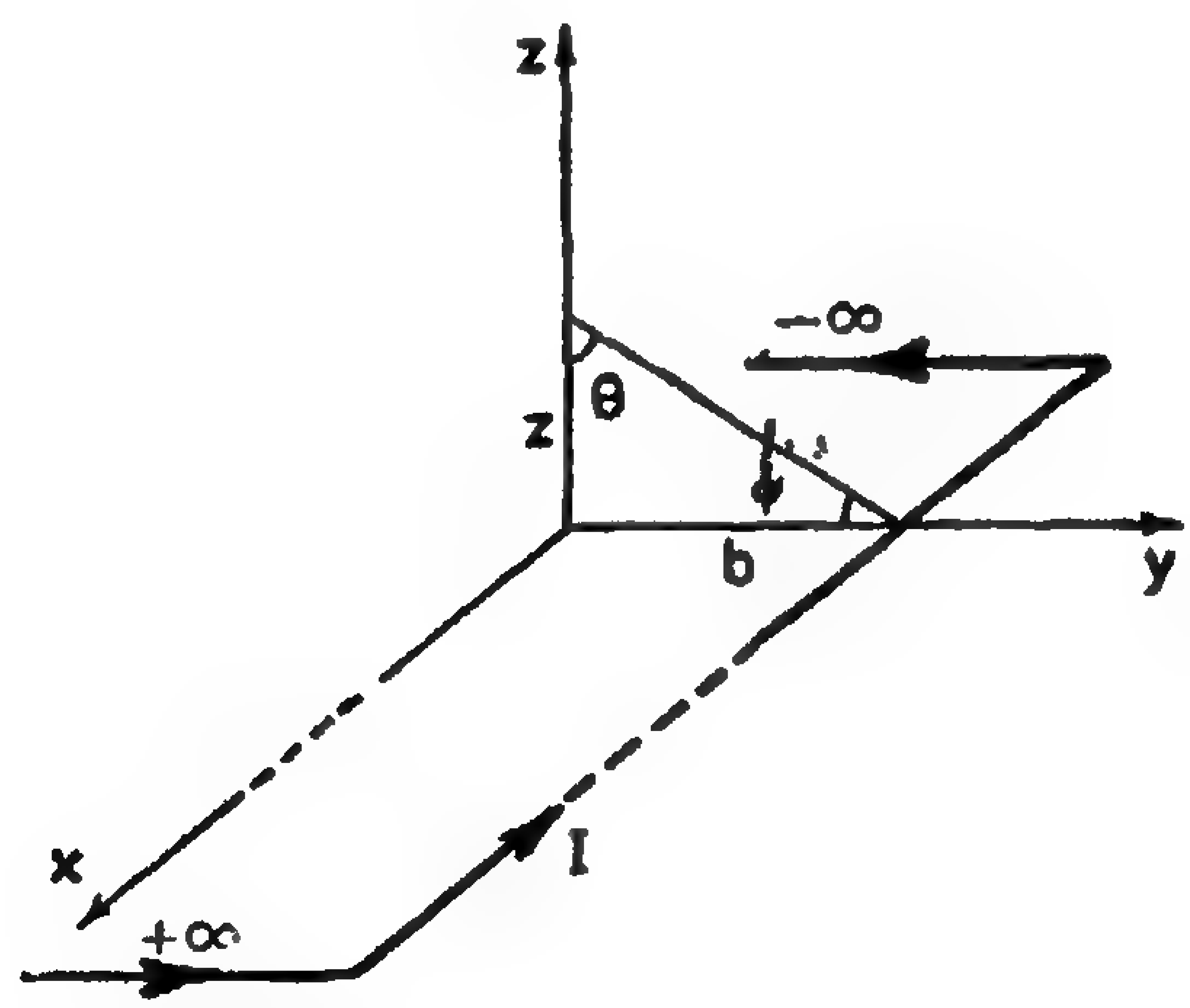


Fig. 7.12.

Substituting in the expression of V_m, we get,

$$V_m = (\mu_0 I / 2\pi)\,(\pi-\phi)$$

The magnetic flux density at P is given by

$$\mathbf{B} = -\nabla V_m = -(\partial V_m/\partial r)\,\mathbf{a}_r - (1/r)\,\partial V_m/\partial\theta\,\mathbf{a}_\phi$$

$$= (\mu_o I/2\pi\, r)\,\mathbf{a}_\phi$$

where $r = \sqrt{(z^2+b^2)}$ and ϕ is shown in Fig. 7.12.

12. *For the loop shown in Fig. 7.13 show that the magnetic flux density at O is given by*

$$B = (\mu_o I/2\pi\, a)\,[(\pi-\alpha) + (1-\cos\alpha)/\sin\alpha]$$

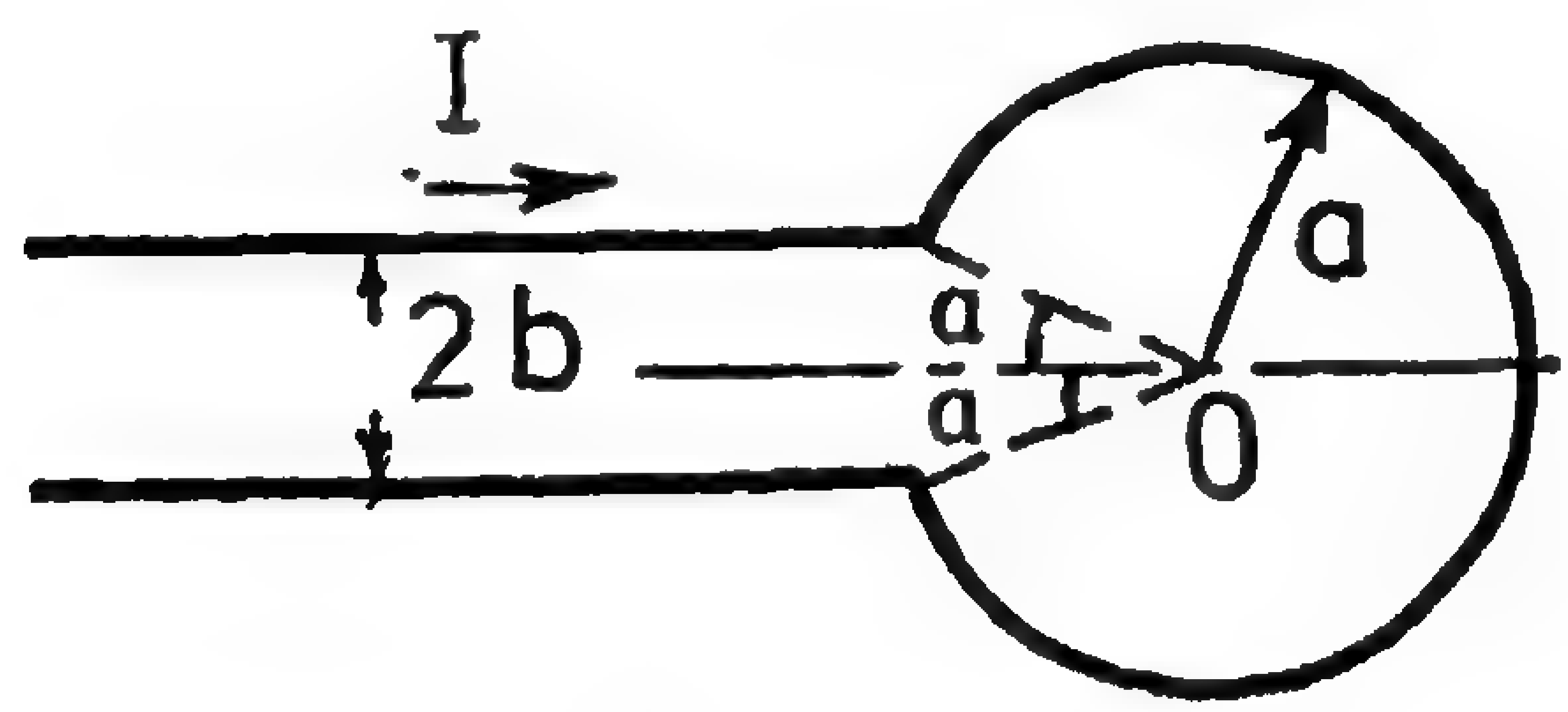


Fig. 7.13.

The magnetic field at the center O will be composed of two parts. The first part is due to that of the curved part which extends from $\theta = \alpha$ to $2\pi-\alpha$. Using Biot-Savart law, we get,

$$B_1 = (\mu_o/4\pi)\int_{\alpha}^{2\pi-\alpha} I\,a\,d\theta/a^2$$

$$= (\mu_o/2\pi)\,(\pi-\alpha)$$

In the direction normal outward from the paper. The second part B_2, is due to the two infinite wires which are parallel and at distance $2b$ apart. The directions of the current in the two wires are such that the magnetic field of one helps that of the other. Thus we have

$$B_2 = 2\,[(\mu_0 I / 4\pi r)(\cos\theta_1 - \cos\theta_2)]$$

where $\theta_1 = 0$ and $\theta_2 = \alpha$. With $r = b$ this expression gives

$$B_2 = (\mu_0 I / 2\pi b)(1 - \cos\alpha)$$

in the same direction as that of B_1. Thus the total magnetic field at O is

$$\begin{aligned} B &= B_1 + B_2 \\ &= (\mu_0 I / 2\pi a)\,[(\pi - \alpha) + (a/b)(1 - \cos\alpha)] \\ &= (\mu_0 I / 2\pi a)\,[(\pi - \alpha) + (1 - \cos\alpha) / \sin\alpha] \end{aligned}$$

13. *Two infinitely long line currents are located at $x = \pm a$, $y = 0$, with their axes parallel to the z–axis. Find the equation of the magnetic scalar equipotentials and the flux lines when*

(i) *The currents flow in the same sense.*

(ii) *The currents flow in opposite senses.*

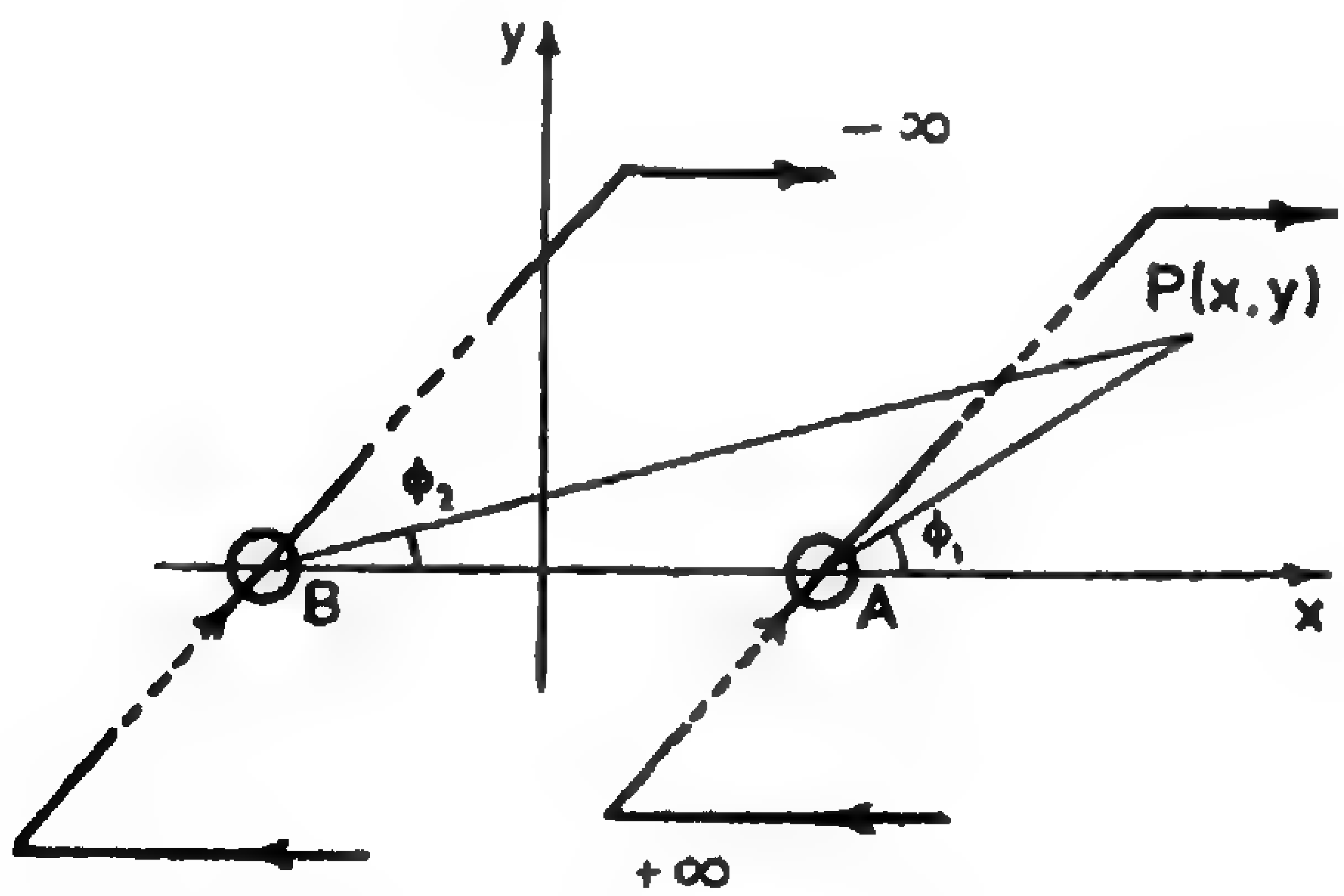


Fig. 7.14.

(i) The solid angle subtended at point P by current A, Fig. 7.14, is (see problem 7.11).

$$\Omega_{pA} = 2(\pi - \phi_1)$$

Similarly

$$\Omega_{pB} = 2(\pi - \phi_2)$$

so that the total solid angle at P is

$$\Omega_p = 2(\pi - \phi_1) + 2(\pi - \phi_2)$$
$$= -2(\phi + \phi_2) + 4\pi$$

Hence the magnetic scalar potential at P is

$$V_p = -(\mu_0 I / 4\pi)\,\Omega$$
$$= (\mu_0 I / 2\pi)(\phi_1 + \phi_2) + \text{Constant} \qquad (1)$$

therefore the equipotentials are

$$\phi_1 + \phi_2 = \text{Const.} = k'$$

so that $tan\,(\phi_1 + \phi_2) = k$

$$= (tan\,\phi_1 + tan\,\phi_2) \,/\, (1 - tan\,\phi_1\, tan\,\phi_2) \qquad (2)$$

But we have

$$tan\,\phi_1 = y/(x-a)\,, \qquad tan\,\phi_2 = y/(x+a)$$

so tnat (2) becomes

$$2xy\,/\,(x^2 - y^2 - a^2) = k$$

This gives, fot $h = 1/k$,

$$x^2 - 2\,h\,xy - y^2 = a^2 \qquad (3)$$

which is a family of rectangular hyperbolas passing through A and B, and h is a parameter. The magnetic flux lines are the family of curves which are or thogonal to the family of rectangular hyperbolas representing the equipotentials. As shown in Problem 2.27 these are the set

of Cassini ovals $r_1 r_2$ = constant. These may also be derived as follows: the equation of the magnetic lines is

$$B_x / dx = B_y / dy$$

or $$dy / dx = B_y / B_x$$

$$B_x = -\partial V/\partial x = 2y\,(x^2+y^2+a^2) / [(x^2-y^2-a^2)^2+4x^2y^2]$$

$$B_y = -\partial V/\partial y = -2x\,(x^2+y^2-a^2) / [(x^2-y^2-a^2)^2+4x^2y^2]$$

so that

$$dy/dx = -x\,(x^2+y^2-a^2) / y\,(x^2+y^2+a^2)$$

or

$$x\,(x^2+y^2-a^2)\,dx+y\,(x^2+y^2+a^2)\,dy = 0$$

This is an exact differential equation and its solution is therefore given by

$$x^4+y^4+2x^2y^2-2a^2x^2+2a^2y^2 = k \tag{4}$$

With $r_1^2 = y^2+(x-a)^2$, and $r_2^2 = y^2+(x+a)^2$, it is easily shown that the equation $r_1 r_2$ = constant is the same as equation (4).

(ii) The currents flow in opposite senses.

In this case the solid angles subtended at P by the currents at A and B are,

$$\Omega_{pA} = 2\,(\pi-\phi_1)$$

$$\Omega_{pB} = -2\,(\pi-\phi_2)$$

and $$\Omega_p = -2\,(\phi_1-\phi_2)$$

The magnetic scalar potential at P is thus given by

$$V_p = \mu_0 I\,(\phi_1-\phi_2) / 2\pi$$

The equation of the equipotentials is therefore

$$\phi_1 - \phi_2 = \text{Constant} = k$$

or $$(\tan\phi_1 - \tan\phi_2) / (1 + \tan\phi_1 \tan\phi_2) = \tan k = 1/h$$

or $$x^2 + 2a\,hy + y^2 = a^2 \tag{5}$$

Which is the equation of a family of circles having their centers on the y-axis and passing through A and B.

The magnetic flux lines are the family of curves which are orthogonal to the circles representing the equipotentials, and are therefore a set of circles with their centers on the x-axis (see Problem 2.25).

14. *Two identical plane circular coils (Helmholtz coils) of radius a carry a current I in the same sense and are placed coaxially at a distance 2d apart. Plot the variation of the flux density along the common axis in the region between the two coils. Find the relations between a and d for which the gradient of the axial field midway between the two coils is minimum (maximum field uniformity).*

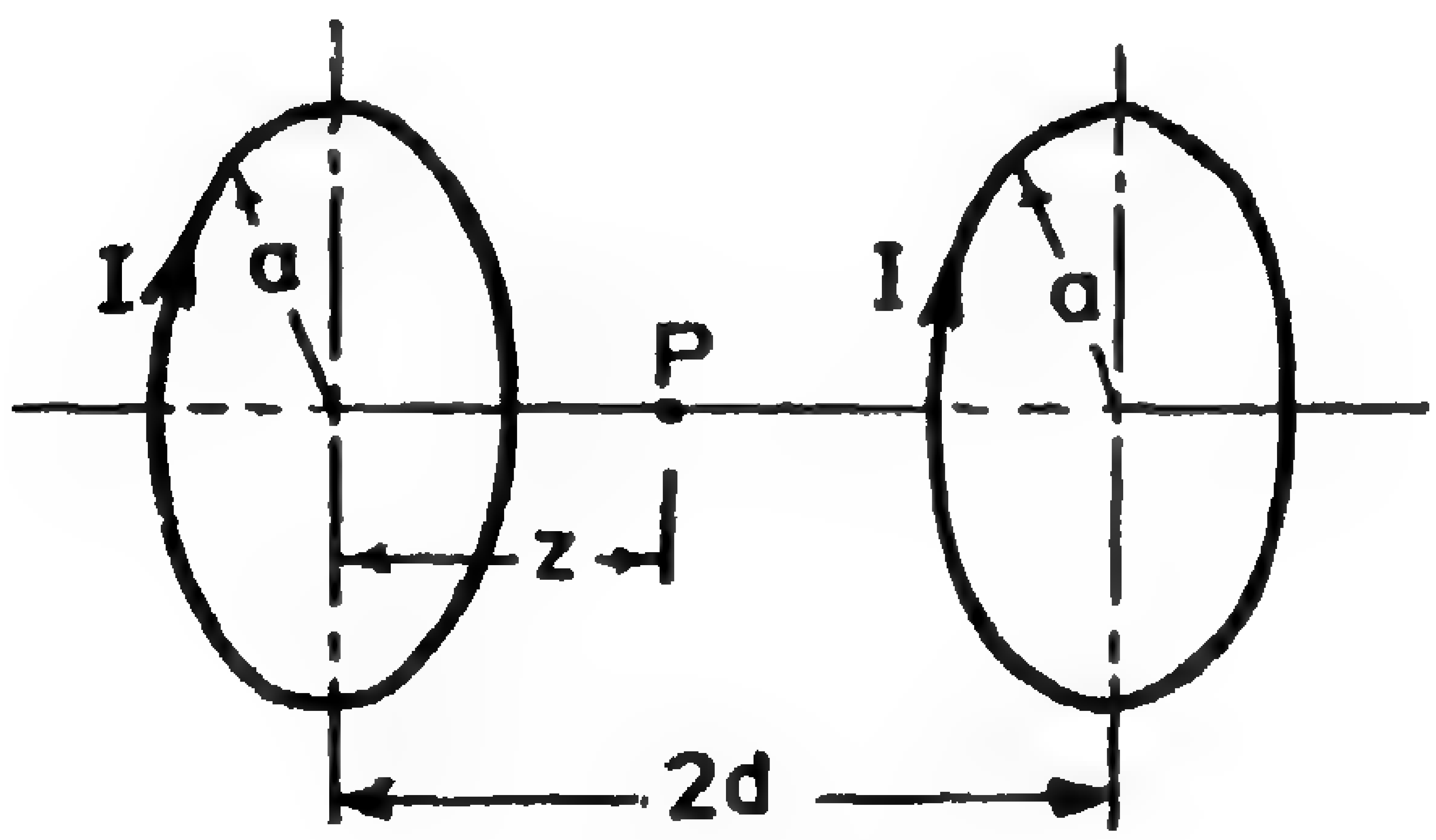


Fig. 7.15.

For a plane circular loop of N turns and radius a, the flux density at a point on the axis at a distance z from the loop center is

$$B = (\mu_0 NI/2)\, a^2 / (z^2+a^2)^{3/2}$$

For the two coils shown in Fig. 7.15. the flux density at point P is

$$B = \tfrac{1}{2}\, \mu_0 N I a^2 \left\{ 1/(z^2+a^2)^{3/2} + 1/[(2d-z)^2+a^2]^{3/2} \right\}$$

If the gradient of B is to be a minimum at $z = d$, then $d^2B/dz^2 = 0$ at $z = d$. Differentiating twice the above expression and then setting $z = d$ gives,

$$d^2B/dz^2 = -3\, \mu_0 I N a^2\, [(a^2-4d^2) / (d^2+a^2)^{7/2}]$$

This is zero when $a^2 = 4d^2$. Thus the required relationship is $a = 2d$. For this separation the flux density at the midpoint on the axis is

$$B = (8/5\sqrt{5})\, \mu_0 I N / a$$

The Helmholtz coils are often used to produce a relatively uniform field over a small region of space.

15. *Two plane circular coils of radii a, b and of M and N turns carry a current I in the same sense and are placed coaxially at a distance $\frac{1}{2}(a+b)$ apart. Show that on the axis near the point distance $\frac{1}{2}a$, $\frac{1}{2}b$ from the two coils the magnetic field is so uniform that its first two differential coefficients vanish at the point if $M/a^2 = N/b^2$ and its third in addition if $a = b$, and $M = N$.*

The origin is taken at distances $\frac{1}{2}a$, $\frac{1}{2}b$ from the two coils as shown in Fig. 7.16. The magnetic flux density at a point P near the origin on the axis is

$$B = \tfrac{1}{2}\,\mu_0 M I a^2 / [a^2 + (\tfrac{1}{2}a - z)^2]^{3/2}$$
$$+ \tfrac{1}{2}\,\mu_0 N I b^2 / [b^2 + (\tfrac{1}{2}b + z)^2]^{3/2}$$
$$= (\mu_0 M I / \sqrt{5}\,a) / (1 - 4z/5a + 4z^2/5a^2)^{3/2}$$
$$+ (\mu_0 N I / \sqrt{5}\,b) / (1 + z/5b + 4z/5b^2)^{3/2} \qquad (1)$$

At point O the magnetic flux density is

$$B = (\mu_0 I / \sqrt{5})\,(M/a + N/b) \qquad (2)$$

Since z is small (1) can be written as

$$B = (\mu_0 I / \sqrt{5})\,[(M/a)\,(1 + 6z/5a - 32z^3/25a^3 - \ldots)$$
$$+ (N/b)\,(1 - 6z/5b + 32z^3/25b^3 - \ldots)] \qquad (3)$$

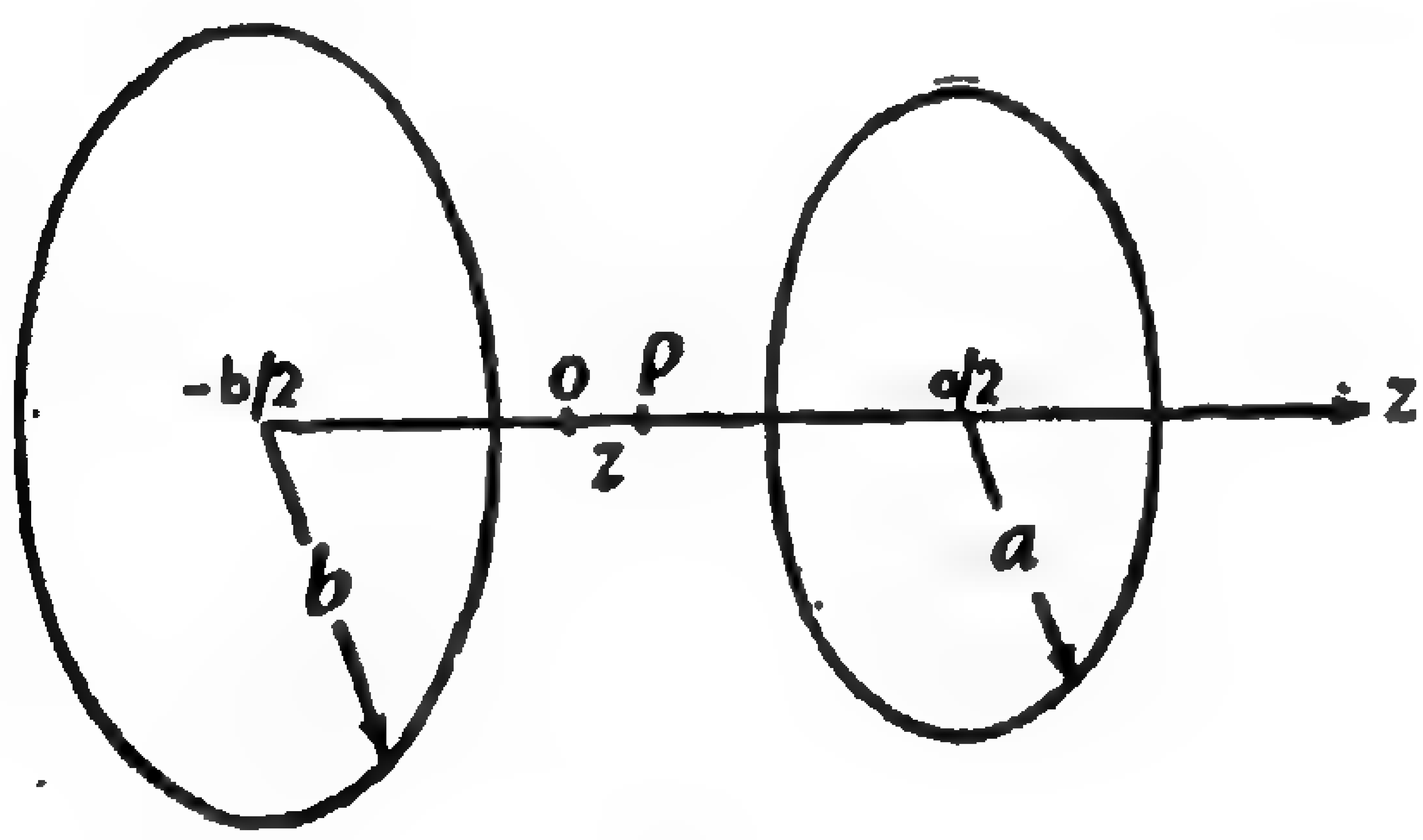


Fig. 7.16.

By taking the derivatives of B from (3) we conclude that dB/dz, and d^2B/dz^2 vanish at $x = 0$ if $M/a^2 = N/b^2$. Also we notice that d^3B/dz^3 vanishes if $M/a^4 = N/b^4$. This is satisfied when $M/a^2 = N/b^2$ if $a = b$ and $M = N$.

16. *An infinitely long conductor has a circular cross section of radius R. The conductor has a cylindrical hole of radius r with its axis parallel to that of the conductor and at a distance d from its center (Fig. 7.17). If the conductor carries a uniform current density*

show that the magnetic flux density inside the hole is uniform and given by $\mathbf{B} = \frac{1}{2}\mu_0\, \mathbf{j}\times\mathbf{d}$, *where* $\mathbf{d}$ *is the vector distance from the center of the conductor to the center of the hole.*

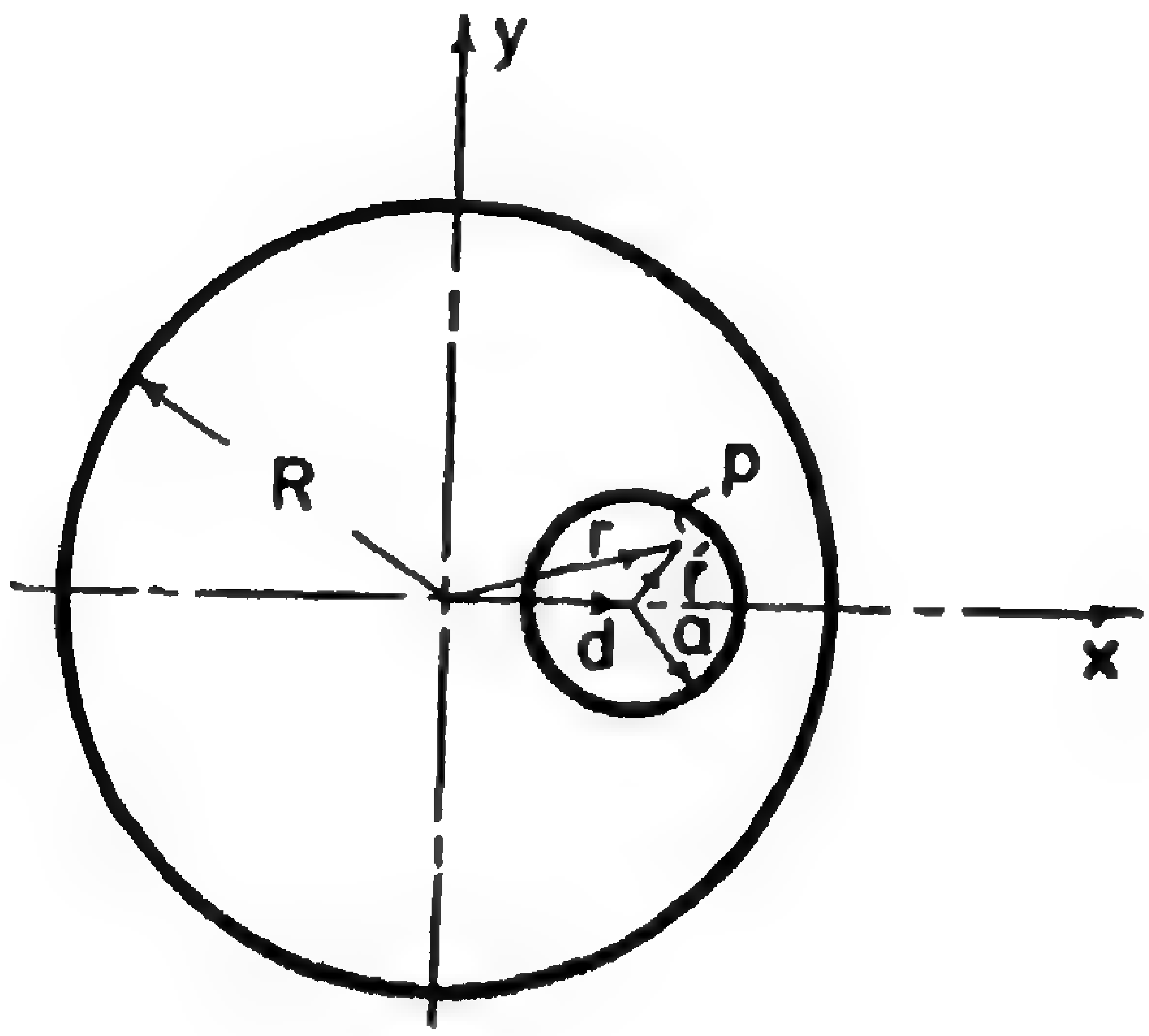


Fig. 7.17.

The simplest method of solving this problem is by making use of the principle of superposition. If there were no hole the flux density at any point P would be constant on the circle of radius r through P. Since the current enclosed by this circle is $\pi r^2 j$,

$$B' = \mu_0 \pi r^2 j \,/\, 2\pi r = \tfrac{1}{2}\mu_0 r j$$

or $$\mathbf{B}' = \tfrac{1}{2}\mu_0\, \mathbf{j}\times\mathbf{r}$$

The flux density at P due to a current density $-\mathbf{j}$ in the hole portion is

$$\mathbf{B}'' = -\tfrac{1}{2}\mu_0\, \mathbf{j}\times\mathbf{r}'$$

Thus the total flux density at P is

$$\mathbf{B}_o = \mathbf{B}' + \mathbf{B}'' = \tfrac{1}{2}\,\mu_o\,\mathbf{j}\times(\mathbf{r}-\mathbf{r}')$$

$$= \tfrac{1}{2}\,\mu_o\,\mathbf{j}\times\mathbf{d} = \tfrac{1}{2}\,\mu_o\,j\,\mathbf{a}_z\times d\mathbf{a}_x$$

$$= \tfrac{1}{2}\,\mu_o\,j\,d\,\mathbf{a}_y = B_y\,\mathbf{a}_y$$

since $\quad j = I/\pi\,(R^2-a^2)$, we have

$$B_y = \mu_o\,I\,d\,/\,2\pi\,(R^2-a^2)$$

17. *Each of two identical coils consists of N turns uniformly wound on the frustrum of a right circular cone (Fig. 7.18). If the coils carry a current I in the same direction, find the magnetic flux density at the common apex of the cones.*

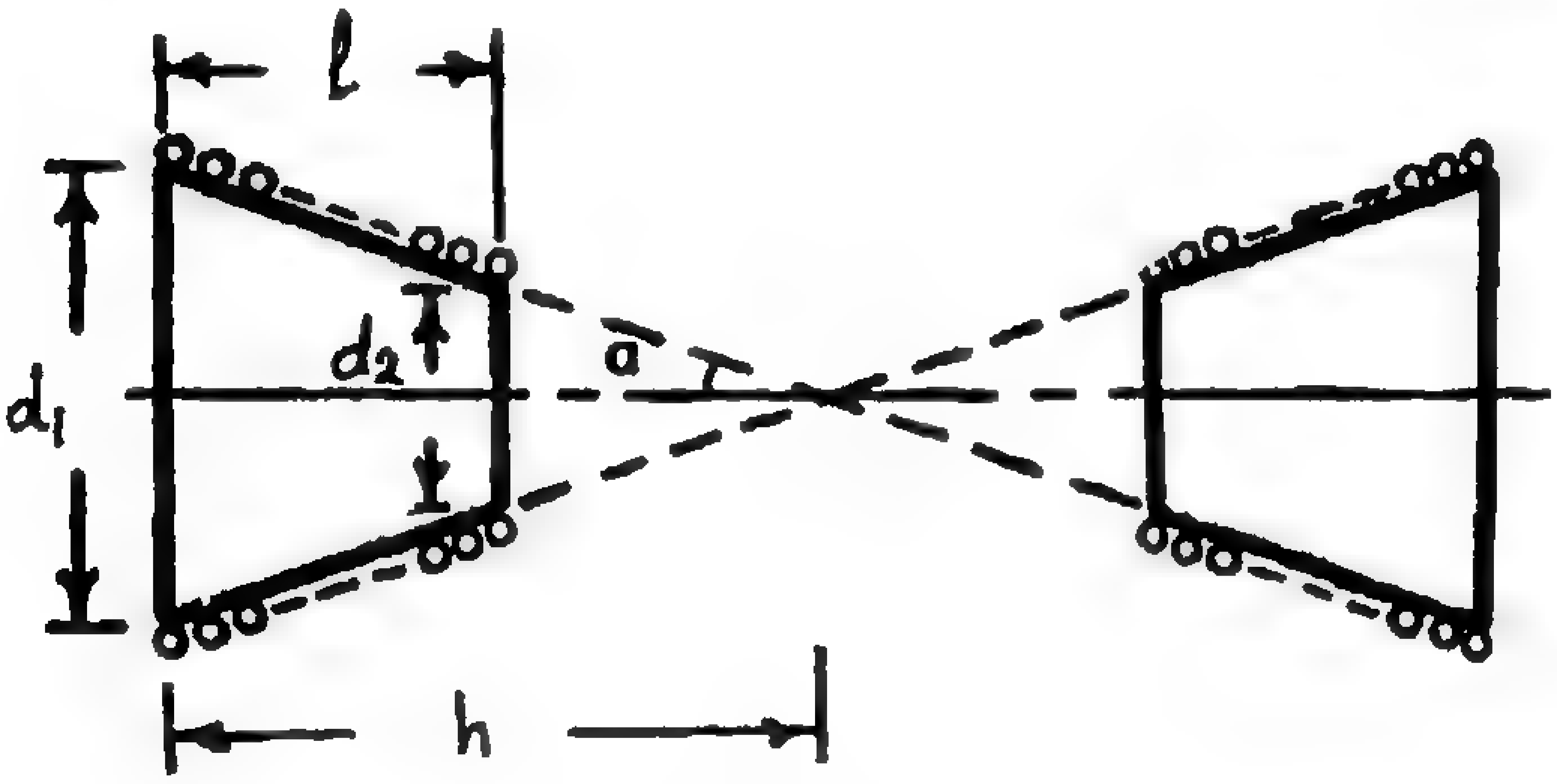


Fig. 7.18.

The flux density on the axis of a circular-coil of radius a is given by (Fig. 7.4),

$$B_z = (\mu_o\,IN/2a)\,\sin^3\alpha$$

Along the frustrum of the coil, the turn density is

$$n = N/S = N\cos\alpha/l.$$

The flux density at the apex of the cone due to an elementary section of height $dz = ds \cos \alpha$ is (Fig. 7.19),

$$dB_z = (\mu_0/2d)\ I\ n\ ds\ \sin^3 \alpha$$
$$= (\mu_0\ I N\ \sin^3 \alpha/2d\ l)\ dz$$

and with $\tan \alpha = d/z$,

$$dB_z = (\mu_0 N I / 2l) \cos \alpha \sin^2 \alpha\ dz/z$$

Integrating between the limits $z = h-l$ and $z = h$ and multiplying by 2 to take into account both frustrums gives

$$B_z = (\mu_0 N I/l) \cos \alpha \sin^2 \alpha \log [h/(h-l)]$$

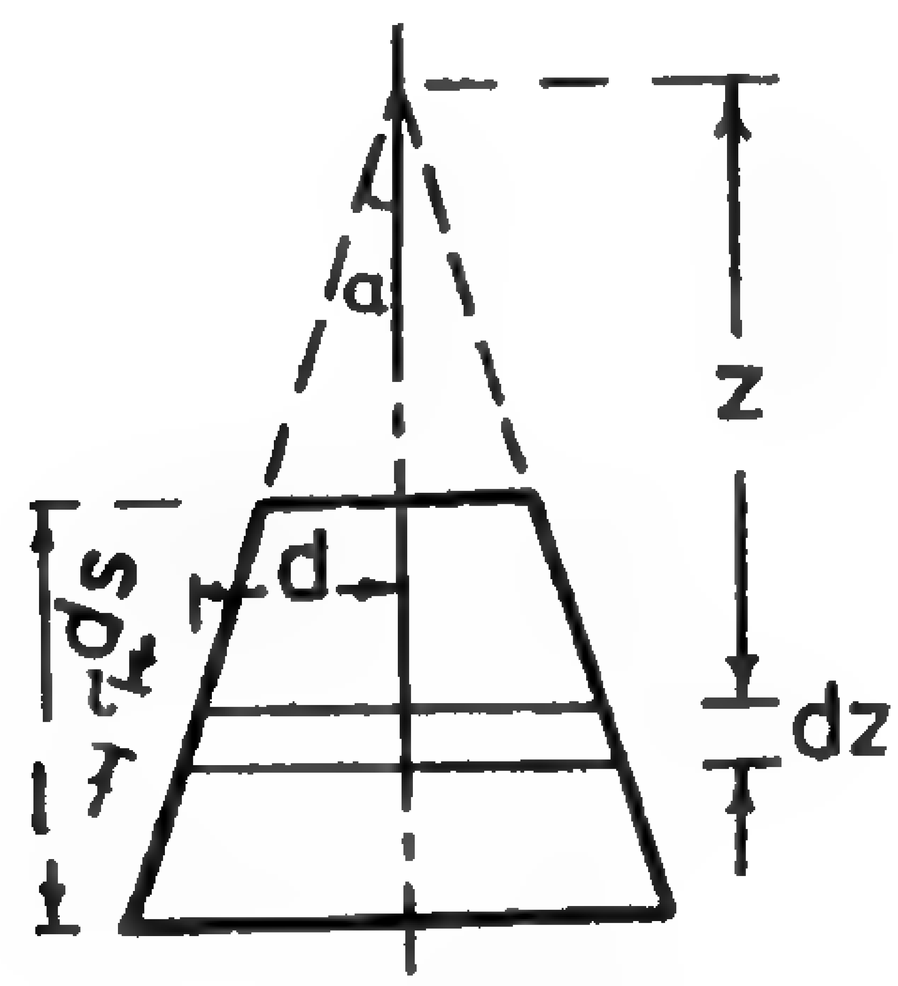


Fig. 7.19.

17. *A thin wire is bent into the form of a circular helix of angle θ, radius a and N complete turns. If a current I flows through the wire show that the axial component of* **B** *at the center of the helix is given by*

$$B = \mu_0 N I / 2a\ (1+\pi^2 N^2 \tan^2 \theta)^{1/2}$$

The equation of the helix is given by

$$x = a \cos \phi$$
$$y = a \sin \phi$$
$$z = a\ \phi \tan \alpha$$

where α is the pitch angle of the winding so that z increases by $2\pi a \tan\alpha$ when ϕ increases by 2π (Fig. 7.20). From the Biot–Savart law, the z–component of B is given by

$$B_z = (\mu_0 I/4\pi) \oint [\mathbf{dS} \times \mathbf{r}]_z \,/\, r^3$$

$$(\mu_0 I/4\pi) \oint (r_y \, d\,S_x - r_x \, dS_y) \,/\, r^3$$

$$r_x = -a \cos\phi \qquad d\,S_x = -a \sin\phi \, d\phi$$

$$r_y = -a \sin\phi \qquad d\,S_y = a \cos\phi \, d\phi$$

$$r_z = -a\,\phi \tan\alpha + c \qquad d\,S_z = a \tan\alpha \, d\phi$$

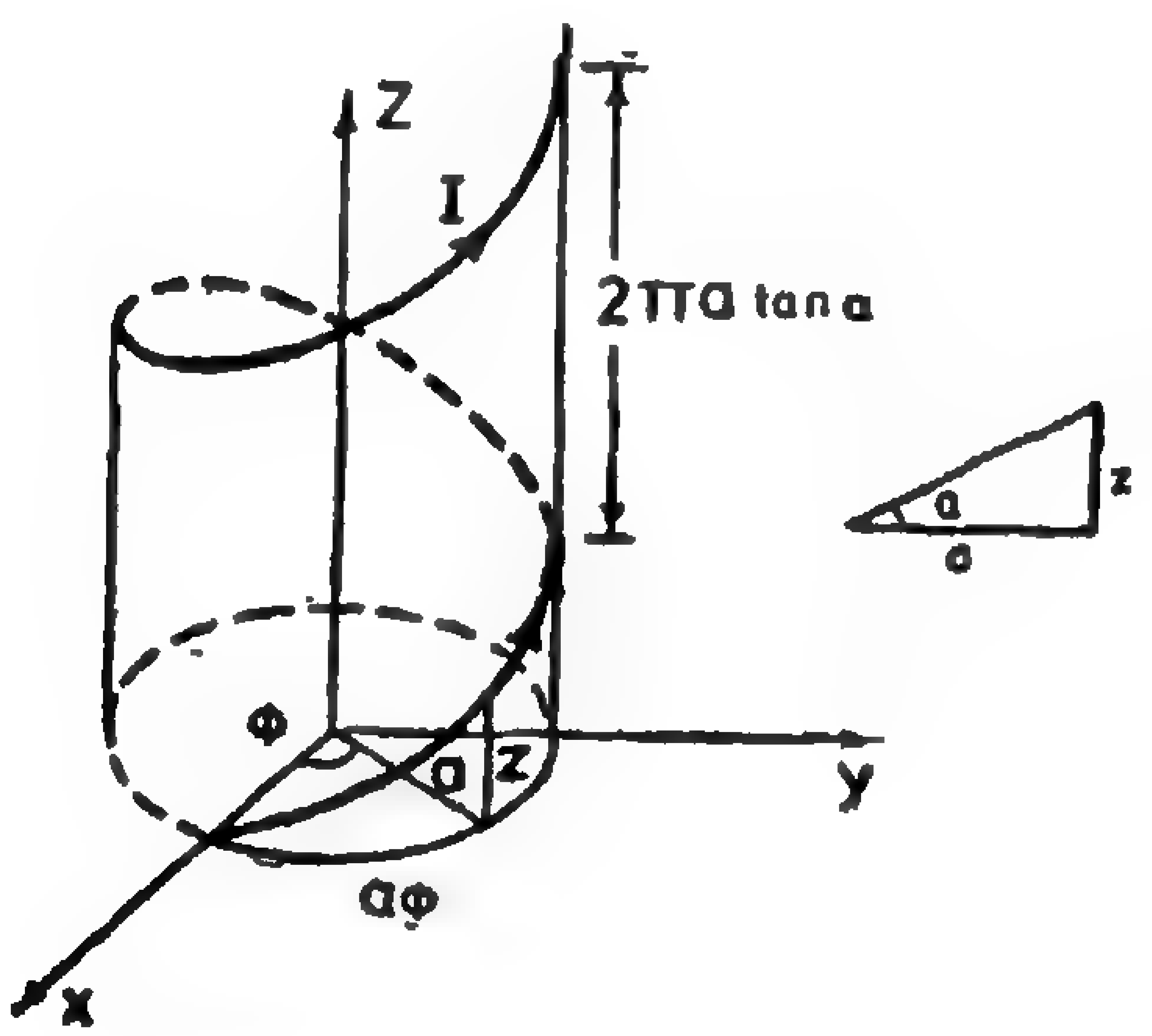


Fig. 7.20.

Taking the origin at the point at which B_z is to be calculated, $c = 0$, and

$$r = a\,(1 + \phi^2 \tan^2\alpha)^{1/2}$$

The limits of integration are evidently $\phi_1 = -N\pi$, and $\phi_2 = N\pi$, thus

$$B_z = (\mu_0 I/4\pi a) \int_{\phi_1}^{\phi_2} d\phi \,/\, (1 + \phi^2 \tan^2 \alpha)^{3/2}$$

$$= (\mu_0 N I/2a) \,/\, (1 + \pi^2 N^2 \tan^2 \alpha)^{1/2}$$

18. *An electron moves in a circular orbit with an angular velocity ω. Find the magnetic moment of the loop and the magnetic flux density at its center.*

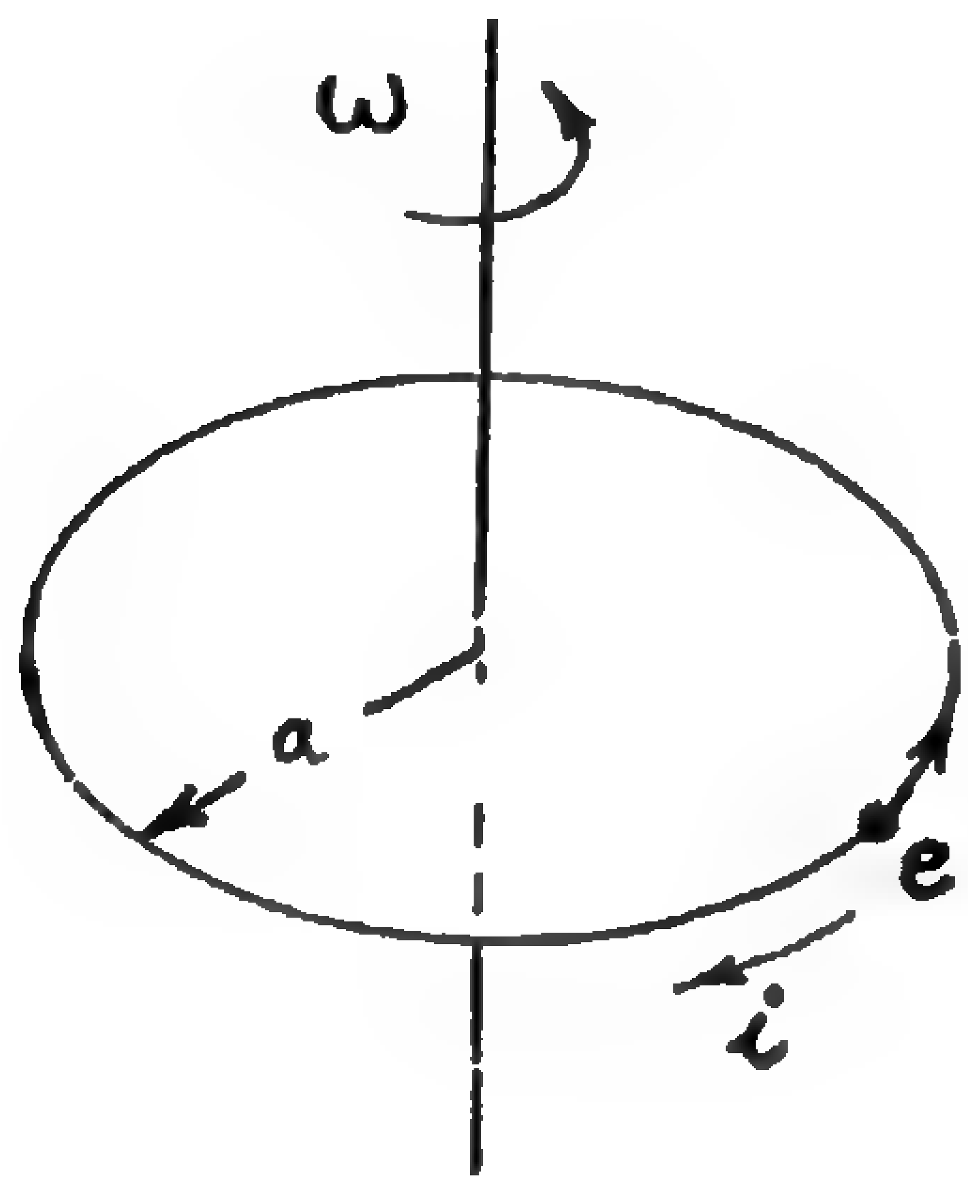


Fig. 7.21.

The rotation of the electron with a frequency $\omega/2\pi$ is equivalent to a current i flowing in the opposite sense and of magnitude (Fig. 7.21),

$$i = e \times \text{number of rotations per second}$$
$$= e\,\omega/2\pi$$

Since by definition, the magnetic moment of a current loop is iS,

$$m = (e\,\omega/2\pi)\,\pi a^2 = \tfrac{1}{2}\, e\,\omega a^2$$

Also the magnetic flux density at the center of a plane circular loop of radius a and carrying a current i is $\mu_0 N i/2a$, hence

$$B = \mu_0 (e\,\omega/2\pi) \,/\, 2a = \mu_0 e\,\omega/4\pi a.$$

19. *A thin disc of radius a carries a uniform surface charge density* σ *and rotates about its axis with angular velocity* ω. *Show that the magnetic dipole moment of the disc is* $1/4\ \sigma\ \omega\ \pi a^4$

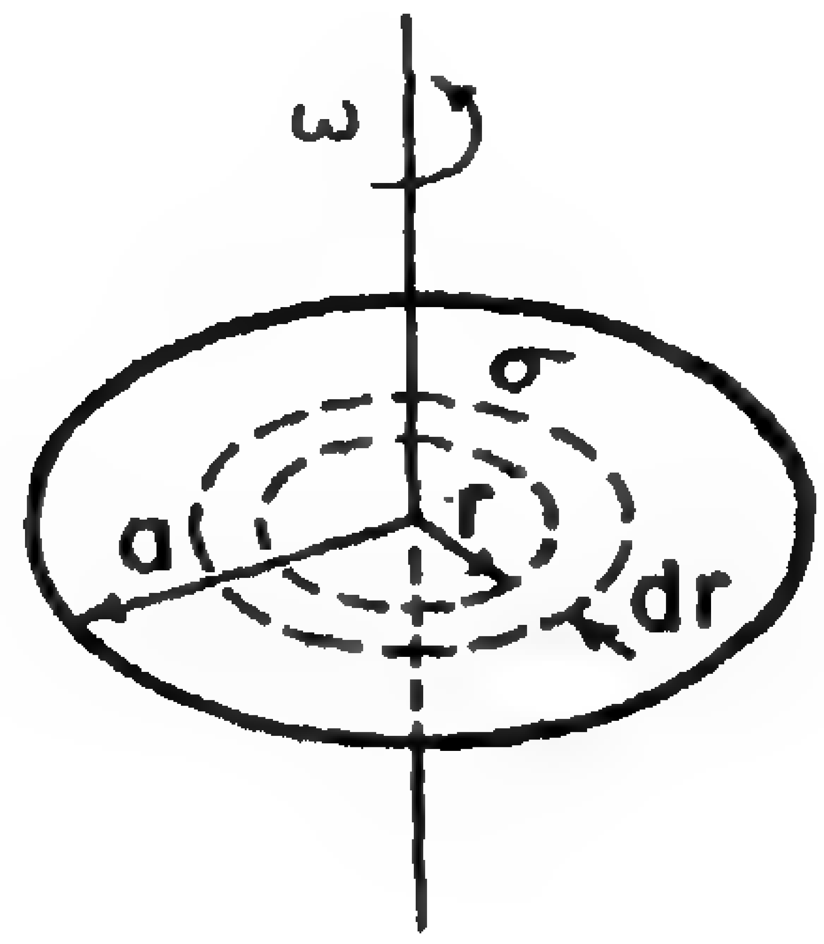


Fig. 7.22.

The charge on an elementary ring is $\sigma\ 2\pi\ r\ dr$. This represents a current, ($\omega/2\pi$ is the revolutions per second), $di = \sigma\ 2\pi\ dr\ \omega/2\pi$

The magnetic moment due to this element is $di\ S'$

$$dm = \sigma\ \omega\ r\ dr\ \pi r^2$$

Hence

$$m = \pi\ \sigma\ \omega \int_0^a r^3\ dr = 1/4\ \sigma\ \omega\ \pi a^4$$

20. *A thin disc of radius a and thickness t carries a uniform volume charge density* δ. *If the disc rotates about its axis with an angular velocity* ω *(Fig. 7.23), show that the magnetic flux density at the center of the disc is* $\frac{1}{2}\ \mu_0\ \delta\ t\ \omega a$.

The flux density at the center of a circular ring of current *I* and radius *r* is

$$B = \mu_0\ I\ /\ 2r$$

In our case the current of an elementary ring is $\sigma\, \omega r\, d r = \delta t\, \omega r\, dr$. Hence

$$dB = \tfrac{1}{2}\, \mu_0\, \delta\, t\, \omega\, dr$$

Integrating we get,

$$B = \tfrac{1}{2}\, \mu_0\, \delta\, \omega\, t\, a$$

21. *A charge Q is uniformly distributed over the surface of a sphere of radius a. If the sphere is rotated about one of its diameters with an angular frequency ω, find its magnetic moment (Fig. 7.23).*

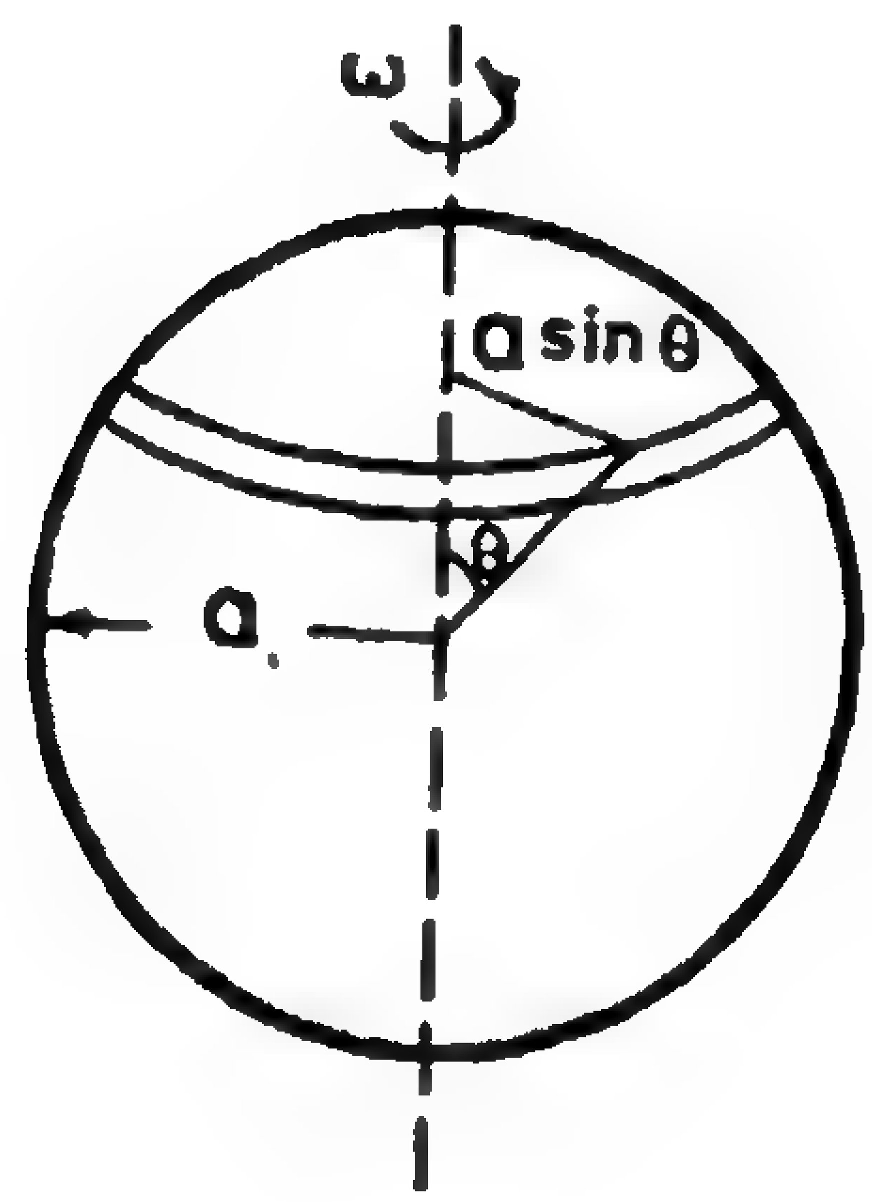


Fig. 7.23.

Consider the contribution to the magnetic moment from the charge moving in the circular ring of radius $a \cos \theta$ and thickness $r\ d\theta$. The current associated with this ring is equal to

$$di = 2\pi\, a^2 \sin \theta\, d\theta\, \sigma\omega/2\pi$$

$$= \sigma\, \omega\, a^2 \sin \theta\, d\theta$$

where, $\sigma = Q/4\, \pi a^2$

Thus the contribution to the magnetic dipole moment is

$$dm = \pi (a \sin \theta)^2 \, di$$

$$= \pi \sigma w a^4 \sin^3 \theta d \theta$$

Hence the total magnetic moment of the system is given by

$$m = \pi \sigma \omega a^4 \int_0^\pi \sin^3 \theta \, d\theta$$

$$= -4 \pi \sigma \omega a^4/3$$

$$= -a^2 \omega Q/3$$

The minus sign means that m points downwards.

22. *An electric charge Q is uniformly distributed throughout the volume of a sphere of radius a. If the sphere rotates about one of its diameters with angular velocity ω, find the magnetic moment of the sphere and the flux density at the center of the sphere (Fig. 7.24).*

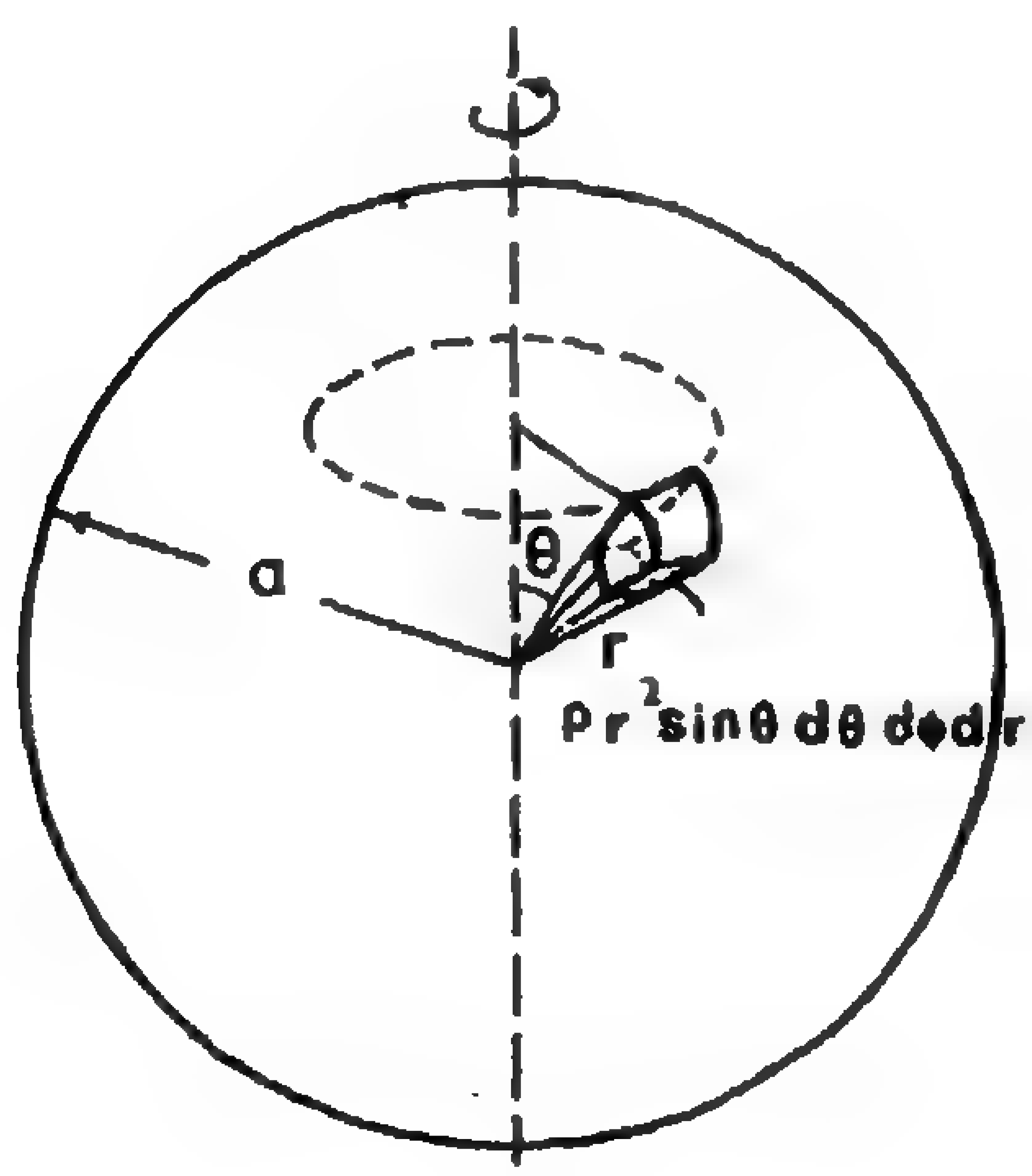


Fig. 7.24.

Consider the contribution to the magnetic moment from the charge within the element of volume $2\pi r^2 \sin\theta \, d\theta \, dr$. The current associated with the rotation of this charge is given by the second order differential

$$d^2 i = 2\pi r^2 \sin\theta \, \rho \, d\theta \, dr \, (\omega / 2\pi)$$

$$= \omega \rho r^2 \sin\theta \, dr \, d\theta$$

where $\rho = 3Q / 4\pi a^3$.

Thus the contribution to the magnetic dipole moment is

$$d^2 m = \pi (r \sin\theta)^2 d^2 i$$

$$= \pi \rho \omega r^4 \sin^3\theta \, d\theta \, dr$$

and the total moment of the system is given by

$$m = \pi\rho\omega \int_0^{2\pi} \int_0^a r^4 \sin^3\theta \, d\theta \, dr$$

$$= - \omega a^2 Q/5$$

The magnetic flux density at the center of the sphere produced by the elementary current ring $d^2 i$ is

$$d^2 B_z = \mu_0 d^2 i \sin^2\theta / 2r \sin\theta$$

$$= \tfrac{1}{2} \mu_0 \omega \rho r \sin^3\theta \, dr \, d\theta$$

$$B_z = 1/4 \, \mu_0 \omega \rho a^2 \int_0^\pi \sin^3\theta \, d\theta$$

$$= - \mu_0 \omega \rho a^2/3 \cdot$$

$$= - \mu_0 \omega Q / 4\pi a$$

23. *A filamentary planar rectangular loop of sides a and b, (b>a) is placed with its longer sides parallel to an infinitely long thin conductor; the distance between the near side of the loop and the conductor*

is c. Show that the mutual inductance between the two circuits is given by

$$M = (\mu_0 b/2\pi) \log (1+a/c)$$

The magnetic intensity at a point distance r from the infinite conductor carrying a current I_1 is

$$B_r = \mu_0 I_1/2\pi r$$

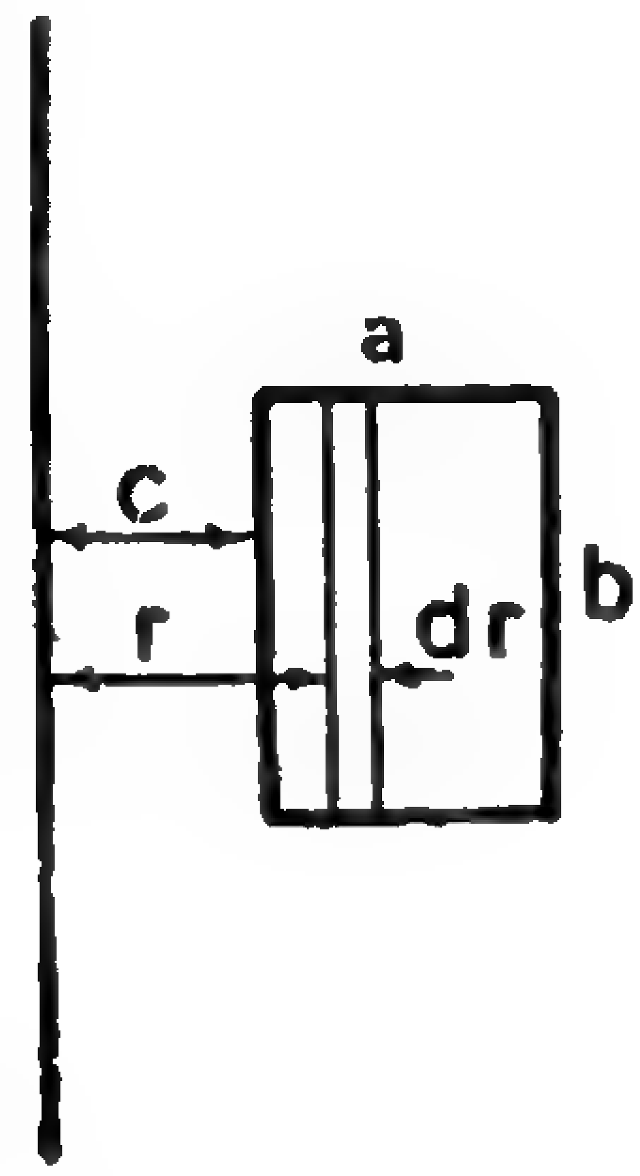


Fig. 7.25.

The flux through an elementary area $b\,d\,r$ of the rectangular loop is (Fig. 7.25),

$$d\phi_{21} = (\mu_0 I_1/2\pi r)\, b\, d\, r$$

Integrating we get,

$$\phi_{21} = (\mu_0 I_1 b/2\pi) \int_{r=c}^{a+c} dr/r$$

$$= (\mu_0 I_1 b/2\pi) \log (1+a/c)$$

The mutual inductance between the two circuits is thus,

$$M = \phi_{21}/I_1 = (\mu_0 b/2\pi) \log (1+a/c)$$

24. *Find the force and couple exerted by one magnetic dipole on another (Fig. 7.26).*

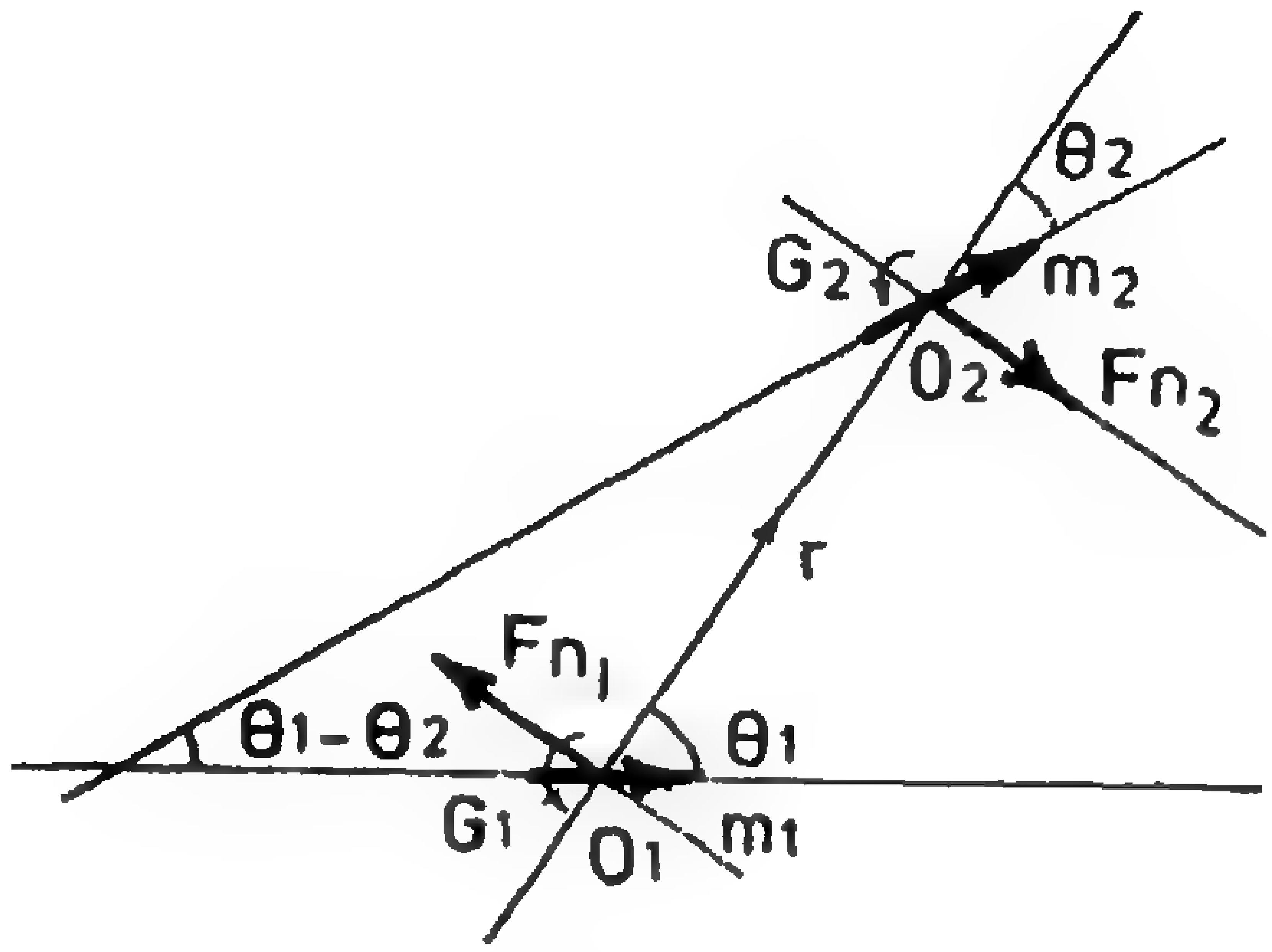


Fig. 7.26.

The mutual potential energy of the dipoles is,

$$U = k\,[\mathbf{m}_1 \cdot \mathbf{m}_2/r^3 - 3\,(\mathbf{m}_1 \cdot \mathbf{r})\,(\mathbf{m}_2 \cdot \mathbf{r})\,/r^5]$$

where, $k = \mu_0/4\pi$.

The translational force exerted by $\mathbf{m}_1$ on $\mathbf{m}_2$ is,

$$\mathbf{F}_2 = -\nabla_2\,(U)$$

$$\nabla\,(\mathbf{m}_1 \cdot \mathbf{m}_2/r^3) = -3\,(\mathbf{m}_1 \cdot \mathbf{m}_2)\,\mathbf{a}_r/r^4$$

$$\nabla\,(\mathbf{m}_1 \cdot \mathbf{r})\,(\mathbf{m}_2 \cdot \mathbf{r})/r^5 = -5\,(\mathbf{m}_1 \cdot \mathbf{r})\,(\mathbf{m}_2 \cdot \mathbf{r})\,\mathbf{a}_r/r^6 + \{(\mathbf{m}_1 \cdot \mathbf{r})\,\nabla(\mathbf{m}_2 \cdot \mathbf{r}) + (\mathbf{m}_2 \cdot \mathbf{r})\,\nabla\,(\mathbf{m}_1 \cdot \mathbf{r})\}\,/\,r^5$$

Now

$$\nabla\,(\mathbf{m}_1 \cdot \mathbf{r}) = \mathbf{m}_1 \times (\nabla \times \mathbf{r}) + \mathbf{r} \times (\nabla \times \mathbf{m}_1) + (\mathbf{m}_1 \cdot \nabla)\mathbf{r} + (\mathbf{r} \cdot \nabla)\,\mathbf{m}_1$$

Since $\nabla \times \mathbf{r} = 0$ and because $\mathbf{m}$ is a constant, $\nabla \times \mathbf{m} = 0$ and $(\mathbf{r} \cdot \nabla)\,\mathbf{m} = 0$.

The term $(\mathbf{m}_1 . \nabla)\ \mathbf{r}$ is]

$$m_x\ \partial \mathbf{r}/\partial x + m_y\ \partial \mathbf{r}/\partial y + m_z\ \partial \mathbf{r}/\partial z = \mathbf{m}_1$$

Thus $\nabla (\mathbf{m}_1 . \mathbf{r}) = \mathbf{m}_1$, $\nabla (\mathbf{m}_2 . \mathbf{r}) = \mathbf{m}_2$ and the final result may be written as

$$F_2 = k\ (\alpha\ \mathbf{r} + \beta\ \mathbf{m}_1 + \gamma\ \mathbf{m}_2) / r^5$$

where,

$$\alpha = 3 \{ (\mathbf{m}_1 . \mathbf{m}_2) - 5\ (\mathbf{m}_1 . \mathbf{r})\ (\mathbf{m} . \mathbf{r}) / r^2 \}$$

$$\beta = 3\ (\mathbf{m}_2 . \mathbf{r})$$

$$\gamma = 3\ (\mathbf{m}_1 . \mathbf{r})$$

The force $\mathbf{F}_1$ exerted on $\mathbf{m}_2$ by $\mathbf{m}_1$ tending to move $\mathbf{m}_1$ parallel to itself is equal to $-\mathbf{F}_2$ since the position vector of $\mathbf{m}_1$ relative to $\mathbf{m}_2$ is $-\mathbf{r}$. The couple tending to rotate $\mathbf{m}_2$ about its center is given by

$$\begin{aligned} \mathbf{G}_2 &= \mathbf{m}_2 \times \mathbf{B}_1 \\ &= \mathbf{m}_2 \times [3\ (\mathbf{m}_1 . \mathbf{r})\ \mathbf{r}/r^5 - \mathbf{m}_1/r^3]\ k \\ &= k\ [3\ (\mathbf{m}_2 \times \mathbf{r})\ (\mathbf{m}_1 . \mathbf{r}) / r^5 - \mathbf{m}_2 \times \mathbf{m}_1/r^3] \end{aligned}$$

For the special case in which the two dipoles are coplanar we have

$$\begin{aligned} \mathbf{F}_{12} = \ & k\ [3 m_1 m_2\ (\sin\ \theta_1\ \sin\ \theta_2 - 12\ \cos\ \theta_1\ \cos \theta_2)\ \mathbf{a}_r / r^4 \\ & + (3 m_2\ \cos\ \theta_2)\ \mathbf{m}_1/r^4 + (3 m_1\ \cos \theta_1)\ \mathbf{m}_2/r^4] \end{aligned}$$

Resolving the forces in the directions of $\mathbf{m}_1$ and $\mathbf{m}_2$ into their components along and perpendicular to $\mathbf{r}$ we have that (Fig. 7.27)

$$F_r{}'_2 = k\ 3\ m_1 m_2\ \cos\ \theta_1\ \cos\ \theta_2/r^4$$

$$F_n{}'_2 = k\ 3\ m_1 m_2\ \cos\ \theta_1\ \sin\ \theta_2/r^4$$

$$F_{r2}'' = k\ 3\ m_1 m_2\ \cos\ \theta_2\ \cos\ \theta_1/r^4$$

$$F_n{}''_2 = k\ 3\ m_1 m_2\ \cos\ \theta_2\ \sin\ \theta_1/r^4$$

The resultant component of force along **r** is therefore

$$F_{r2} = k\, 3\, m_1 m_2 (\sin\theta_1 \sin\theta_2 - 2 \cos\theta_1 \cos\theta_2) / r^4$$

and the component of force perpendicular to **r** is,

$$F_{n2} = k\, 3\, m_1 m_2 \sin(\theta_1 + \theta_2) / r^4 \qquad (1)$$

Since the energy is given by

$$U = k\, m_1 m_2 (\sin\theta_1 \sin\theta_2 - 2 \cos\theta_1 \cos\theta_2) / r^3$$

The torque acting on $\mathbf{m}_2$ is given by

$$G_2 = \partial U/\partial\theta_2 = k\, m_1 m_2 (\sin\theta_1 \cos\theta_2 + 2\cos\theta_1 \sin\theta_2)/r^3 \qquad (2)$$

and the torque acting on $\mathbf{m}_1$ is

$$G_1 = \partial U/\partial\theta_1 = k\, m_1 m_2 (\cos\theta_1 \sin\theta_2 + 2 \sin\theta_1 \cos\theta_2) / r^3 \qquad (3)$$

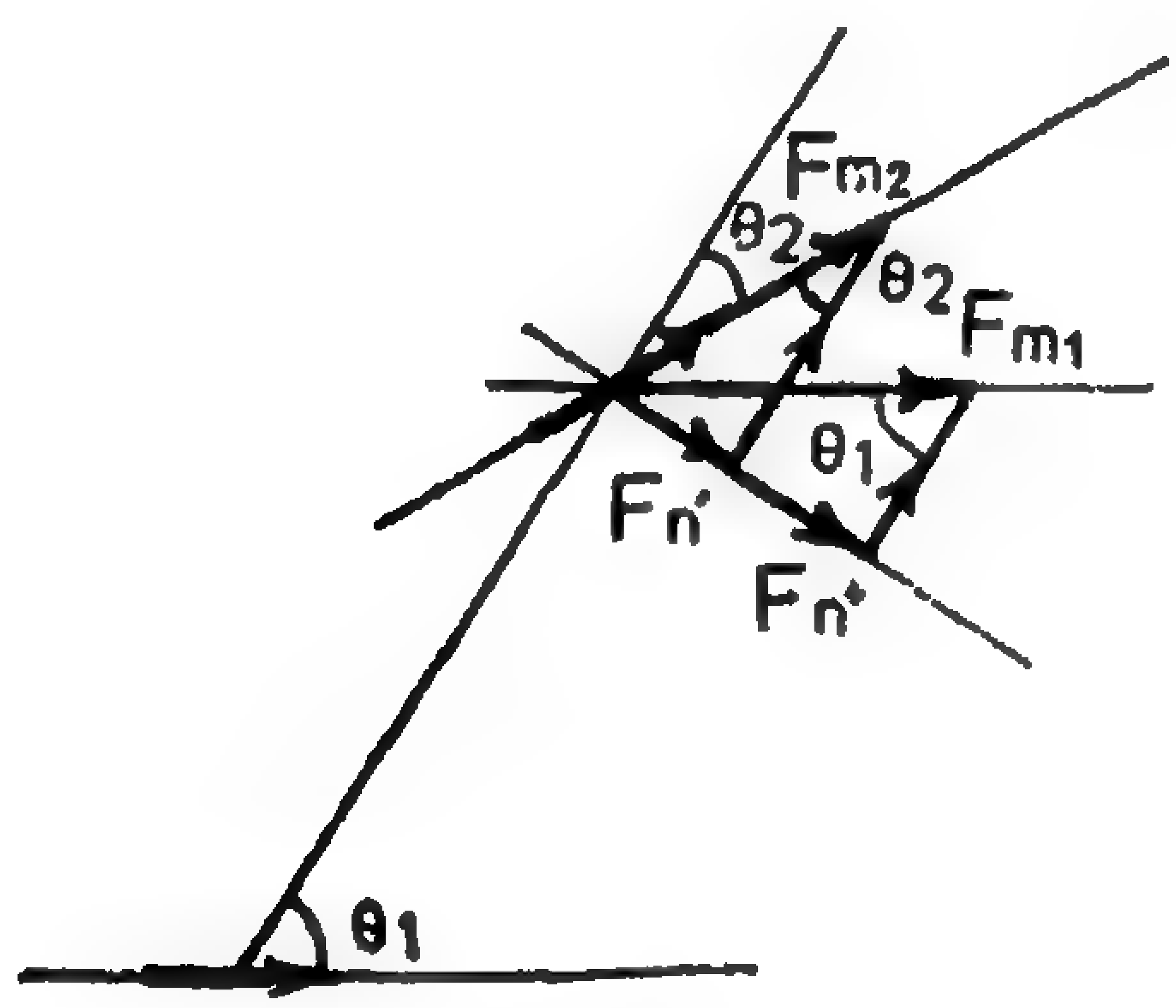


Fig. 7.27.

Clearly G_2 is not equal to G_1. This is because the forces F_n are not along the same straight line. By taking moments about O_1 and O_2 we have that

$$G_1 + G_2 - r\, F_n = 0$$

or

$$G_1 + G_2 = r\, F_n$$

This result is readily verified by adding (2) and (3) to give (1).

25. *Show that for an electron travelling in a magnetic field of axial symmetry, its angular momentum changes in proportion to the magnetic flux which passes between the two concentric circles drawn through the initial and final positions (Busch's theorem).*

The angular momentum of a mass m moving at distance r about a point of reference O is

$$\mathbf{G} = m\, r^2\, \omega\, \mathbf{a}_z$$

where ω is the angular velocity about O. If $\mathbf{F}$ is the force acting on m then the moment of this force is equal to the rate of change of angular momentum *i.e.*

$$d\,(mr^2\,\omega\,\mathbf{a}_z)/dt = \mathbf{F}\times\mathbf{r}$$

If the magnetic field is given by $\mathbf{B}\,(r, z)$, then the Lorentz force acting on the electron is,

$$\mathbf{F} = -e\,(\mathbf{v}\times\mathbf{B}) = -\,e\,[(dr/dt)\,B_z - (dz/dt)\,B_r]\,\mathbf{a}_\theta$$

where B_z and B_r are the axial and radial components of the flux density. Hence,

$$m\,d\,(r^2 w)/dt = -e\,[(dr/dt)\,B_z - (dz/dt)\,B_r]\,r \tag{1}$$

Consider a circle drawn at right angles to the *z-axis*. If this circle is shifted by a distance dz along the *z-axis* and at the same time its radius is increased by dr, the flux through the circle will change by (Fig. 7.28),

$$d\phi = (B_z \sin\theta - B_r \cos\theta)\;2\pi\, r\, d\,l$$

or, since $d\,l \sin\theta = dr$ and $d\,l \cos\theta = dz$,

$$d\phi = (B_z\, dr - B_r\, dz)\;2\pi\, r \tag{2}$$

Comparing equations (1) and (2) it is evident that

$$m\,d\,(r^2\,\omega) = -\,(e/2\pi)\,d\phi$$

Integrating with the initial and final positions as limits, we obtain

$$m\,(r_2^2\,\omega_2^2 - r_1^2\,\omega_1^2) = (e/2\pi)\,(\phi_1 - \phi_2)$$

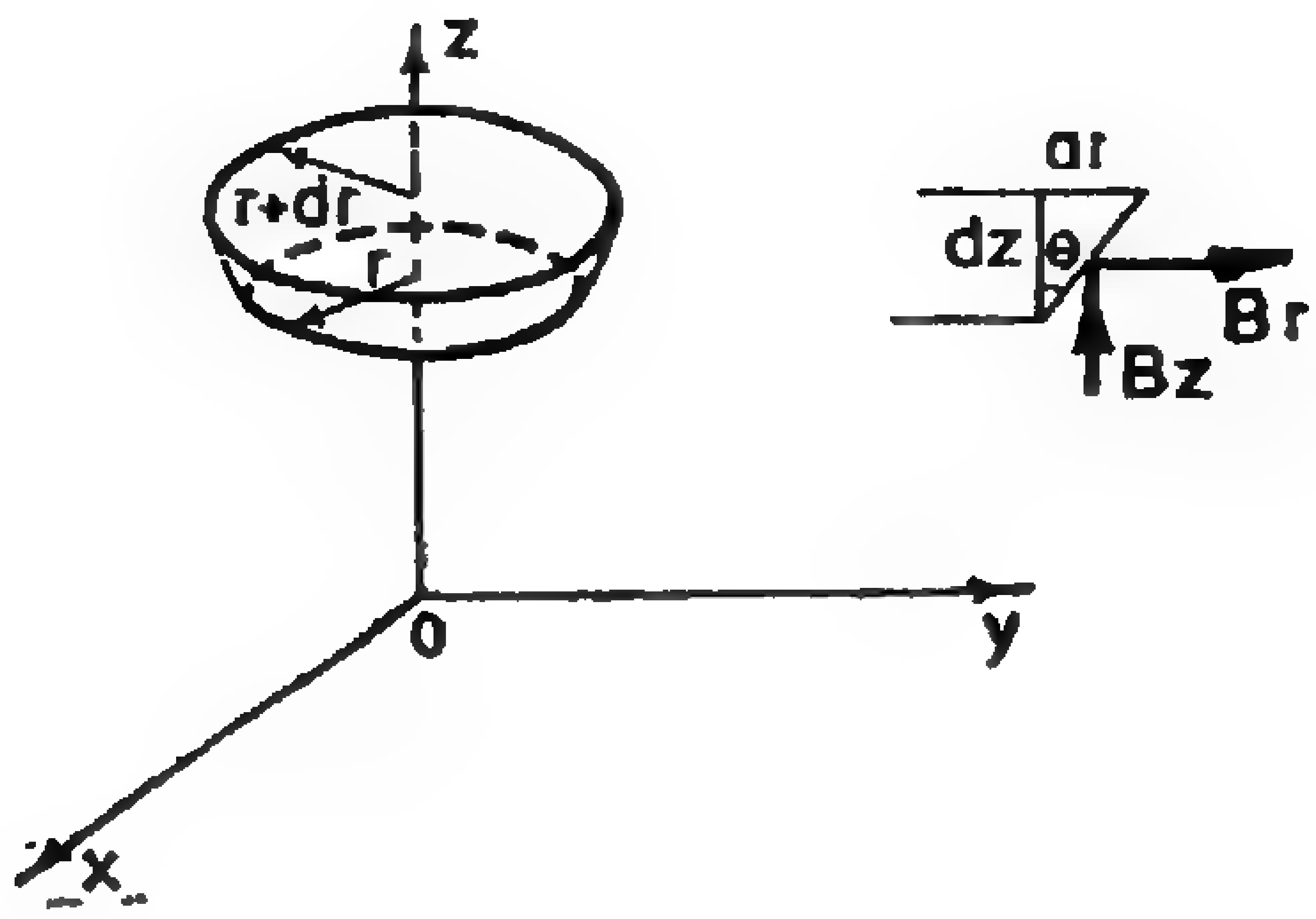


Fig. 7.28.

26. *An ion beam of radius R consists of positively charged particles of charge* Q C *and mass m kg, moving with velocity v m/sec parallel to the beam axis. Assuming that the current density is constant throughout the cross section of the beam, find the radial acceleration of an ion at the periphery of the beam, taking into account the electric and magnetic fields produced by the beam. Is the assumption of uniform current density justifiable ?*

Since the current density is constant throughout the cross section of the beam,

$$I = \pi R^2 N Q v = \pi R^2 \delta v$$

where, N is the number of particles per unit volume. The magnetic flux density at a point on the prephery of the beam is (Fig 7.29),

$$\mathbf{B} = (\mu_o I/2\pi R)\,\mathbf{a}_\phi$$

$$= \tfrac{1}{2}\,\mu_o R\,\delta\,v\,\mathbf{a}_\phi$$

The electric field intensity at P is

$$\mathbf{E} = \pi R^2 \delta \, \mathbf{a}_r \, / \, 2\pi \epsilon_o R = (\delta R/2 \epsilon_o) \, \mathbf{a}_r$$

The Lorentz force acting on a particle at P is therefore

$$= Q \, [\mathbf{E} + \mathbf{v} \times \mathbf{B}]$$

$$= Q \, [(\delta R/2 \epsilon_o) \, \mathbf{a}_r + \tfrac{1}{2} \mu_o R \delta v^2 \, \mathbf{a}_z \times \mathbf{a}_\phi]$$

$$= Q \, [(\delta R/2 \epsilon_o) - \tfrac{1}{2} \mu_o R \, \delta v^2] \, \mathbf{a}_r$$

The acceleration is therefore $\mathbf{F}/m$,

$$\mathbf{a} = (Q\delta R/2 \epsilon_o m) \, (1 - v^2/c^2) \, \mathbf{a}_r$$

where $c = 1/\sqrt{\mu_o \epsilon_o} = 3 \times 10^8 \; m/sec.$

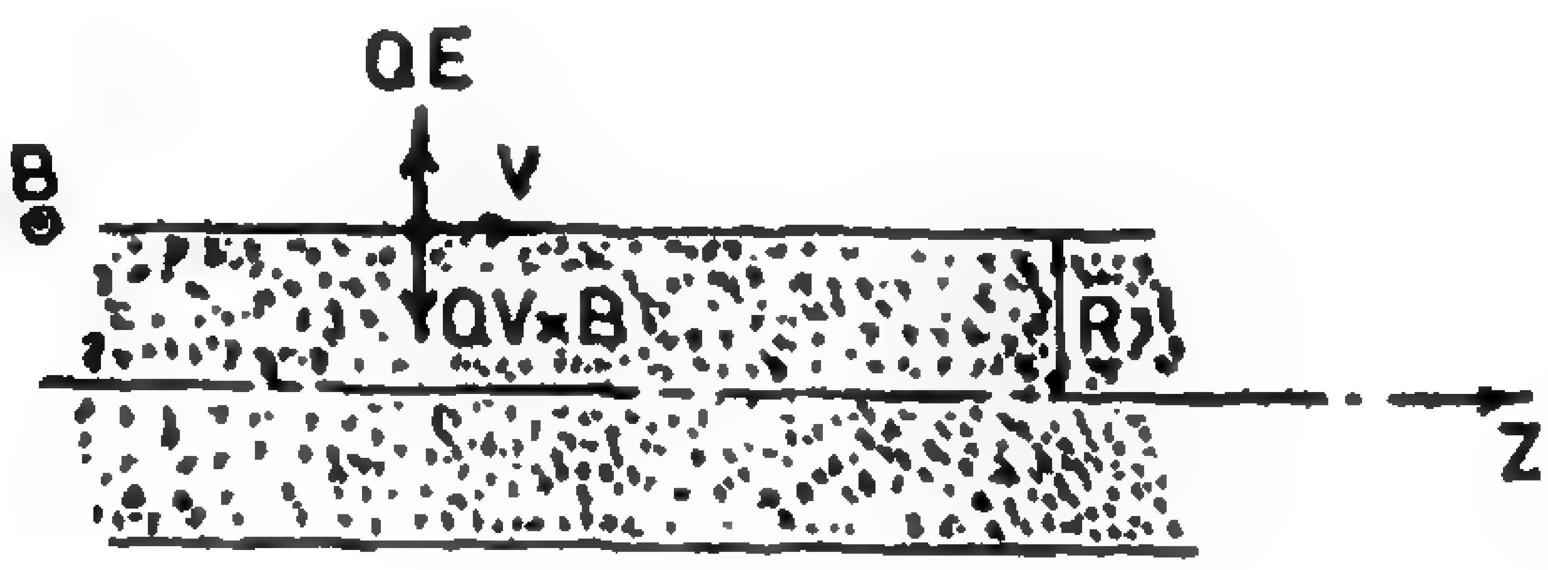


Fig. 7.29.

Now since the radial acceleration of an ion depends on its distance from the beam axis it is concluded that the assumption of uniform current density is not justifiable.

27. *Two long coaxial cylinders of radii a and b ($b>a$) are placed with their axis parallel to a uniform magnetic field of density B (Fig. 7.30). If a voltage V is now applied between the two cylinders show that a particle of charge q and mass m, initially at rest on the surface of the inner cylinder, will reach the anode at grazing incidence if*

$$V = (q B^2 b^2 \, / \, 8m) \, (1 - a^2/b^2)^2$$

The Lorentz force acting on the particle is

$$\mathbf{F} = q\,(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

Now since **E** and **B** are at right angles, then

$$\mathbf{F}.\mathbf{B} = q\,(\mathbf{E}.\mathbf{B} + \mathbf{v} \times \mathbf{B}.\mathbf{B}) = 0$$

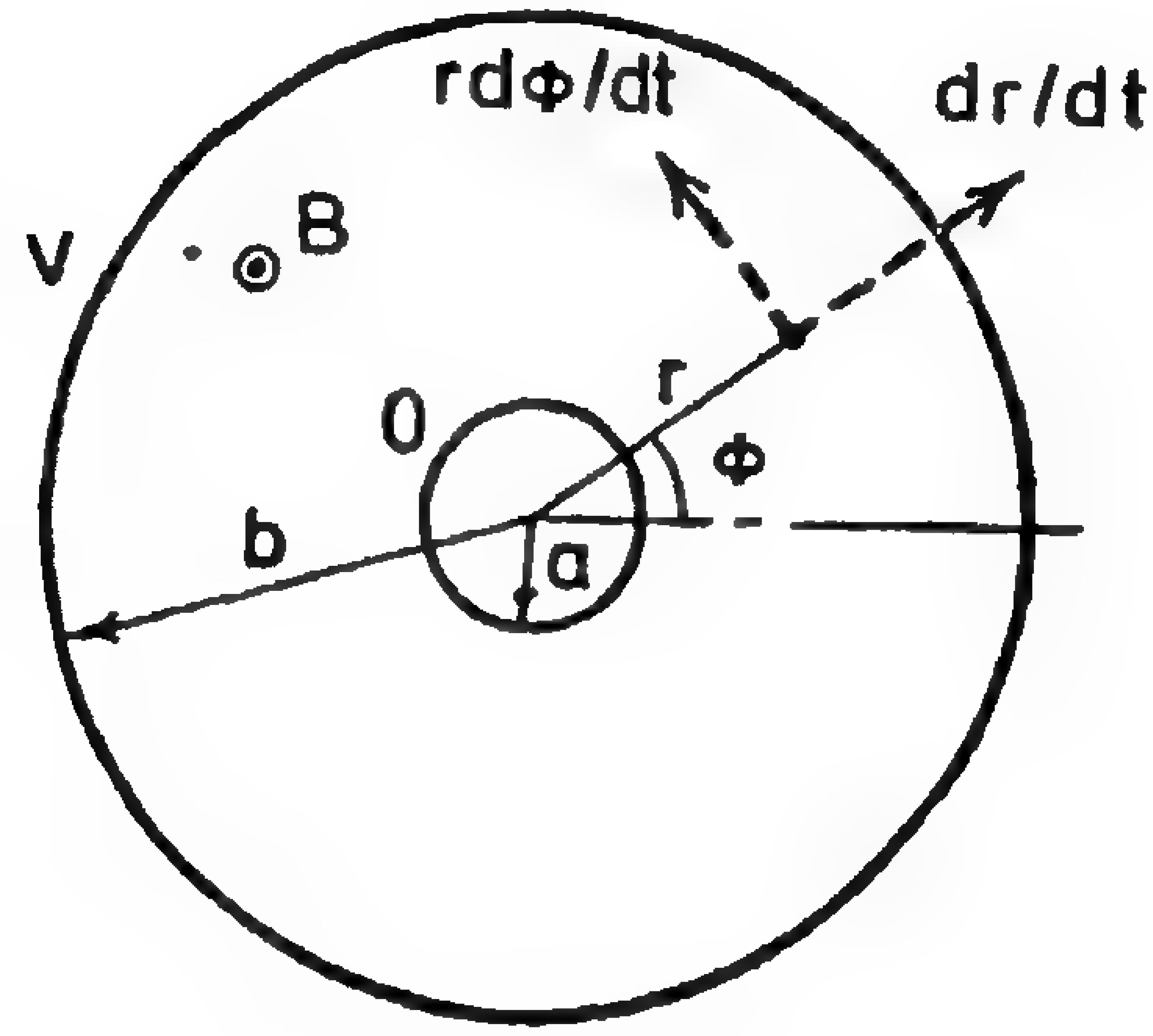


Fig. 7.30.

Hence the force is always at right angles to the direction of **B** and it follows that the velocity component of the particle parallel to **B** is constant throughout the motion and equal to its initial value in this direction. In the present case this initial value is zero since the charge was initially at rest. The velocity vector **v** is thus in the plane of the paper; the radial component is dr/dt and the transverse component is $r d\phi/dt$.

Since the rate of change of angular momentum is equal to the torque, we have that,

$$d\,(mr^2\, d\phi/dt)/dt = r\, F_\theta$$
$$= rq\, B\, dr/dt$$

Integrating and using the fact that the angular velocity is zero at the cathode surface,

$$mr^2\, d\phi/dt \quad = \quad \tfrac{1}{2}\, q \;\; B(r^2—a^2) \qquad (1)$$

The kinetic energy of the particle is

$$K.E. \;=\; \tfrac{1}{2}\, m \;\; [(dr/dt)^2 \;+\; (rd\phi/dt)^2$$
$$-\;\; q \;V\;(r) \qquad (2)$$

where $V\,(r)$ is the potential at a distance r. It is important to note that the kinetic energy comes only from the electric field since no work is done by the static magnetic field, since the magnetic force is perpendicular to the velocity. The condition for grazing incidence (cutoff condition) is that at $r=b$ the radial velocity is zero.

Thus $\quad \tfrac{1}{2}\, m\; (b\, d\phi/dt)^2 \;=\; q\; V$

and from equation (2)

$$mb^2\; d\phi \;/\; dt \;=\; \tfrac{1}{2}\; q\; B\;\; (b^2—a^2)$$

Solving the above two equations for V we get,

$$V \;\;=\;\; (q\;\; B^2b^2 \;/\; 8m)\;\; (1—a^2/b^2)^2$$

7 - 17 مسائل إضافية

SUPPLEMENTARY PROBLEMS

1. A two-wire line has a capacitance of 0.005 μF/m. Show that the ratio between the voltage and the current in the wire when the electrostatic attraction between the wires is compensated by the electrodynamic repulsion is 15×10^{-4} A/V. The voltage and the current are assumed constants and the dielectric is air.

2. A thin wire is bent into a loop to form a circular arc $2\pi - 2\alpha$ of radius a which is closed by its chord, $\alpha < \pi/2$. The wire loop is suspended from a point on the arc diametrically opposite to the mid-point of the chord. A long straight conductor passes through the center of the arc perpendicular to the plane of the loop. Show that the torsional moment acting on the loop around a vertical axis passing through the point of suspension is, $(\mu_0 \; a I_1 I_2 / \pi)$ $(\sin \alpha - \alpha \cos \alpha)$ where I_1 is the current in the wire and I_2 the current in the loop.

3. Four long thin straight wires are mounted in air with their length parallel and horizontal. They intersect a vertical plane perpendicular to their length in the vertices of a rectangle of horizontal length a and vertical length b. The upper pair of wires is used as a two wire transmission line with a steady current I_1 along one wire and back along the other. The lower pair is used similarly, the current being I_2. Wires in the same vertical plane carry currents in opposite directions. Calculate the magnetic flux density at a point distant y vertically above the center of the rectangle. If $I_1 = I_2$ show that the magnetic flux density vanishes at the center of the rectangle, and sketch the lines of magnetic flux.

4. Three long parallel conductors 1, 2, and 3, which may be considered linear, lie in a horizontal plane at equal intervals d. The conductors carry the currents $I_1 = I \sin \omega t$, $I_2 = I \sin (\omega t - 120^\circ)$, and $I_3 = I \sin (\omega t - 240^\circ)$, respectively show that the largest force per meter acting on conductor 1 is $\mu_0 I^2 (3/2 + \sqrt{3}) / 8\pi d$, ($\omega t = 75^\circ$) away from conductors 2 and 3. Find this force for $I = 10000$A and $d = 20$ cm.

5. A wire loop is in the form of a square of 3 meters a side. Find the magnetic field strength at the center when the wire carries a current of 10 A.

6. A long thin straight wire in free space carries a current I. In a plane through the wire an equilateral triangle of side a is drawn with a vertex on the wire and the opposite side parallel to the wire. Calculate the magnetic flux linking the triangle if $I = 1$A and $a = 10$ cm.

7. One of two long metal strips are bounded by the lines $x = \pm a$, $z = 0$, the other by the lines $y = \pm a$, $z = b$. Both carry a direct current I in the positive direction of the x and y axes. Show that the magnetic field strength at a point on the z-axis is,

$$\mathbf{B} = (\mu_0 I/2\pi a) \{ \arctan (a/z)\, \mathbf{a}_x - \arctan [a/(z-b)]\, \mathbf{a}_y \}$$

8. A long straight conductor in air is of uniform wall thickness, the internal radius being a and the external radius being b. A current I flows along the conductor and is uniformly distributed over the cross section. Calculate the magnetic flux density at distance r from the axis of the conductor (a) when $r \geq b$, (b) when $a \leq r \leq b$, and (c) $r \leq a$. On the assumption that $a = 3b/4$ plot as a function of r the ratio of the magnetic flux density at distance r to the magnetic flux density at the outer surface of the conductor.

9. A long straight hollow conductor has a square cross section $ABCD$ of side $2a$ and negligible wall thickness. If the conductor carries a current $-I\,\mathbf{a}_z$, show that, for a rectangular coordinate system with the origin at the center of the conductor and with axes parallel to the sides of the conductor with [$A\,(-a, a)$, $B\,(-a, a)$ ... etc.] the magnetic flux density at an arbitrary point outside the conductor is given by

$$\mathbf{B} = (\mu_0 I/32\pi a)\ \{[\log r_A r_D/r_B r_C + 2\ (\phi_{BA} + \phi_C)]\ \mathbf{a}_x$$
$$+\ [\log r_A r\ /r_B r\ + 2(\phi\ + \phi\ )]\ \mathbf{a}_y\}$$

where r_A, r_B, ... denote the distance from the field point to the corners A, B, ... and ϕ_{BA}, ϕ , ... the angles suspended by the sides BA, CD, ... these angles are positive in the case of the horizontal sides if they are suspended from above and negative if they are suspended from below.

10. Two long straight wires are perpendicular to each other and carry currents I_1 and I_2. The smallest distance between the conductors is a. Show that the torsional moment acting on a length $2b$ of the second wire, such that the length $2b$ is symmetrical with respect to the point which is closest to the other wire, is given by $(\mu_0 I_1 I_2\ a/\pi)\ (b/a - \arctan b/a)$.

11. The interior of a thin hollow spherical conductor has radius a. A thin wire of length $2a$ is fitted so as to run along a diameter of the sphere. A steady current I flows along the wire and returns uniformly through the conducting sphere. Calculate the magnetic flux density at any point inside the sphere at a distance r from the axis of the wire. Explain why the result is independent of the radius of the sphere.

12. Two hollow conducting circular cones are constructed one with semivertical angle α and the other with semivertical angle β. They are arranged coaxially with the vertices in contact. A total current I flows symmetrically along the generators of one cone towards the common vertex and symmetrically away from the vertex on the other cone. Show that the magnetic flux density between the cones is the same as that due to a current I along the common axis in the absense of the cones. At a point distant R from the common vertex calculate the current per unit width on each cone.

13. An equilateral triangle of side a is rotated about a line in its plane that is parallel to one side and does not intersect the other two sides. The distance from the axis of rotation to the parallel side is b and to the opposite vertex of the triangle is $b+a\sqrt{3}/2$. The toroidal surface thus formed is used as a toroidal inductor with an air core and with a total current I distributed uniformly round the toroid. Calculate the total magnetic flux linking the toroid.

14. The coil of a moving coil instrument for measuring steady current is rectangular, with a pair of edges parallel to the axis of rotation and perpendicular to the magnetic field across which they move. The magnetic flux density at these edges is B, and the coil has N turns each of area S. At full scale delfection the needle is turned from zero through an angle θ_{max} and the current through the coil is then I_{max}. Calculate the torque of the spring per unit angular deflection. Evaluate this quantity in newton meter per radian if $I_{max} = 100$ mA, $B = 0.1$ tesla, $S = 1$ sq cm, and $\theta_{max} = 90^{\circ}$.

15. The coil of *d'* Arsonval's instrument described in the previous problem carries a steady current I. The movement has a moment of inertia G about the axis of rotation, and the torque constant

of the spring is k. If θ is the angular deflection of the needle from the equilibrium position, show that the equation of motion of the needle is

$$G\, d^2\theta/dt^2 + k\theta = NIBS$$

Show that the needle oscillates about its true reading with frequency $(k/G)^{1/2}/2\pi$, and suggest a method for preventing this.

16. The instrument described above is used as a ballistic instrument to measure the total flux of a charge Q involved in discharging a condenser. It may be assumed that the charge Q passes through the winding of the instrument so quickly that the needle does not suffer any appreciable displacement until the discharge is practically complete. By integrating the equation of motion over this short interval of time, show that the initial angular velocity of the needle is (NBS/G) Q. Calculate the total angular throw of the needle and show that it is proportional to Q.

17. A string galvanometer consists of a thin wire held under a tension T by means of a helical spring and set vertically in a horizontal magnetic field. A current I flows in the wire and the horizontal displacement d of the center of the wire is measured with a microscope. Show that the wire assumes the form of a parabola and that d is given by $Il^2B/8T$, where B is the induction and l is the distance between the points of support of the wire.

18. A solenoid 20 cm long and 8 cm in mean diameter carries a given current. What the ratio between the magnetic field strength at one end and in the middle on the solenoid axis

19. A solenoid 20 cm long and 5 cm in diameter has one terminal of a battery connected to a tap on the winding 5 em from one of its ends, while the other terminal is connected to the two ends. Show that the magnetic field disappears 1.3 cm from the tap inwards.

20. A coil consisting of 500 turns is uniformly wound on the outer surface of a circular frustrum of a right cone, the height of which is 10 cm and the radii of the basis 10 and 2 cm respectively. Show that in order to produce a flux density of 0.1 m Wb/m^2 at the vertex of the cone the current in the coil must be 64.8 mA.

21. Two solenoids have both the same length L and the same number of turns N while the diameter of one is insignificantly smaller than that of the other. They are telescoped into each other and connected in series so that the currents in the coils have opposite directions. Show that the flux density at an arbitrary point on the mutual axis at a distance z from their mutual center is

$$(\mu_o\, a\, b\, I N/2L)\ \{(z-\tfrac{1}{2}L)\ /\ [a^2+(z-\tfrac{1}{2}L)^2]^{3/2} - (z+\tfrac{1}{2}L)\ /\ [a^2+(z+\tfrac{1}{2}L)^2)]^{3/2}\}$$

where a is the mean radius of the solenoids and b is the difference between their radii. Determine the points on the axis at which the flux density disappears if $a<<L$.

22. On the surface of a wooden sphere of radius a a coil has been wound which consists of N parallel turns, the projections of which on the diameter of the sphere perpendicular to the planes of the turns are all equally large. Show that the flux density at any point on the above diameter is

$$B = (\mu_o I N/3a) \qquad x \leq a$$

$$B = (\mu_o I N/3a)\,(a/x)^3 \qquad x \geq a$$

where x is the distance from the center of the sphere.

23. Three identical circular coils are mounted with a common diameter and with angles $\pi/3$ between their planes. The current in

each coil oscillates sinusoidally with the same angular frequency ω and the same complitude I_0. The currents in the three coils at time t are $I_0 \cos(\omega t)$, $I_0 \cos(\omega t + 2\pi/3)$, and $I_0 \cos(\omega t - 2\pi/3)$. Show that if these currents are suitably arranged, the direction of the magnetic field at the common center rotates with uniform angular velocity ω.

24. A solenoid 50 cm long and 2 cm in diameter has 1000 turns. It is over-wound near its center with a second coil of 100 turns. What is the mutual inductance between the coils.

25. A fine insulated wire is wound in a close spiral forming a circular disc, the ends being at the center and at the circumference. Show that the flux density at a point on the axis of this disc is given by,

$$\tfrac{1}{2}\,\mu_0\, n\, I\,[\operatorname{arccosh}(\sec\theta) - \sin\theta]$$

where I is the current in the wire, n is the number of turns per unit length and θ is the angle subtended by a radius of the disc at the point on the axis.

26. A sphere of radius a is charged to a uniform surface density σ and rotated about an axis through its center with an angular velocity ω. Show that the flux density at the center is $(2/3)\,\mu_0\,\sigma\, a\,\omega$ along the axis of rotation. Show that the flux density has the same value at any point on the axis. Show by applying the equation $\nabla \times \mathbf{B} = 0$ inside the sphere that this is the value of B throughout the spherical volume.

27. A pair of closely wound circular coils each have N turns of radius a. They are arranged coaxially in air with their centers at a distance $2b$ apart, and a steady current I is passed through each coil. Calculate the magnetic scalar potential at a point on the common axis distant $(z-b)$ from the center of one coil and $(z+b)$ from the

center of the other, (i) if the currents in the coils are in the same direction and (ii) if they are in opposite directions. In both cases deduce the magnetic flux density at the center of each coil and at a point mid-way between the centers.

28. A hydrogen ion enters a uniform magnetic field at right angles to the flux lines and is deflected in a circular path. What must be the value of B in order that the ion should transverse a complete circle in 0.01 μ sec.

29. Calculate the natural period of rotation of an electron in the earth's field if it is taken as 10^{-4} tesla. What is the radius of curvature of the path of a 100 volt electron moving normally to this field

30. Show that the drift velocity of an ion with a charge to mass ratio e/m in a region of flux density **B** and gravitational acceleration **g** is given by $(\mathbf{g} \times \mathbf{B})/e\ B^2$.

31. A beam of electrons after falling through a potential of 100 volts spreads out in a cone of small apex angle.

What current must be passed through a solenoid of 10 turns per cm coaxial with the cone in order to refocus the electrons in a distance of 10 cm.

32. A cathode ray tube has a gun voltage V and the spot is capable of being deflected simulatenously in two directions at right angles. The deflection in one direction is by means of an electric field and that in the other by means of a magnetic field. The deflecting electric and magnetic fields influence the electron beam over the same path length and at the same distance from the screen. Both deflecting fields vary in strength with time t periodically. The

periodic time is $2\pi/\omega$, the deflecting electric and magnetic fields are $E_0 \cos(\omega t)$ and $B_0 \sin(\omega t)$ respectively. With suitable approximations show that the spot on the screen describes an ellipse and calculate the axis ratio If $V = 1000$ volts evaluate the ratio E_0 to B_0 required to produce a circle on the screen

33. Two fairly long and linear cathode rays which are at a mutual distance of 5 cm from each other carry a current of 100 μA. The electrons of one of the beams have a kinetic energy of 100 electron volts and the second 1000 electron volts. Determine the resulting electrostatic field strength due to the volume charge of the rays at a point 5 cm from both of them.

34. A cathode ray tube has a beam velocity v and its deflecting field (electric or magnetic) influences the beam over a length L. The deflecting field oscillates sinusoidally with time with frequency f. Prove that the deflection of the spot on the screen cannot follow the variation of the deflecting field faithfully unless f is small compared with $v/2\pi L$. Evaluate this limiting frequency if $L = 3$ cm and the gun potential is 1000 volts.

35. If n parallel straight wires each carrying a current I in the same sense are placed at equal intervals around a circle $r = a$, $z = o$, and normal to its plane, prove that the vector potential at the point $P(r, \theta, z)$ can be expressed in the form

$$\mathbf{A} = -(\mu_0 I/4\pi) \log (r^{2n} + a^{2n} - 2r^n a^n \cos n\theta)$$

36. A current I flows in a wire in the form of a circle of radius a. Show that at a point P distance z from the plane of the circle and r from the axis of the circle, the vector potential is in a direction perpendicular to the axial plane through P and is of magnitude

$$(\mu_0 Ia/2\pi) \int_0^{2\pi} \cos\theta \, d\theta/D \text{ where } D = a^2+r^2+z^2-2ar\cos\theta.$$

37. A wire is in the form of that part of the curve $2\pi r \quad a\theta$ for which $o \leq \theta \leq 2\pi$. Prove that when a current I flows in the wire the magnetic flux density at a point distant z from O (the pole of coordinates) on the normal through O to the plane of the wire, has a component along the normal of magnitude

$$\tfrac{1}{2}\ \mu_0\ I\ [(1/a)\ \text{arcsinh}\ (a/h) - 1/(a^2 + z^2)^{\frac{1}{2}}]$$

الفصل الثامن

المجال المغناطيسي الساكن في الأوساط المغناطيسية

THE MAGNETOSTATIC FIELD IN MAGNETIC MEDIA

8 - 1 مقدمة :

يتكون أي وسط سواء كان غازاً أو سائلاً أو صلباً من ذرات . وهذه الذرات لها تركيب الكتروني تتحرك فيه الالكترونات في مدارات مستوية حول النواة ، وتدور هذه الالكترونات كذلك حول نفسها . وينتج عن ذلك حلقات تيار (current loops) ذات أبعاد ذرية تعرف بالتيارات الأمبيرية (Amperian Currents) . وتكافىء هذه الحلقات مغناطيسيات متناهية الصغر لكل منها عزم مغناطيسي ويتم الكشف عن هذه الحلقات بقياس هذا العزم المغناطيسي . والعزم المغناطيسي لكل ذرة هو محصلة العزم المداري وعزم الدرور (Spin) لكل الكترونات الذرة . وعند وضع مادة لها محصلة عزم مغناطيسي في مجال مغناطيسي خارجي يعمل هذا المجال على تغيير التركيب المغناطيسي للمادة بحيث يزيد من العزم المغناطيسي المحصل ويعمل على تغيير إتجاهه ليصبح في إتجاه المجال الخارجي .

من الناحية العامة يمكن وصف المادة المغناطيسية عند وضعها في مجال مغناطيسي خارجي بالقول أنه حدث لها إستقطاب مغناطيسي ولكن هذا يعتبر غير كافي في بعض الأحوال وذلك لأن الخواص المغناطيسية لمعظم المواد

الحديدية المعروفة هي غير خطية وغير متجانسة ولها إنشوطات تخلفية (Hysteresis Loops) ولذا يجب دراستها بعناية ولا يتأتى ذلك إلا بالبحث بدقة عن الخواص الفيزيائية لهذه المواد .

2-8 متجه الاستقطاب المغناطيسي : (Magnetization Vector)

من الناحية الفيزيائية نعبر عن التيارات الامبيرية والعزوم المغناطيسية وذلك بما يسمى متجه الاستقطاب المغناطيسي M وهو مناظر للاستقطاب الكهربي P في العوازل التي يؤثر عليها مجال كهربي خارجي . إعتبر حجم صغير Δv من المادة المغناطيسية التي يؤثر عليها مجال مغناطيسي خارجي . المجموع الاتجاهي لعزوم الذرات في هذا الحجم هو ،

$$\sum_{i=1}^{N \Delta v} m_i = N \Delta v\, m = M \Delta v \qquad (1\text{-}8)$$

حيث m هي متوسط العزم المغناطيسي لذرة واحدة وN عدد الذرات في وحدة الحجوم . ويمكن تعريف M عند نقطة على أنه العزم المغناطيسي لوحدة الحجوم حول هذه النقطة أو ما يسمى متجه الاستقطاب المغناطيسي . وتكون وحدات M هي أمبير لكل متر . ويمكن كتابة M على الصورة الآتية :

$$M = Lt_{\Delta v \to 0} \frac{1}{\Delta v} \sum_{i=1}^{N \Delta v} m_i$$

$$= Lt_{\Delta v \to 0} \frac{\Delta m}{\Delta v} = \frac{dm}{dv} \qquad (2\text{-}8)$$

وإذا لم ينعدم الاستقطاب المغناطيسي M عند كل نقطة داخل الجسم الموضوع في المجال المغناطيسي الخارجي فإنه يقال أن الوسط ممغنط .

3-8 المتأثرية المغناطيسية والانفاذية المغناطيسية : (Magnetic Susceptibility and Magnetic Permeability)

من الناحية العامة يمكن كتابة العلاقة بين كثافة الفيض المغناطيسي B

والمجال المغناطيسي H في المادة الممغنطة على الصورة .

$$B = \mu_0 \ (H + M)$$

$$= \mu_0 \ (1 + M/H) \ H \qquad (3\text{-}8)$$

فإذا كانت العلاقة بين M و H هي علاقة خطية فإن النسبة بينهما تكون عدد ثابت X_m يسمى المتأثرية المغناطيسية *(Magnetic Susceptibility)* .

$$X_m = M/H$$

وتصبح المعادلة *(3-8)* على الصورة :

$$B = \mu_0 \ (1 + X_m) \ H$$

$$= \mu_0 \ \mu_r \ H \qquad (4\text{-}8)$$

حيث μ_r هي الانقاذية النسبية للوسط المغناطيسي . ويلاحظ أنه في الفراغ $M=o$. ويمكن كتابة M كدالة من كثافة الفيض المغناطيسي B والمجال الكهربي H داخل المادة المغناطيسية على الصورة .

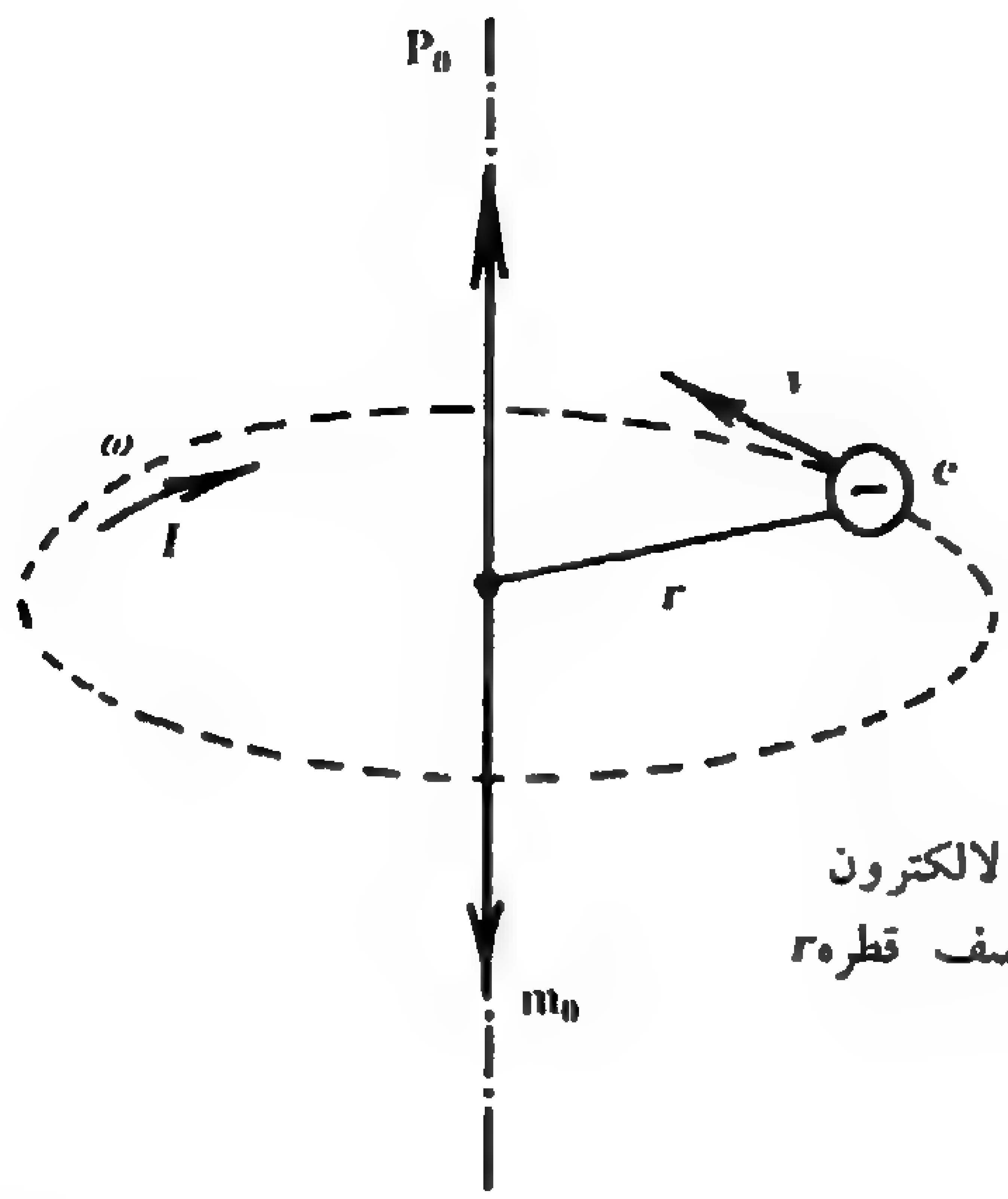
$$M = \frac{B}{\mu_0} - H \qquad (5\text{-}8)$$

4-8 المجالات المغناطيسية الميكروسكوبية :

يمكن تعيين الخواص الطبيعية للمواد المغناطيسية وذلك بايجاد العزم المغناطيسي m_o الناشىء عن دوران الالكترون حول النواة وكذلك العزم المغناطيسي m_s الناشىء عن درور الالكترون .

(.1) العزم المغناطيسي لدوران الالكترون في مدار مستوى حول النواة :

نفرض أن الالكترون يدور كما في شكل *(a-8)* بسرعة v أو بسرعة



شكل (8 - a) العزم المغناطيسي لالكترون يدور في مدار دائري مستوى نصف قطره r

زاوية $\omega = v/2\pi r$ حيث r هو نصف قطر المدار . يمكن تعيين التيار المكافىء في الحلقة الدائرية التي يدور بها الالكترون من المعادلة

$$I = -e\omega = -ev/2\pi r$$

ويكون العزم المغناطيسي المكافىء لهذا الاكلترون هو :

$$m_0 = \pi r^2 I \, a_z$$

$$= -\frac{1}{2} e v r \, a_z \qquad (6\text{-}8)$$

كمية الحركة الزاوية لهذا الالكترون هي :

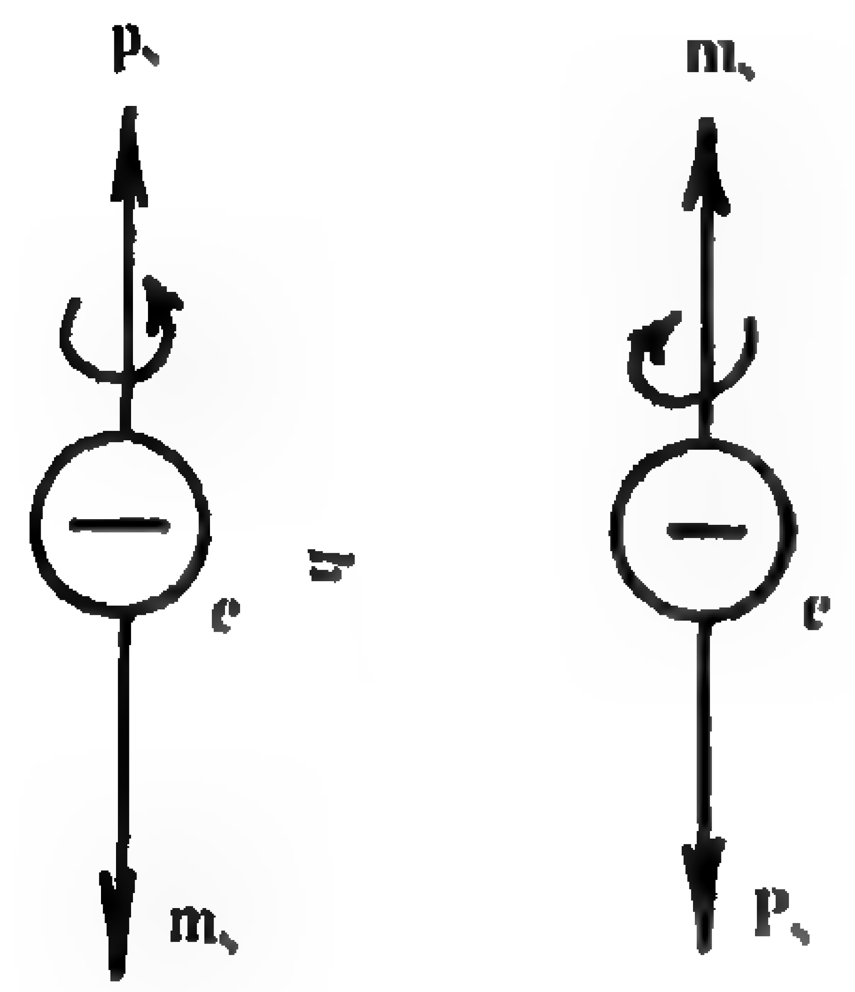
$$p_0 = m \omega \, a_z \qquad (7\text{-}8)$$

وبذلك تكون العلاقة بين m_o وp_o هي :

$$.m_0 = -\frac{e}{2m} P_o$$

حيث $e = 1.6 \times 10^{-19}$ C. ، m هي كتلة الالكترون .

(2.) العزم المغناطيسي الناتج عن درور (spin) الالكترون حول نفسه يمكن تعينه من الخواص الكمية للالكترون (quantum mechanical properties) حيث كمية الحركة للالكترون هي شكل (8 - b) .



شكل (8 - b) العزم المغناطيسي للدور الالكترون

$$P_s = \frac{h}{4\pi} a_n$$

حيث h هو ثابت بلانك ، a_n هو متجه وحدة موازي لمحور درور الالكترون وتحدد نظرية بوهر (Bohr) العزم المغناطيسي لدرور الالكترون بالعلاقة .

$$m_s = -\frac{e}{2m}\frac{h}{2\pi} a_n \qquad (9\text{-}8)$$

من المعادلات السابقة نستنتج أن كمية الحركة الزاوية P_s للالكترون وعزمه المغناطيسي الناشيء عن الدرور يكونان في إتجاهين مختلفين .

$$m_s = -\frac{e}{m} P_s \qquad (10\text{-}8)$$

وتعرف النسبة e/m بالنسبة الجيرو مغناطيسية لدرور الالكترون . المعادلات (6-8) إلى (9-8) تعطي العزوم المغناطيسية للالكترون وكذلك كمية

الحركة الناشئة عن درووه ودورانه حول النواة وهي معادلات أساسية بواسطتها يمكن تفسير الخواص المغناطيسية للمواد الممغنطة ، وعندما تكون المادة المغناطيسية معزولة وغير واقعة تحت تأثير مجال مغناطيسي خارجي فإن العزوم المغناطيسية لذرات هذه المواد لها إتجاهات عشوائية وبحيث يكون عزمها المغناطيسي المحصل منعدم .

وهناك ثلاثة ظواهر مغناطيسية وهي :

(أ) الديا مغناطيسية (Diamagnetism)

(ب) البارا مغناطيسية (Paramagnetism)

(حـ) المغناطيسية الحديدية (Ferromagnetism)

(أ) الديا مغناطيسية :

بعض المواد المغناطيسية تكون ذراتها متعادلة مغناطيسياً في غياب أي مجال مغناطيسي خارجي بمعنى أن m_s, m_0 يلاشيان بعضهما تماماً . وعند التأثير بمجال مغناطيسي خارجي فإنه يحدث تفاعل بين هذه العزوم والمجال بحيث تكون X_m سالبة في الاشارة وصغيرة جداً في خدود 10^{-5} – وبالتالي فإن النفاذية النسبية μ_r تكون قريبة جداً من الواحد الصحيح ولكنها أصغر منه وبالتالي تكون كثافة الفيض الكهربي B أصغر من قيمتها في الفراغ ومن أمثلة هذه المواد النحـاس والفضة والسليكـون والذهب والـزئبق والصوديوم . . .

(ب) البارا مغناطيسية :

بعض المواد المغناطيسية تكون ذراتها لها عزم مغناطيسي محصل في غياب أي مجال مغناطيسي خارجي . ولكن هذه العزوم تكون عشوائية التوزيع نتيجة للحركة الحرارية لذراتها . وعند التأثير عليها بمجال مغناطيسي

خارجي فإن العزوم المغناطيسية للذرات تعمل على أن تتحاذى في إتجاه واحد مع المجال الخارجي وتعمل بالتالي على تقوية هذا المجال . وتتوقف هذه العملية على درجة الحرارة T فهي تقل بزيادة درجة الحرارة حيث تهتز هذه الالكترونات وتتصادم نتيجة لحركتها الحرارية مما يؤدي إلى ازدياد عشوائية عزوم ذرات المادة وبالتالي إلى خفض مغنطتها . ونلاحظ في هذه المواد أنه عند درجة الحرارة المنخفضة X_m تتناسب مباشرة مع $\frac{1}{T}$ فإذا انخفضت درجة الحرارة فإن μ_r تزيد والعكس صحيح .

(حـ) المغناطيسية الحديدية :

تعتبر المادة أنها ذات مغناطيسية حديدية إذا كانت لها قابلية اكتساب المغنطة تلقائياً مثلما هو الحال في المغناطيسيات الدائمة . وتحدث هذه المغنطة الدائمة عند درجات حرارة أقل من قيمة معينة T_c تسمى درجة حرارة كوري للمغناطيسية الحديدية (Ferromagnetic Curie Temperature) وعند درجات حرارة أعلى من T_c فإن هذه المواد تكون خواصها بارا مغناطيسية أي تقل مغنطها مع زيادة درجة الحرارة T (حيث $T > T_c$) .

في هذه المواد يكون العزم المغناطيسي للذرات كبيراً جداً الى الحد الذي تتوحد فيه إتجاهات هذه العزوم بسهولة وإذا توحدت في اتجاهها فإنها تظل كذلك وتصبح للمادة مغناطيسية دائمة حتى بعد إزالة المجال الخارجي الممغنط . ويمكن الآن تفسير ما يحدث إذا ارتفعت درجة الحرارة عن T_c . ففي هذه الحالة تكون الطاقة الحرارية عالية بما فيه الكفاية لتمزيق الترتيب العزمي للذرات داخل المادة . ومن أمثلة هذه المواد الحديد والنيكل والكوبلت (T_c لهذه المواد هي 1043°K ، 631°K ، 1393°K على التوالي) .

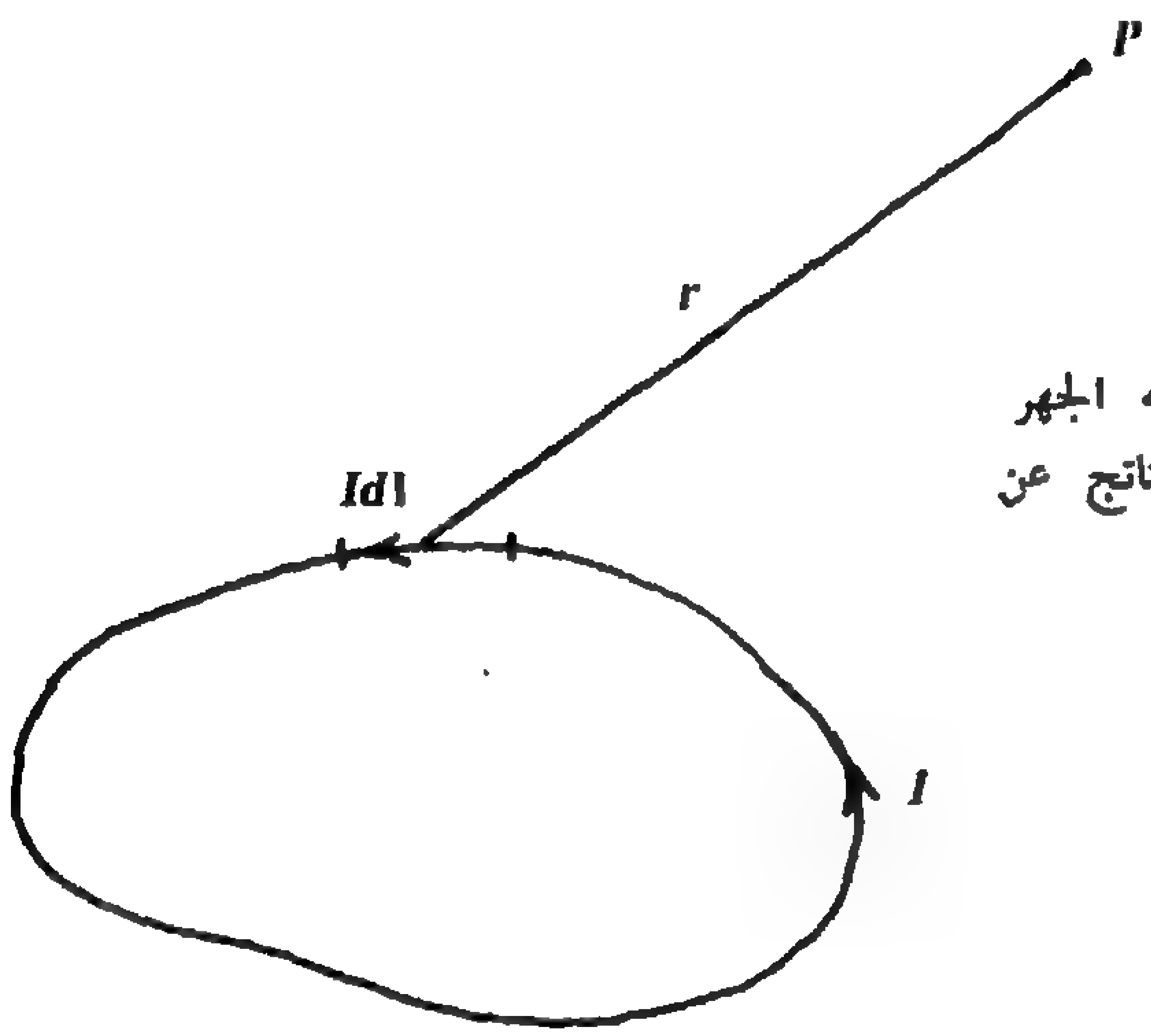
5-8 نماذج التيار - الامبيري والقطب - المكافىء : (Amperian-Current and Equivalent-Pole Models)

توجد طريقتان لتعيين المجال المغناطيسي الناشىء عن وجود جسم

ممغنط وذلك عند نقطة خارجة عنه وهما نموذج التيار الامبيري ونموذج القطب المكافىء . ولفهم هذين النموذجين اعتبر أولاً متجه الجهد المغناطيسي لحلقة تحمل تيار .

8 - 5 a متجه الجهد المغناطيسي لحلقة تحمل تيار :

إفرض حلقة تحمل تياراً I كما هو مبين بشكل (8 - c) . متجه الجهد المغناطيسي الناتج عند أي نقطة P هو :



شكل (8 - c) تعيين متجه الجهد الكهربي عند نقطة P الناتج عن حلقة بها تيار كهربي

$$A = \frac{\mu_0 I}{4\pi} \oint_c \frac{dl}{r} \qquad (11\text{-}8)$$

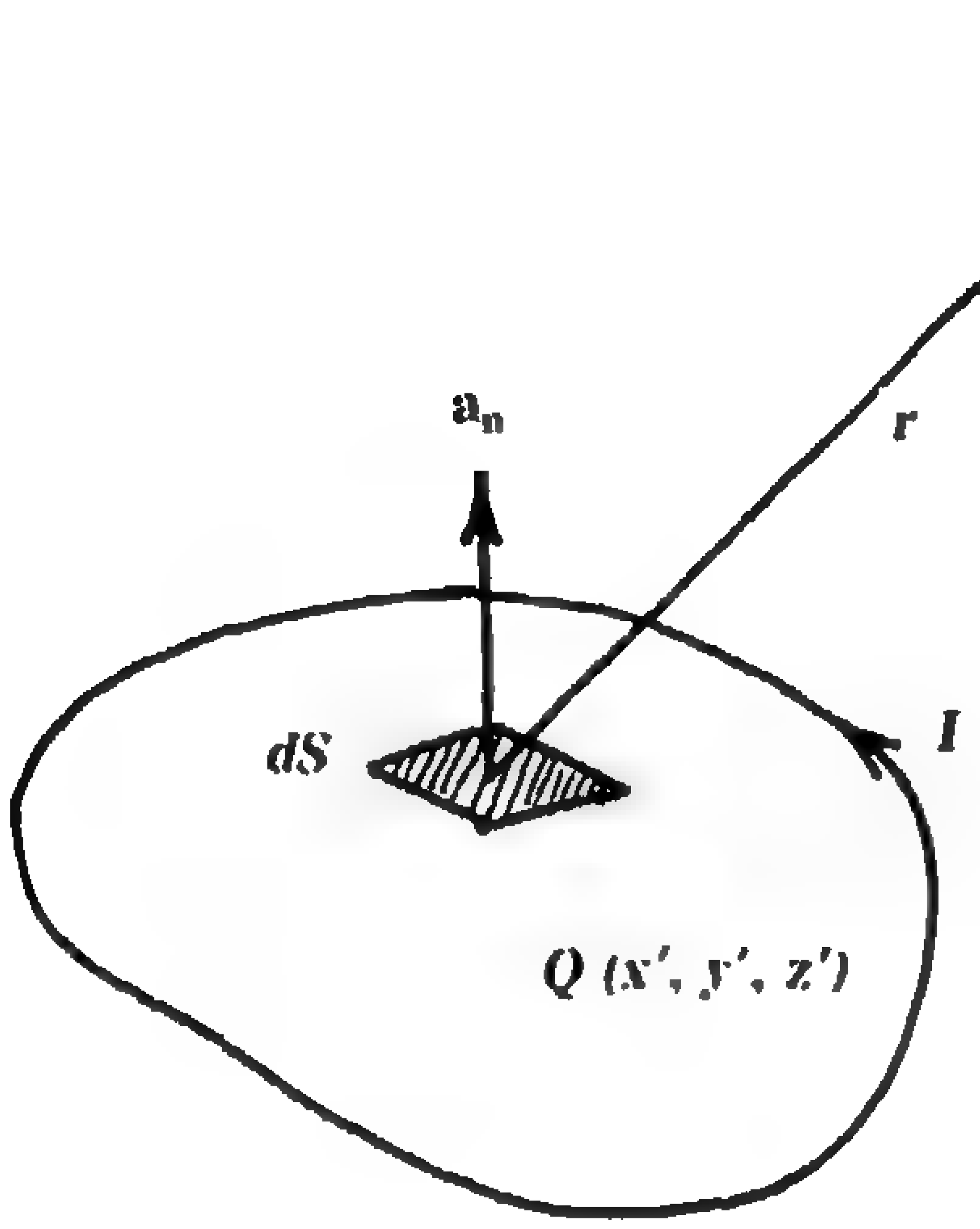
باستخدام نظرية ستوك يمكن ان نستنتج أنه إذا كانت هي الدالة قياسية فإن :

$$\oint_c \Psi \, dl = \int_S dS \times \nabla \Psi \qquad (12\text{-}8)$$

وباعتبار $\Psi = \frac{1}{r}$ يمكن تحويل المعادلة (8-11) الى الصورة الآتية :

$$A = \frac{\mu_0 \; I}{4\pi} \int_S a_n \times \nabla_Q \left(\frac{I}{r} \right) dS \qquad (13\text{-}8)$$

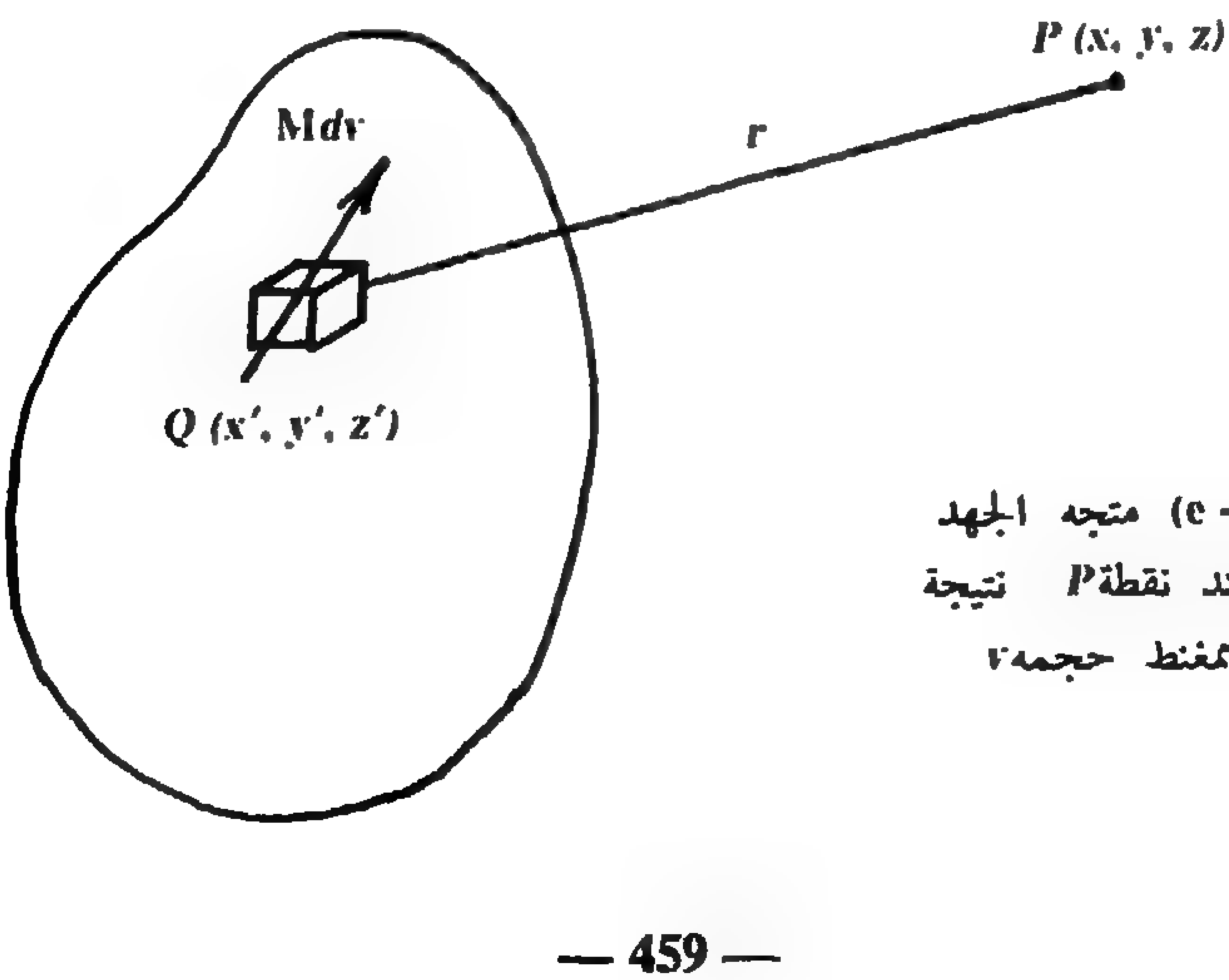
حيث S هي مساحة أي سطح محاط بالحلقة C . والمسافة r هي في هذه الحالة من المساحة dS الى النقطة P . شكل (d-8) .



شكل (d - 8) التكامل السطحي المكافىء للتكامل الخطي بشكل (c - 8)

b5-8 إنموذج التيار الامبيري :

إعتبر جسم ذات مغنطة غير منتظمة حجمه v . إفرض أن المغنطة عند نقطة $Q\ (x', y', z')$ داخل هذا الجسر (شكل e-8) هي M يمكن تعيين



شكل (e - 8) متجه الجهد الكهربي عند نقطة P نتيجة لجسم ممغنط حجمه v

متجه الجهد المغناطيسي dA الناشيء عند نقطة $P(x,y,z)$ خارج v وعلى بعد r من النقطة Q من العلاقة .

$$\mathbf{M}\, dv = I\, dS\, a_n$$

وبذلك يكون :

$$\mathbf{A} = \frac{\mu_0}{4\pi} \int_v \mathbf{M} \times \nabla_Q \left(\frac{1}{r}\right) dv \qquad (14\text{-}8)$$

ويمكن كتابة هذه المعادلة على الصورة البسيطة التالية

$$A = \frac{\mu_0}{4\pi} \int_v \frac{M}{r}\, dv + \frac{\mu_0}{4\pi} \oint_S \frac{K_M}{r}\, dS \qquad (15\text{-}8)$$

حيث :

$$J_M = \nabla_Q \times \mathbf{M}\,,\ K_M = \mathbf{M} \times a_n \qquad (16\text{-}8)$$

وa_n هو متجه وحدة عمودي على السطح S الذي يحدد الجسم الممغنط . من العلاقة (15-8) نستنتج أن متجه الجهد المغناطيسي عند نقطة P ينشأ من تيار حجمي كثافته J_M وتيار سطحي K_M موزعة داخل وعلى سطح الجسم الممغنط على الترتيب . وتعرف هذه التيارات بالتيارات الامبيرية المغناطيسية وذلك لتمييزها عن التيارات J ، K . ويمكننا القول أن التيارات J_M ، K_M هي تيارات مرتبطة تنشأ من دوران ودرور الالكترونات المرتبطة بذراتها داخل المادة المغناطيسية .

8 - 5 ج إنموذج القطب المكافىء :

في هذا النموذج يستبدل الجسم الممغنط بتوزيع أقطاب مغناطيسية مكافئة مكونة من كثافة حجميه ρ_m وكثافة سطحية σ_m وهذا التكافؤ هو

تكافؤ رياضي فقط لأنه لا يوجد أقطاب مغناطيسية مفردة . وهذا النموذج مفيد كل كثير من المسائل المتعلقة بالمغناطيسيات الدائمة . وفي هذا النموذج تعرّف دالة الجهد المغناطيسي القياسية V_m بالمعادلة .

$$V_m = \frac{\mu_0}{4\,\pi} \int_v \frac{\rho_m}{r}\, dv + \frac{\mu_0}{4\,\pi} \oint_S \frac{\sigma_m}{r}\, ds \qquad (17\text{-}8)$$

حيث :

$$\rho_m = -\nabla_Q . \mathrm{M}, \sigma_m = \mathrm{M}.\ \mathrm{a_n} \qquad (18\text{-}8)$$

$\mathrm{a_n}$ هو متجه الوحدة العمودي على السطح S ، M هو الاستقطاب المغناطيسي عند النقطة التي يؤخذ عندها dv أو dS ويرتبط الجهد المغناطيسي V_m بكثافة الفيض المغناطيسي B بالمعادلة .

$$\mathrm{B} = -\nabla V_m \qquad (19\text{-}8)$$

حيث $\mathrm{B} = \mu_0\ \mathrm{H}$ هي كثافة الفيض المغناطيسي عند نقطة P خارج الجسم المغنط .

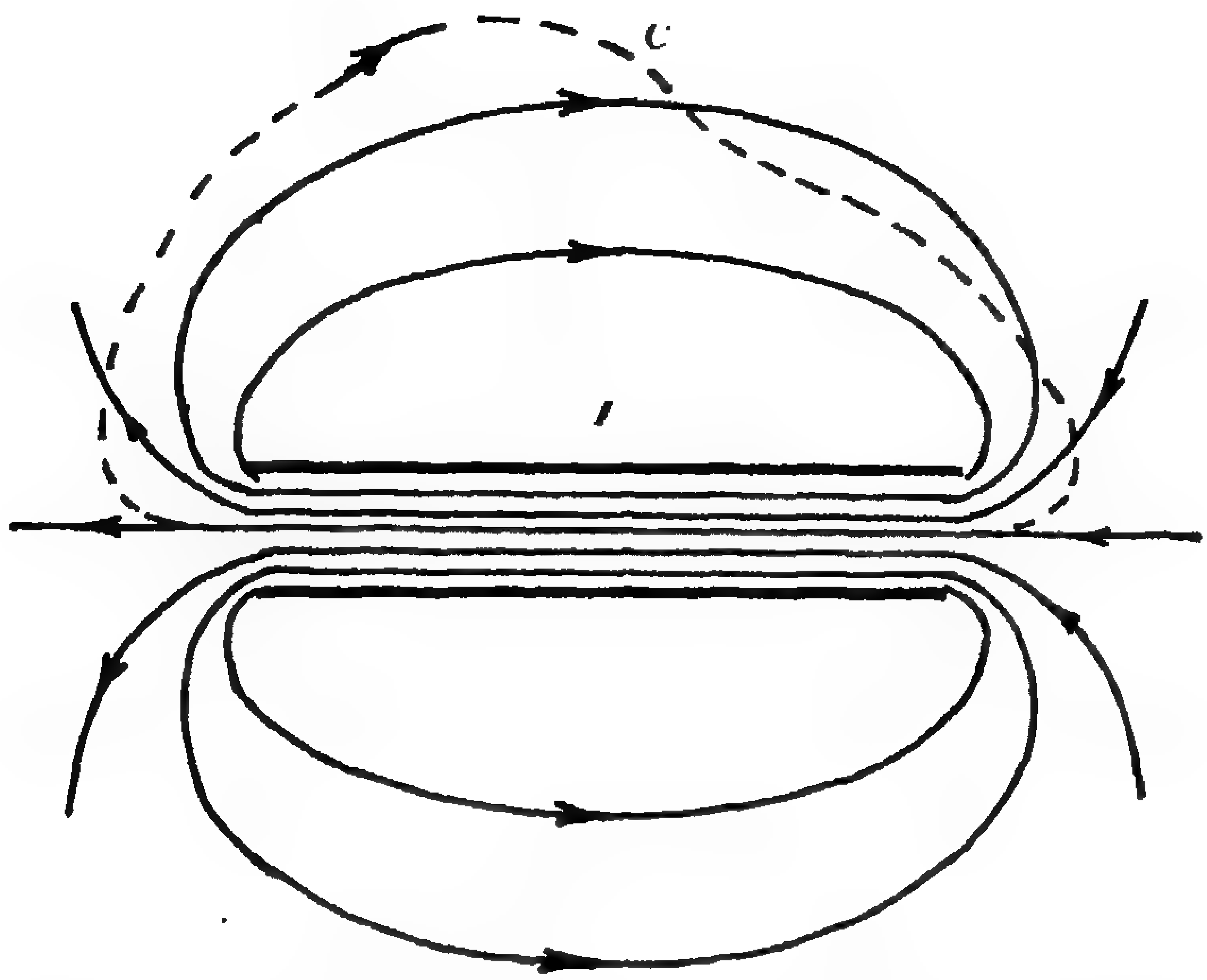
6-8 الدوائر المغناطيسية : (Magnetic Circuits)

بتطبيق قانون أمبير على ملف حلزوني طوله l (شكل f-8) وعدد لفاته N نجد أن :

$$\oint_C \mathrm{H}.d\mathrm{l} = NI \qquad (20\text{-}8)$$

المنحنى المغلق C في هذه الحالة يمر خلال عدد لفات N كما هو مبين . وحيث أن خطوط القوى المغناطيسية تتناثر خارج الملف فإن المجال المغناطيسي H خارج الملف يكون ضعيف وبالتالي يمكن تقريب التكامل الخطي إلى lH وبذلك نجد أن :

$$H = NI/l$$



شكل (F - 8) تطبيق قانون أمبير على ملف حلزوني عدد لفاته N وطوله l

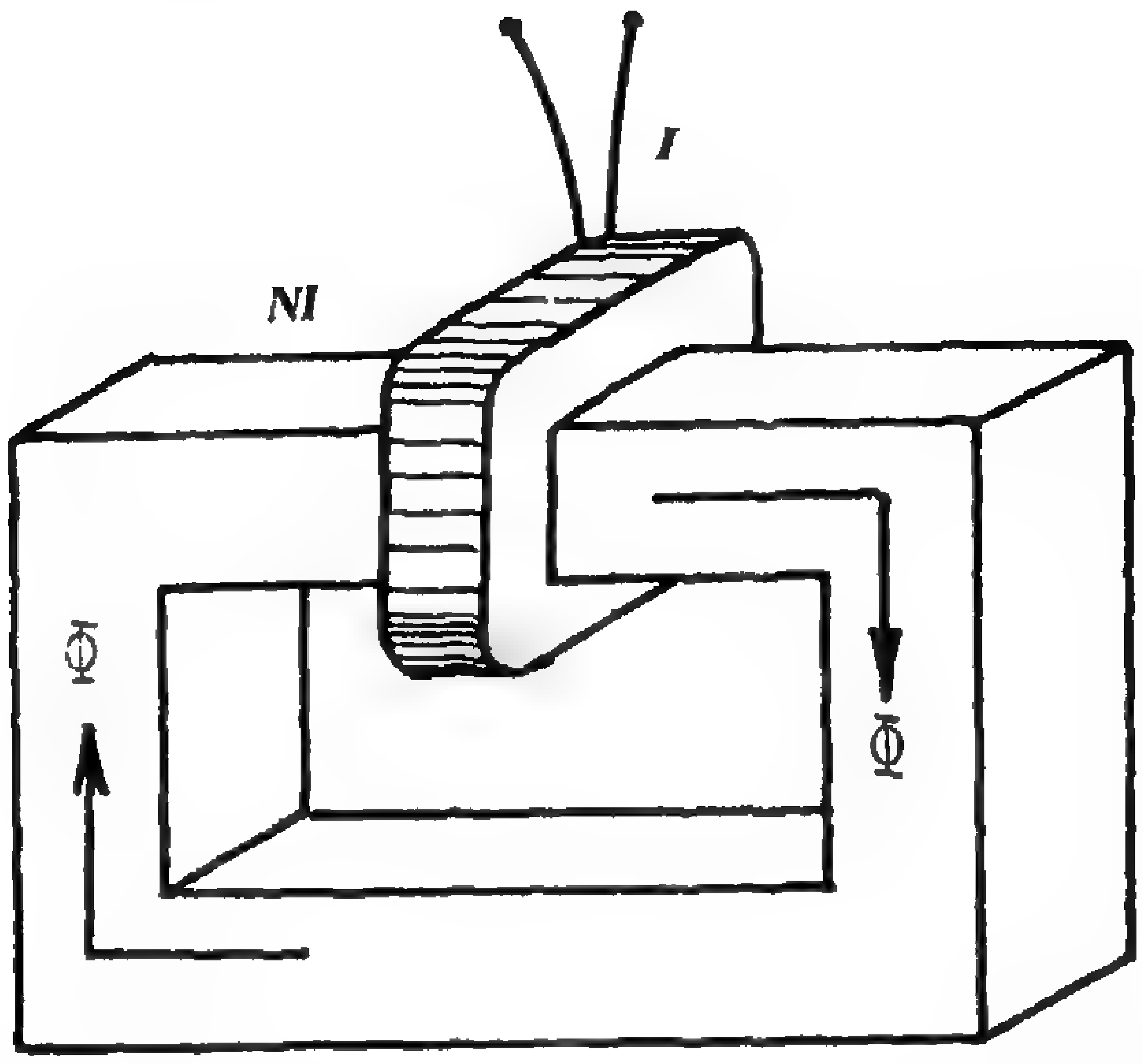
ومن المواد الحديدية تكون μ_r كبيرة جداً وبالتالي تكون كثافة الفيض المغناطيسي كبيرة $(B = \mu_0 \ \mu_r H)$ ويجب ملاحظة أن قيمة μ_r غير ثابتة إذ تتوقف على مقدار المجال المغناطيسي H وتسمى العلاقة بين B وH بمنحنى B – H ويعطي شكل (8-18) بعض منحنيات B – H لمجموعة من المواد المغناطيسية . ويلاحظ أن الفروض الآتية تتحقق إلى حد كبير وخاصة في حالة المجالات المغناطيسية الثابتة والمتغيرة ببطء .

(i) يكون معظم الفيض المغناطيسي داخل حدود المادة ذات المغناطيسية الحديدية حيث الإنفاذية عالية جداً بالمقارنة بانفاذية الهواء $(\mu_r = 1)$.

(ii) يفترض أن الفيض المغناطيسي موزع بالتساوي على أي من المقاطع المكونة للدائرة المغناطيسية .

(iii) في المواد ذات المغناطيسية الحديدية اللينة (soft ferromagnetic

materials) . تعمل أنشوطة التحلف (hysteresis loop) . والعلاقة اللاخطية بين B و H تعرف بمنحنى المغنطة للمادة . فمثلاً في شكل (8 - g)



شكل (8 - g) قلب حديدي يؤثر عليه قوة دافعة مغناطيسية NI

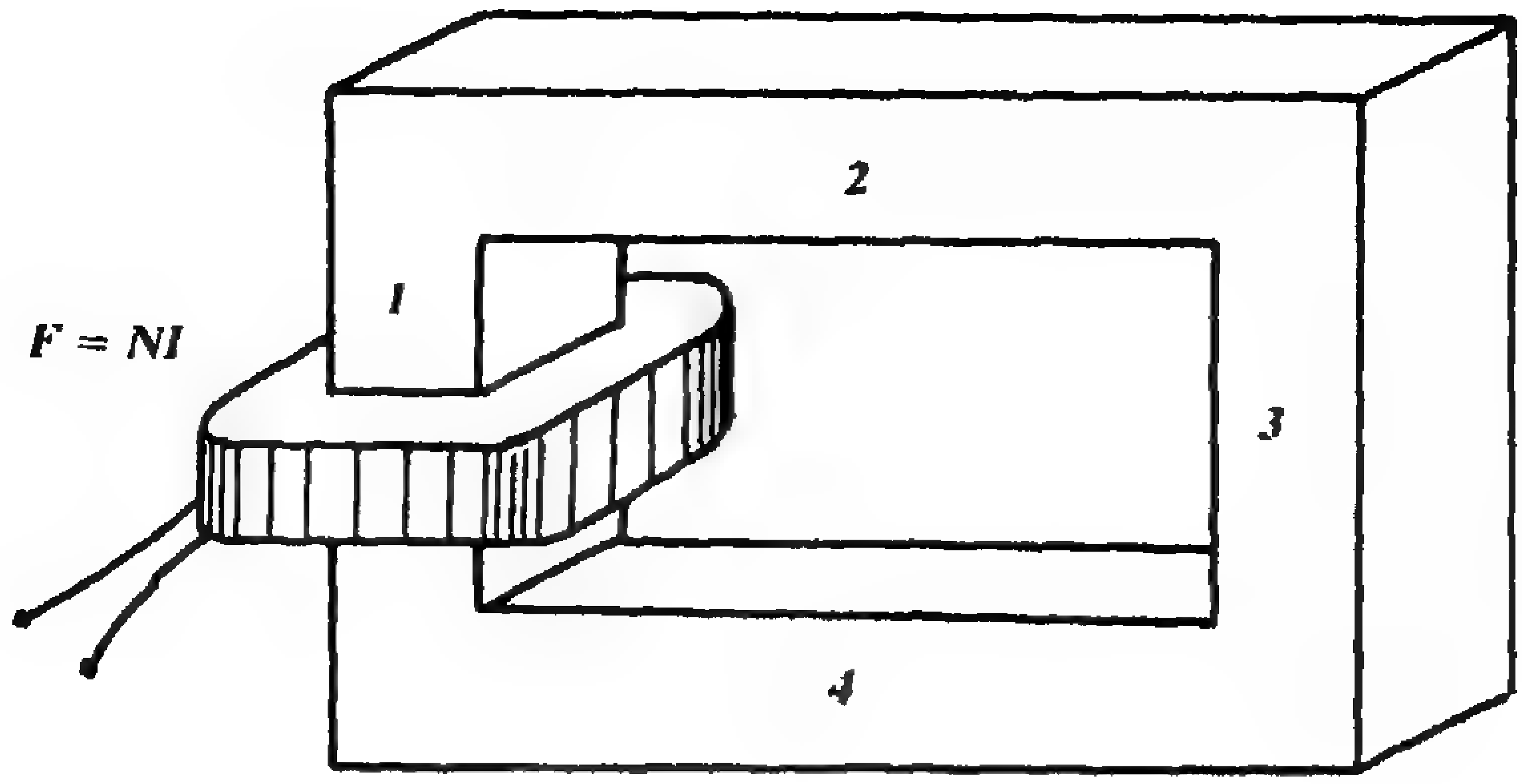
الملف غير موزع بالتساوي على كل القلب الحديدي وبالرغم من ذلك فإن القوى الدافعة المغناطيسية NI تسبب فيض مغناطيسي Ø مساره في القلب الحديدي . وعند حساب طول المسار الذي يمر به هذا الفيض يكتفي بحساب متوسط هذا الطول .

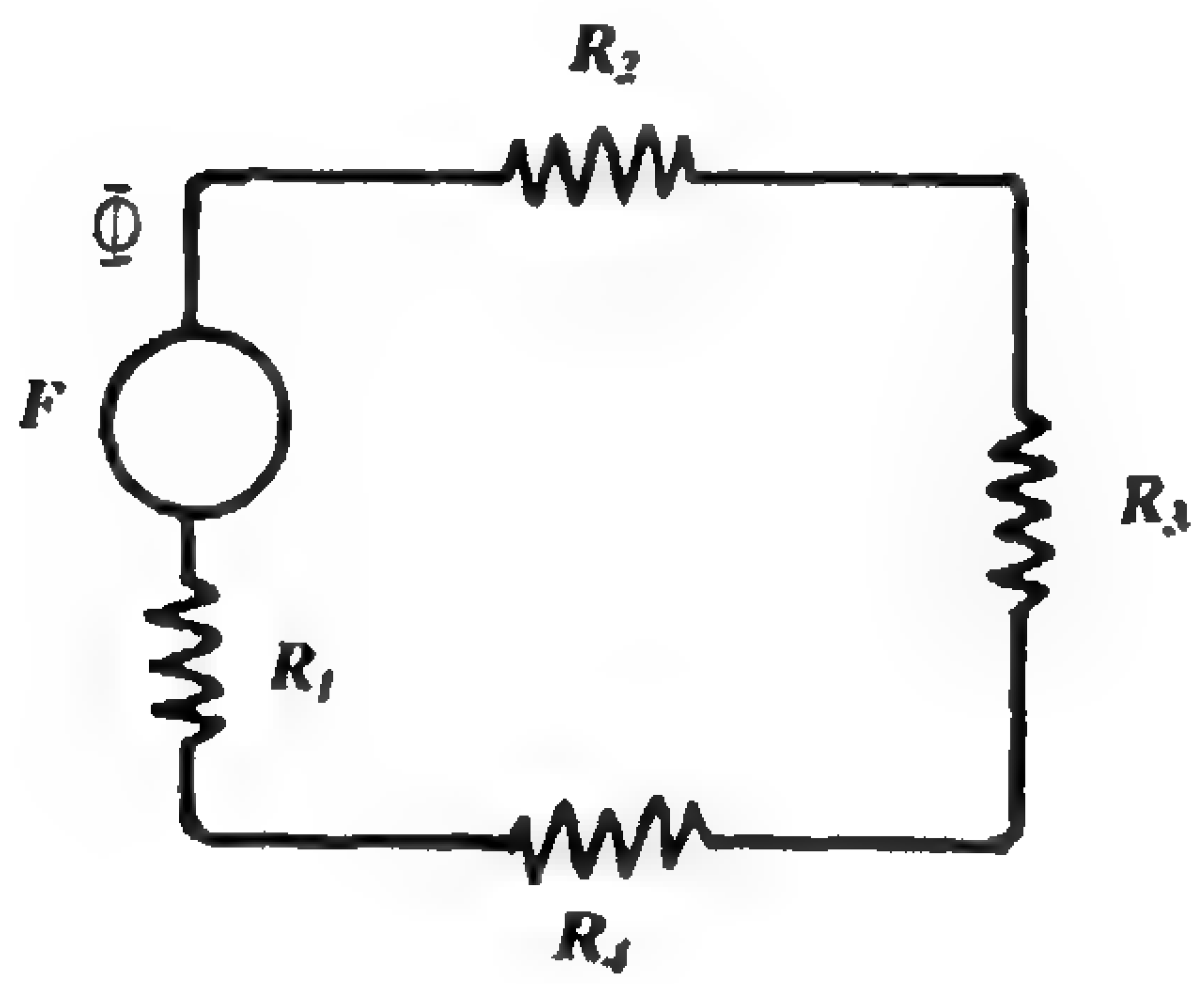
7-8 قانون أمبير للدوائر المغناطيسية : (Amperés Law for Magnetic Circuits)

أفرض ملف مكون من N لفة يمر به تيار I وملفوف حول قلب حديدي . القوة الدافعة المغناطيسية (magneto motive force) هي $F = NI$ ووحداتها أمبير- لفة وتعين من قانون أمبير (شكل 8-h)

$$\dot{F} = NI = \oint_c \mathrm{H}.d\vec{l}$$

$$= H_1 l_1 + H_2 l_2 + H_3 l_3 + H_4 l_4 \qquad (22\text{-}8)$$





شكل (h - 8)

ويمكن مقارنة هذه العلاقة بقانون كيرشوف لدوائر التيار المستمر وبذلك نكتب المعادلة (22-8) على الصورة .

$$NI = R\,\varnothing$$

حيث R هي الممانعة المغناطيسية (Magnetic Reluctance) للدائرة ويمكن كتابة هذه العلاقة على الصورة .

$$NI = Hl = S\,\mu\,H\,(l/\mu S)$$

$$= \varnothing\,(l/\mu S) \qquad (23\text{-}8)$$

حيث S هي مساحة مقطع القلب الحديدي بفرض أنه منتظم . وعلى ذلك يمكن كتابة المعادلة (8-22) على الصورة .

$$NI = \varnothing R_1 + \varnothing R_2 + \varnothing R_3 + \varnothing R_4 \qquad (24\text{-}8)$$

يلاحظ أنه لحساب الممانعة المغناطيسية R لأي دائرة مغناطيسية يجب معرفة إنفاذية كل جزء من أجزاء هذه الدائرة .

8 - 8 مسائل محلولة

SOLVED PROBLEMS

1. *Show that the magnetic scalar potential at a point outside a uniformly magnetized body is given by*

$$V_m = \mathbf{M}.\mathbf{R}$$

where $\mathbf{R} = -\nabla V$ *and V is the potential due to a uniform distribution of charge of density* $\mu_o \epsilon_o$ *distributed over the volume.*

The general expression for the magnetic scalar potential produced by a magnetized body is,

$$V_m = \frac{\mu_o}{4\pi} \int_v \frac{(-\nabla_Q . \mathbf{M})}{r} dv + \frac{\mu_o}{4\pi} \oint \frac{\mathbf{M}.\mathbf{a}_n}{r} dS$$

where $\nabla . \mathbf{M}$ is evaluated at the source point. Since **M** is uniform the volume integral is zero. Transforming the surface integral into a volume integral we have,

$$V_m = (\mu_o/4\pi)\,\mathbf{M}. \int_v \nabla_Q\,(1/r)\,dv$$

$$= (-\mu_o/4\pi)\,\mathbf{M}. \int_v \nabla_P\,(1/r)\,dv$$

$$= (-\mu_o/4\pi)\,\mathbf{M}.\nabla_P \int_v dv/r$$

where ∇P is evaluated at the point outside the magnetized body. Hence we have,

$$V_m = \mathbf{M}.\mathbf{R}$$

where, $\mathbf{R} = -\nabla V$

and

$$V = (\mu_0/4\pi) \int_v dv/r$$

which is the potential produced by a uniform volume distribution of charge of density $\epsilon_0 \mu_0$.

2. *Show that the magnetic potential at points outside a uniformly magnetized sphere of radius a and magnetization* **M** *is the same as that of a magnetic dipole of moment* $4/3 \pi a^3$ **M** *placed at the center of the sphere.*

Making use of the result of the previous problem we have that

$$V_m = \mathbf{M}.\mathbf{R}$$

where $\mathbf{R} = -\nabla V$. Since the point lies outside the sphere, the whole charge of density $\epsilon_0 \mu_0$ may be regarded as concentrated at the center of the sphere so that,

$$V = (\epsilon_0 \mu_0)(4/3\pi a^3)/(4\pi \epsilon_0 r)$$

$$= \mu_0 a^3/3r$$

Thus $\mathbf{R} = (\mu_0 a^3/3r)\, \mathbf{a}_r$

and

$$V_m = \tfrac{1}{3} \mu_0 a^3 (\mathbf{M}.\mathbf{a}_r) / r^2$$

and the magnetic potential of a dipole **m** is given by

$$V_m = (\mu_0/4\pi)(\mathbf{m}.\mathbf{a}_r) / r^2$$

It follows that the equivalent dipole moment of the sphere is

$$\mathbf{m} = {}^4/_3 \pi\, a^3\, \mathbf{M}$$

3. *The magnetization inside a permeable sphere of radius a is uniform and given by* $\mathbf{M} = M\mathbf{a}_z$. *Find the equivalent amperian current densities and the equivalent magnetic pole densities.*

The amperian current densities are given by,

$$\mathbf{J}_m = \nabla \times \mathbf{M}$$

$$\mathbf{K}_m = \mathbf{M} \times \mathbf{a}_n$$

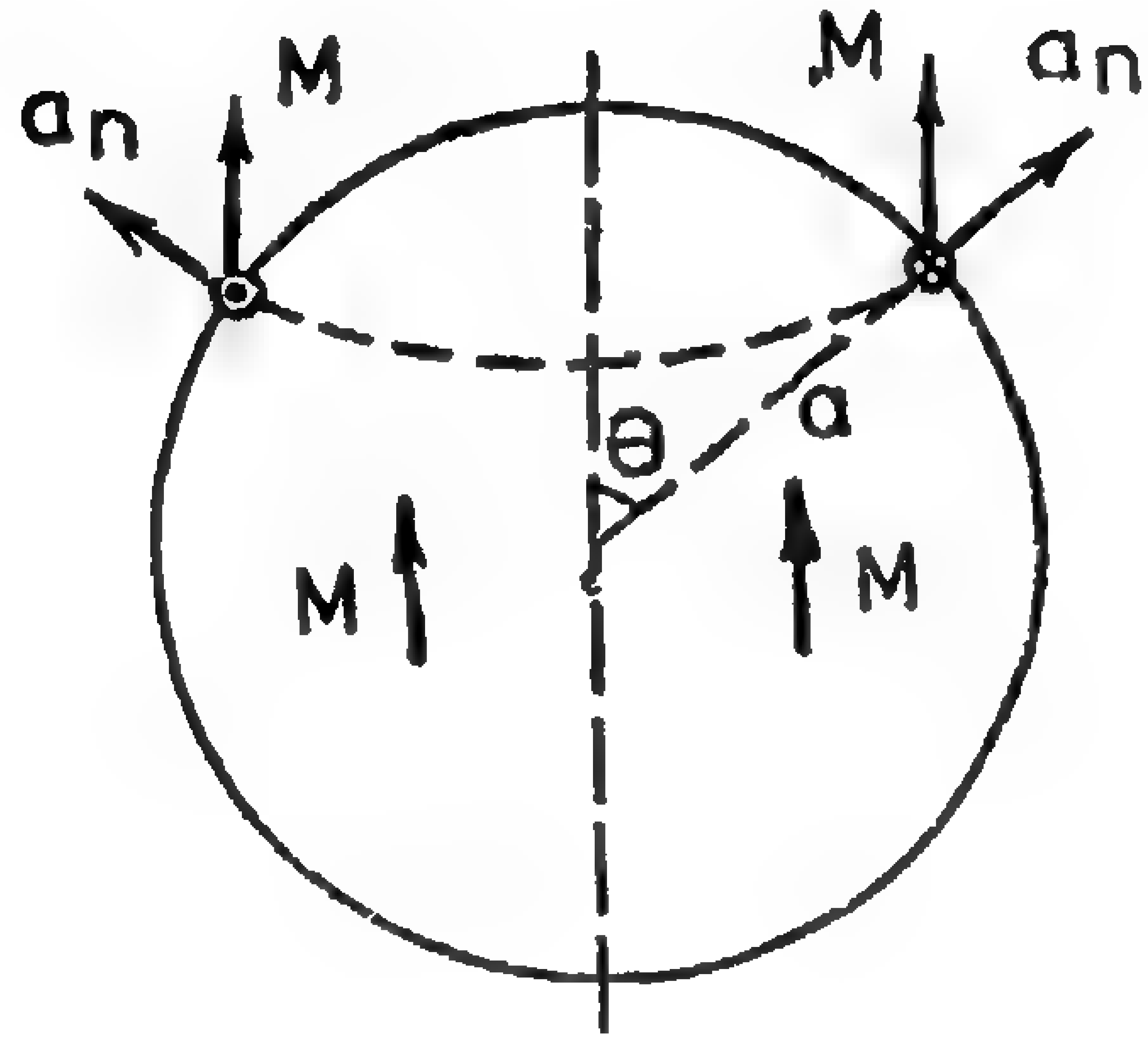


Fig. 8.1.

Thus the volume current density $\mathbf{j}_m = 0$ since $\mathbf{M}$ is uniform, and the surface current density is, (see Fig. 8.1),

$$\mathbf{K}_m = M\, \mathbf{a}_z \times \mathbf{a}_r = M \sin\theta\, \mathbf{a}_\phi$$

The equivalent pole densities are given by,

$$\rho_m = -\nabla . \mathbf{M}$$

$$\sigma_m = \mathbf{M} . \mathbf{a}_n$$

The volume pole density $\rho_m = 0$ since **M** is uniform, and the surface pole density is,

$$\sigma_m = \mathbf{M}\ \mathbf{a}_z.\mathbf{a}_r = M \cos \theta$$

4. *In the previous problem find the magnitizing force and magnetic flux density at the center of the sphere*

(a) Equivalent pole model.

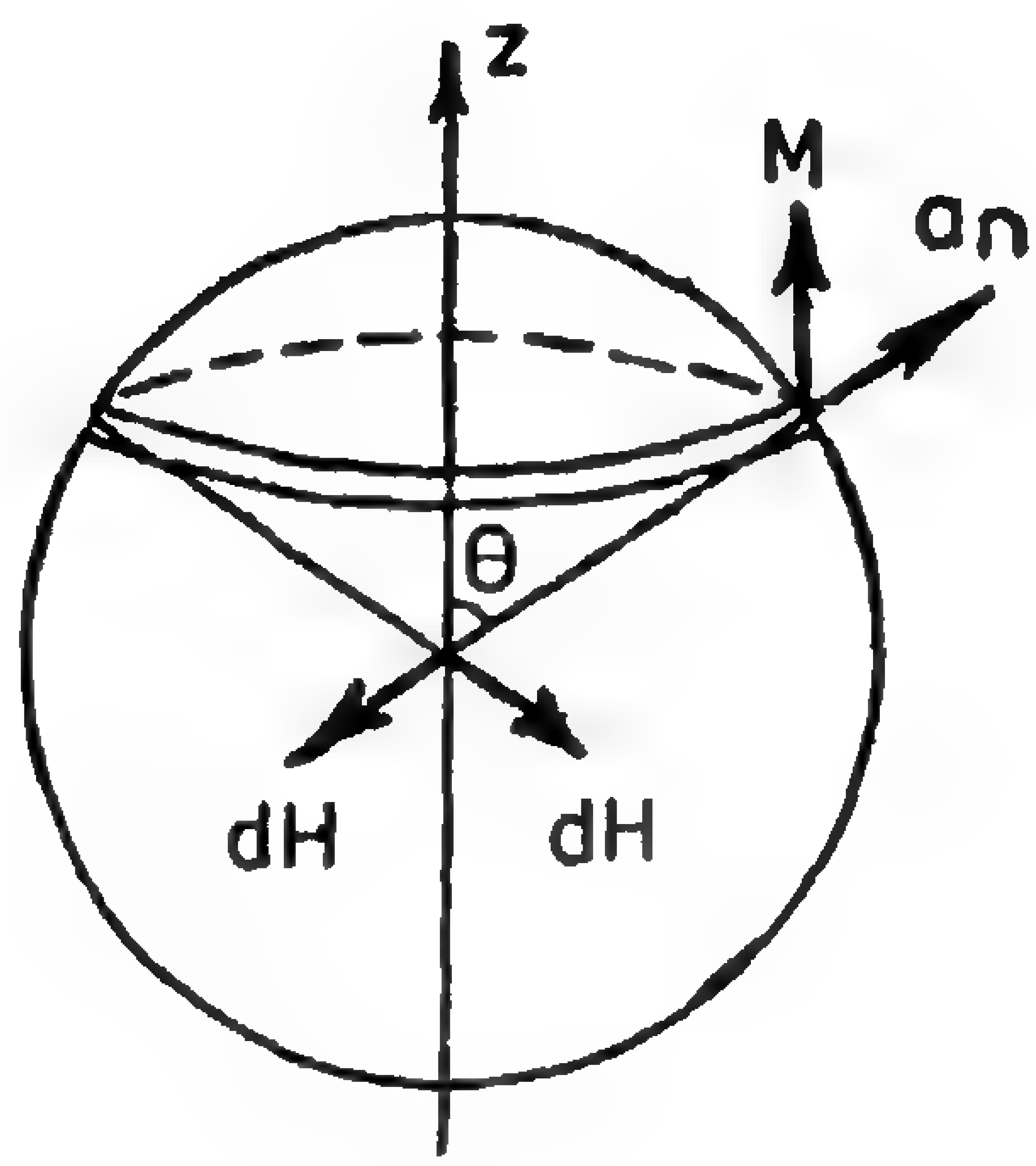


Fig. 8.2.

The sphere may be replaced by a surface charge of density (see Fig. 8.2),

$$\sigma_m = M \cos \theta$$

Also since,

$$dH = (1/4\pi)\ (\mathbf{M}.\mathbf{a}_n/a^2)\ dS$$

$$= (1/4\pi)\ M \cos \theta \sin \theta\ d\theta\ d\phi$$

Because of symmetry it is evident that the horizontal components of dH will cancel out. The resultant vertical component is thus,

$$H = \int dH \cos\theta = (M/4\pi) \int_0^{\pi} \int_0^{2\pi} \cos^2\theta \sin\theta \, d\phi \, d\theta$$

$$= M/3$$

or $\quad \mathbf{H} = -\frac{1}{3} M \, \mathbf{a}_z$

Hence, $\mathbf{B} = \mu_0 (\mathbf{H}+\mathbf{M}) = \mu_0 (-\frac{1}{3} M+M) \, \mathbf{a}_z$

$$= {}^2/_3 \, \mu_0 M \, \mathbf{a}_z$$

It will be noted that **B** and **H** point in opposite directions. This is always so inside magnetized matter.

(b) Amperian current model.

The sphere is replaced by a surface current of density

$$K_m = M \sin\theta$$

The magnetic flux density at the center is given by

$$B = \tfrac{1}{2} \mu_0 M a^3 \int_0^{\pi} \sin^3\theta \, d\theta \, / \, [a^2\cos^2\theta + a^2 \sin^2\theta]^{3/2}$$

$$= \tfrac{1}{2} \mu_0 M \left[\tfrac{1}{3} \cos^3\theta - \cos\theta\right]_0^{\pi}$$

$$= {}^2/_3 \, \mu_0 M$$

5. *The magnetization* **M** *is uniform inside a cylindrical disc of radius a and thickness d (d<<a) and is directed along the disc axis. Find the magnetizing force and magnetic flux density on the axis of the disc at points outside and inside the disc using (a) the equivalent pole model and (b) the equivalent current model.*

(a) Equivalent pole model.

Since **M** is uniform,

$$\rho_m = \nabla . \mathbf{M} = 0$$

Also, $\quad \sigma_m = \mathbf{M}.\mathbf{a}_n = \pm M$

so that the cylinder may be replaced by two equally and oppositely charged discs at a distance d apart (Fig. 8.3a). At a point at a distance z from the positively charged disc, the magnetizing force due to this disc is,

$$H' = -\tfrac{1}{2} M \left[z/(z^2+a^2)^{1/2} - 1\right]$$

and that due to the negatively charged disc,

$$H'' = \tfrac{1}{2} M \left[z' / (z'^2 + a^2)^{1/2} - 1\right]$$

where $z' = z+d$. The resultant external magnetizing force is therefore,

$$H_e = \tfrac{1}{2} M \left\{ (z+d) / [(z+d)^2 + a^2]^{1/2} - z / (z^2+a^2)^{1/2} \right\}$$

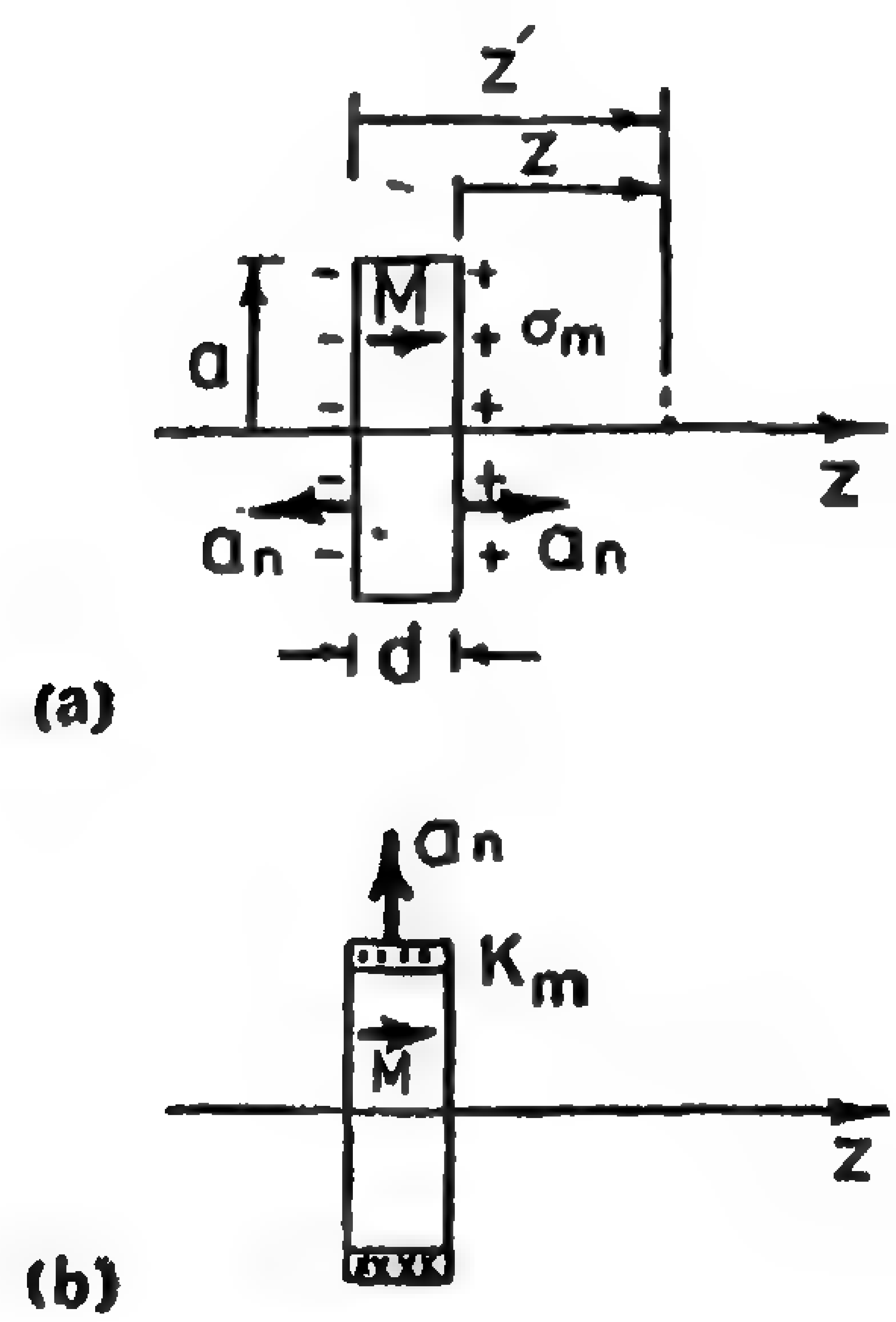


Fig. 8.3.

If the distance d is very small, the charges may be replaced by a uniform distribution of dipoles of strength Md such that in the limit as

d tends to zero and M tends to ∞ the product Md remains constant. It can be shown that in this case

$$H = \tfrac{1}{2}\, Md\, a^2 / (z^2+a^2)^{3/2}$$

The magnetic flux density is then given by $B = \mu H$. For a point on the axis inside the disc we have that

$$H' = \tfrac{1}{2}\, M\, [z / (z^2+a^2)^{1/2} - 1]$$

$$H'' = \tfrac{1}{2}\, M\, \{ (d-z) / [(d-z)^2+a^2]^{1/2} - 1 \}$$

Since at such a point $z << a$, we may write, remembering that $d << a$,

$$H_t = H'+H'' = \tfrac{1}{2}\, M\, [z/a + (d-z)/a - 2]$$

$$\simeq -M$$

Inside the disc the magnetic flux density is given by

$$\mathbf{B} = \mu_o\, (\mathbf{H}+\mathbf{M})$$

so that

$$B = \mu_o\, [\tfrac{1}{2}M\, (d/a - 2) + M]$$

$$= \tfrac{1}{2}\, \mu_o\, Md/a$$

(b) Equivalent current model.

The amperian current densities are (Fig. 8.3*b*)

$$\mathbf{j}_m = \nabla \times \mathbf{M} = 0$$

$$K_m = |\mathbf{M} \times \mathbf{a}_n| = M$$

This is equivalent to a surface current $I = Md$. Since d is small the disc may be represented by a circular current ring. The magnetic flux density at any point on the axis is given by

$$B = \tfrac{1}{2}\, \mu_o\, Md\, a^2 / (z^2+a^2)^{3/2}$$

at the center

$$B = \tfrac{1}{2}\, \mu_o\, Md/a$$

The magnetizing force outside is given by B/μ_0 and inside by

$$H = B/\mu_0 - M = \tfrac{1}{2} M d/a - M$$
$$= M(d/2a - 1) \simeq - M$$

6. *The magnetization* **M** *inside a sphere is uniform. Show that if there is a cubical cavity concentric with the sphere and with two faces perpendicular to* **M**, *the magnetizing force at the center of the sphere (and cavity) is zero.*

As shown in Problem 8.4. the magnetizing force at the center of the uniformly magnetized sphere is

$$\mathbf{H}_s = -\tfrac{1}{3} M \mathbf{a}_z$$

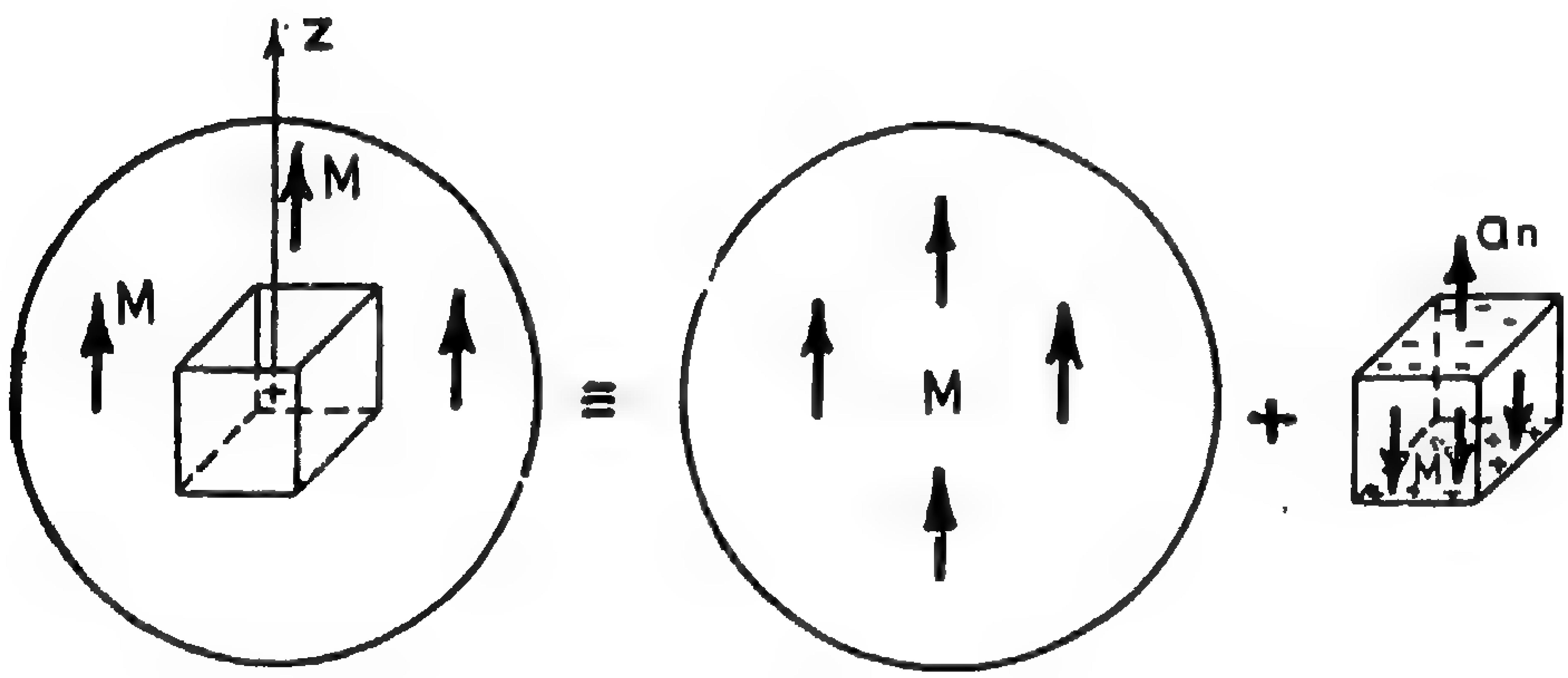


Fig. 8.4. a

The magnetized cube may be replaced by a surface charge density $\sigma_m = \mathbf{M}.\mathbf{a}_n$ which evidently gives $+M$ on the upper face and $-M$ on the lower face (Fig. 8.4*a*). The z-component of the magnetizing force at a point P on the axis normal to the plane is

$$H_z = (M/4\pi) \int dS \cos\theta/r^2 = (M/4\pi) \int d\,\Omega$$
$$= M\,\Omega\,/\,4\pi$$

where Ω is the solid angle subtended by the plane at P. The solid angle subtended at the center of a cube by one of the cube faces is evidently 1/6 of the total solid angle. Hence $\Omega = 4\pi/6 = 2\pi/3$. Thus the magnetizing force at the center of the cube due to both charged faces is

$$\mathbf{H}_c = 2\,(M/4\pi)\,(2\pi/3)\,\mathbf{a}_z = \tfrac{1}{3}\,M\,\mathbf{a}_z$$

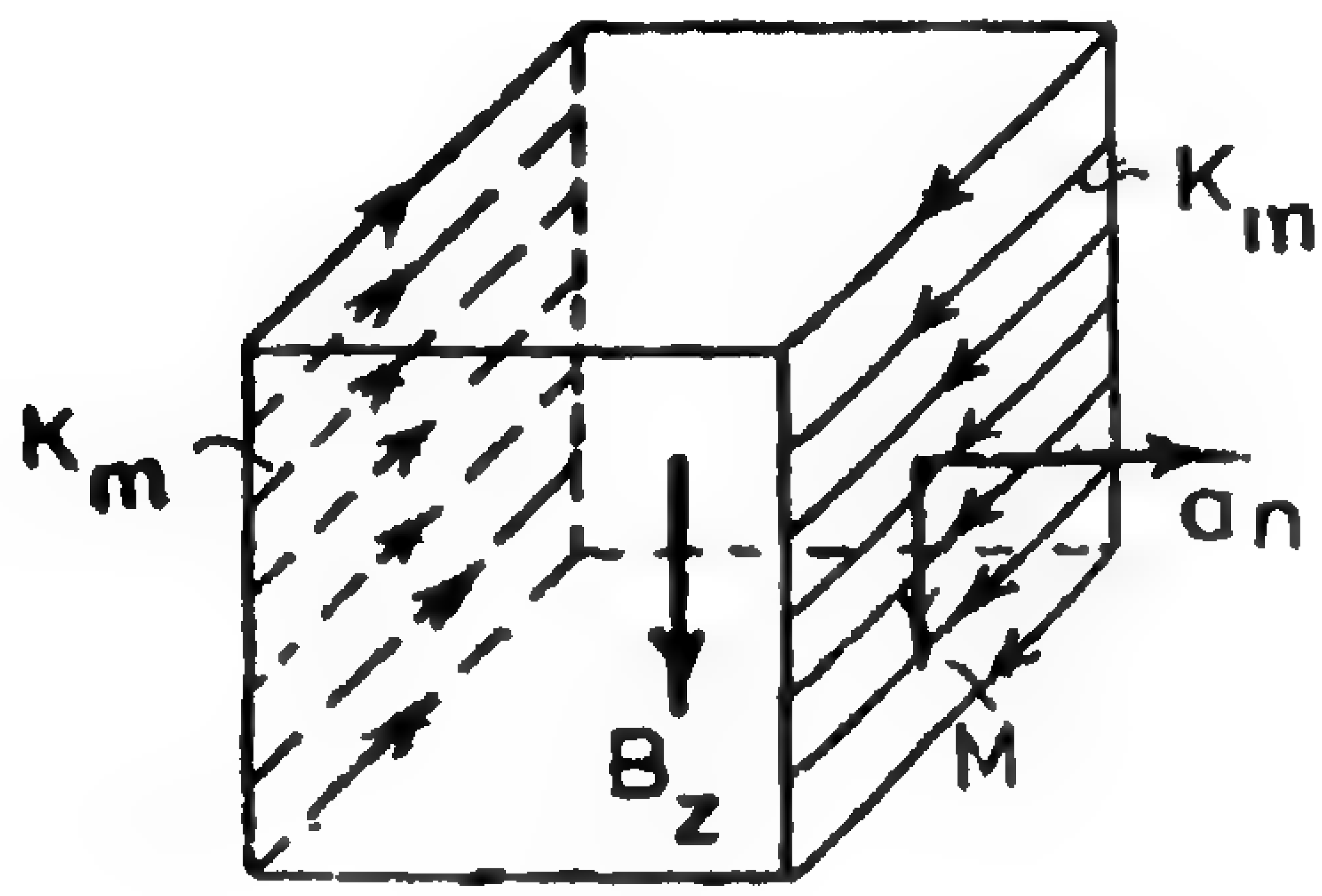


Fig. 8.4. b

The resultant magnetic force at the center is evidently zero.

The magnetized cube may also be represented by its equivalent amperian current. In this case the cube is replaced by a solenoid with a square cross section and carrying a current $K_m = M$ A/m (Fig. 8.4b) From problem 7·10 we know that the flux density at a point on the solenoid axis is

$$B_z = (\mu_0 M/4\pi)\,(\Omega_1 - \Omega_2)$$

In the present problem $\Omega_1 = 4\pi - 4\pi/6$, and $\Omega_2 = 4\pi/6$ so that

$$B_z = (\mu_0 M/4\pi)\;.\;8\pi/3 = {}^2/_3\,\mu_0 M$$

$$\mathbf{B} = -\,{}^2/_3\,\mu_0 M\,\mathbf{a}_z$$

and $\quad \mathbf{H} = \mathbf{B}/\mu_0 - \mathbf{M} = \tfrac{1}{3}\,M\,\mathbf{a}_z$

7. *A cast steel toroid has a uniform cross–sectional area of 10 cm^2 and a mean circumference of 0.7 m (Fig. 8.5). A coil of 30 turns is wound uniformly around the toroid; if the coil carries a direct current of 2 A, find the flux in the toroid. Find the current in the coil to produce a flux of 8×10^{-4} Wb in the toroid.*

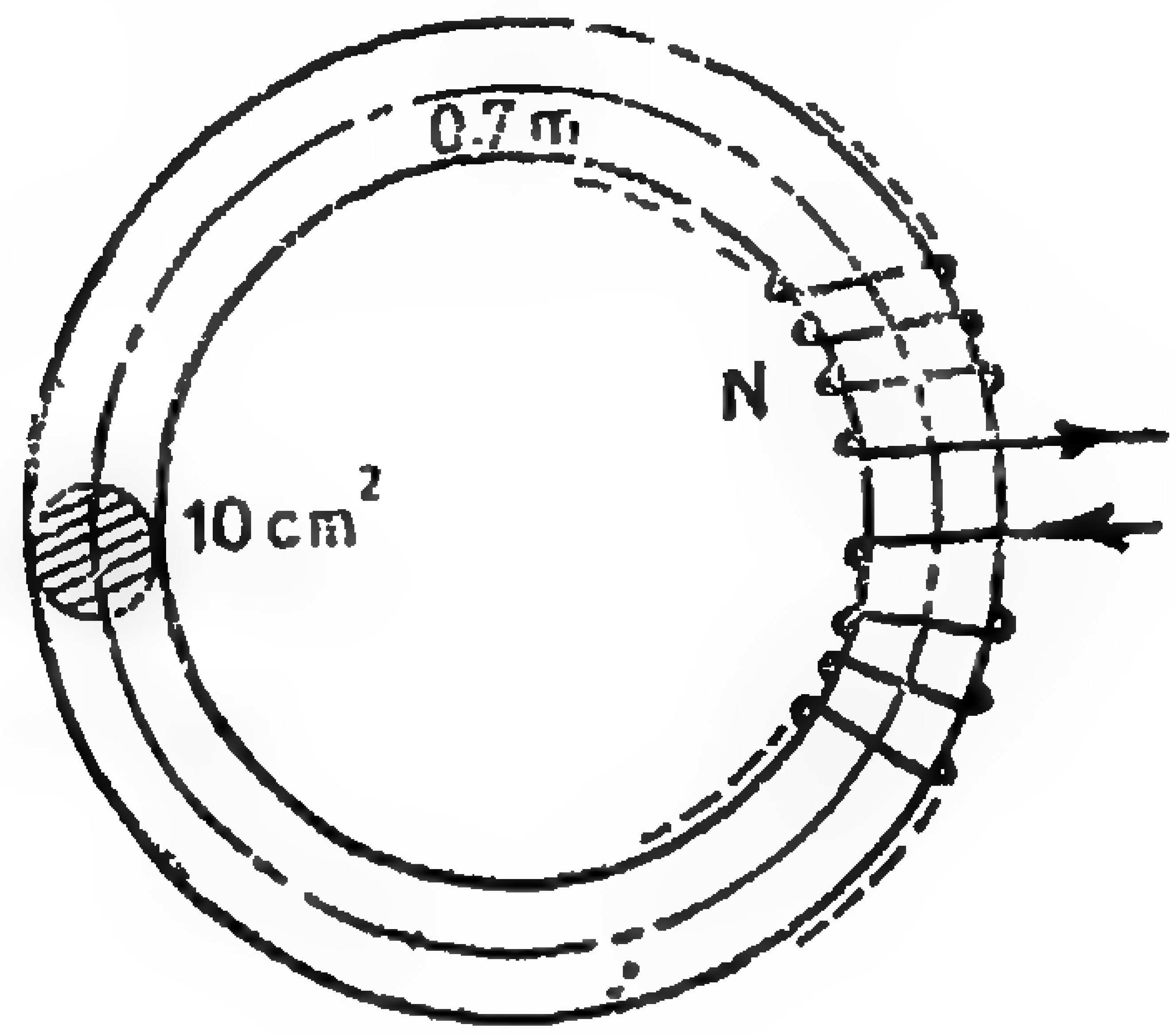


Fig. 8 5.

If l is the mean circumference then

$$H\,l = N\,I$$

so that

$$0.7H = 600$$

and $$H = 857 \quad AT/m$$

From the magnetization curve of the cast steel* the corresponding value for the flux density is $B = 1.15$ tesla; hence the flux is,

$$\phi = 1.15 \times 10/10^4 = 1.15 \times 10^{-3} \; Wb$$

If the flux is 8×10^{-4} Wb then,

$$B = 0.8 \quad \text{tesla}$$

---

* The magnetization curves of some commonly used magnetic materials are given in Fig. 8.18.

and $\quad H = 520 \quad AT/m$

so that

$$520 \times 0.7 = 300\ I$$

this gives

$$I \simeq 1.21 \quad A.$$

8. *The magnetic circuit shown in Fig. 8.6 consists of a cast iron portion of mean length 0.2 m and a cast steel portion of mean length 0.4 m. The cross sectional area is uniform throughout and equals to 15 sq cm. If the exciting winding has 100 turns find the current required to produce a flux of 1.5×10^{-4} Wb in the circuit. Neglect any leakage flux*

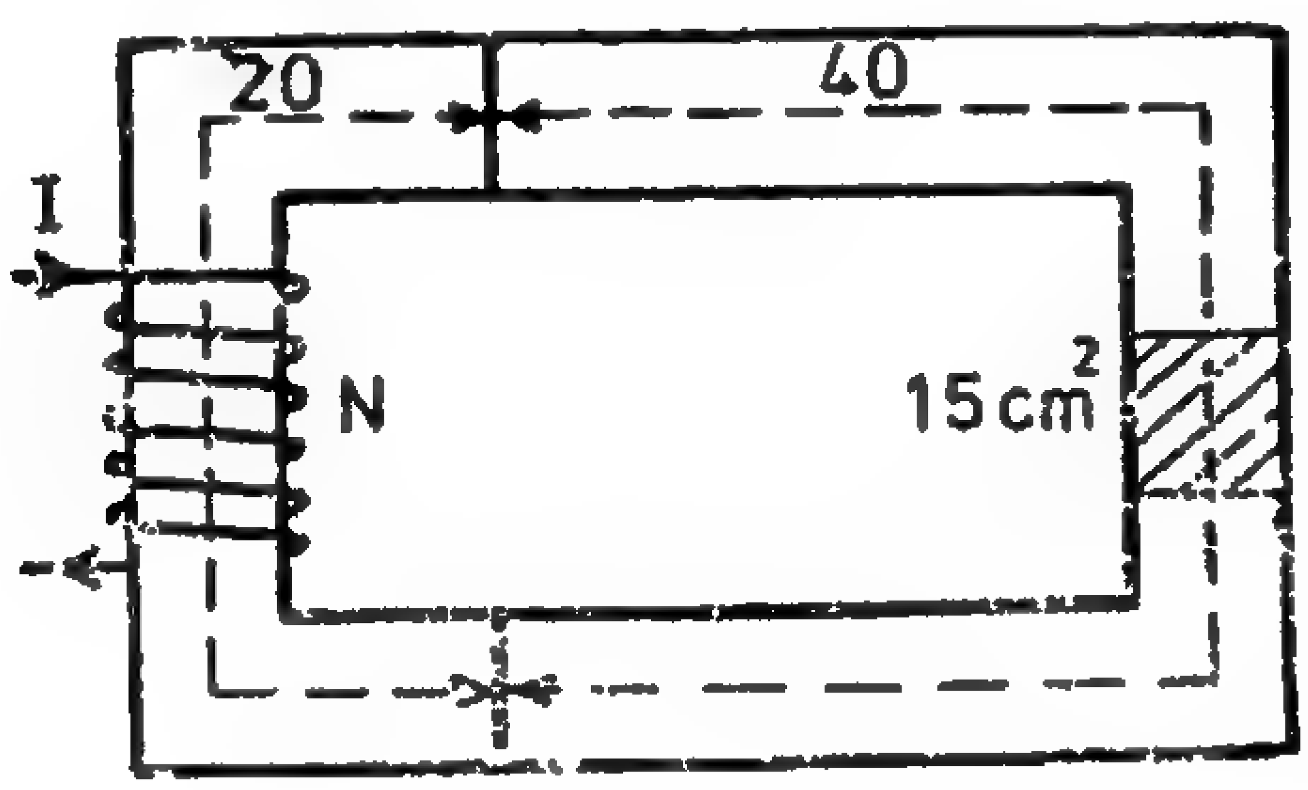


Fig. 8.6.

The circuit equation is

$$NI = H_1\, l_1 + H_2\, l_2$$

$$\phi_1 = \phi_2$$

we have that $B_1 = B_2 = 1.5/15 = 0.1$ tesla.

The corresponding values of H_1 and H_2 are obtained from the magnetization curves for cast iron and cast steel respectively,

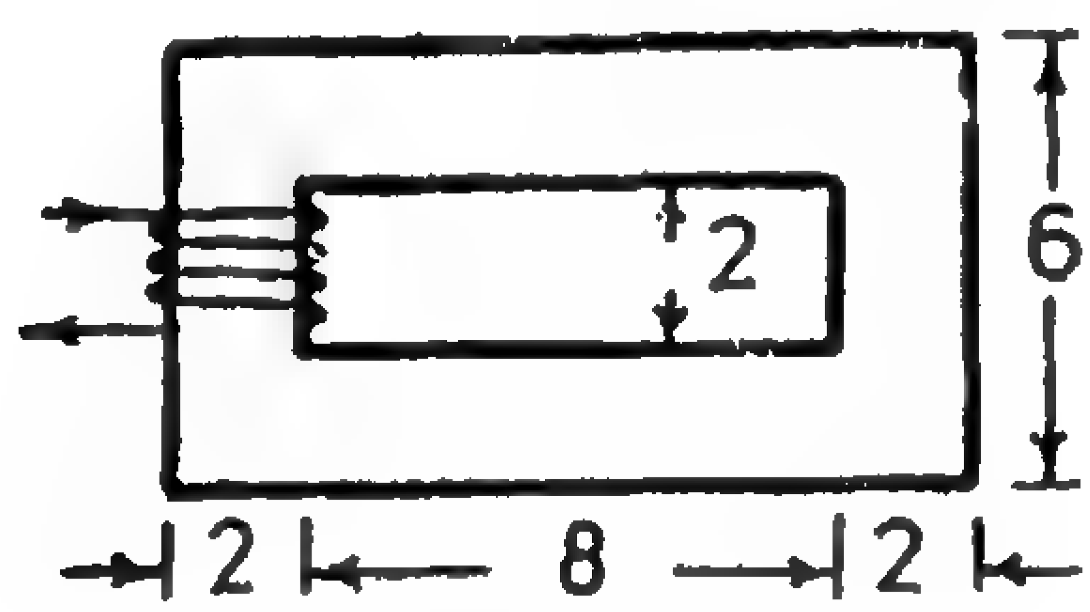
$$H_1 = 260 \ AT/m \ , \ \text{and} \ H_2 = 150 \ AT/m$$

Hence $100I = 260 \times 0.2 + 150 \times 0.4$

This gives

$$I = 1.12 \ A.$$

9. *The core shown in Fig 8.7. is made of USS transformer steel. Find the current in the 150-turn winding to produce a flux of 10^{-3} Wb in the core which has a uniform cross section of 2×5 cm^2. What would this current be if the core were made of cast steel. Neglect leakage flux. (Dimensions shown in Fig. 8.7 are in cm).*



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The average length of path $l = 28 \ cm = 0.28 \ m$. The flux through the core is,

$$\phi = BS = B \times 10 \times 10^{-4} = 10^{-3} \qquad Wb$$

and $B = 1.0$ *tesla*. The corresponding value of H, obtained from the magnetization curve is,

$$H = 210 \ AT/m$$

Since

$$NI = Hl$$

$$150\ I = 210 \times 0.28$$

and $$I = 0.394\ A.$$

If the core is of cast steel the value of H, corresponding to B 1, would be 670 and the required current would be $I = 1.25\ A$.

10. *The core shown in Fig. 8.8 is made of USS transformer steel. Find the current in the central 100 turn coil to produce a flux density of 1 tesla in each of the outer legs A and B. The mean length of the branches and their cross-sectional areas are shown in the figure (dimensions in cm). Neglect leakage flux.*

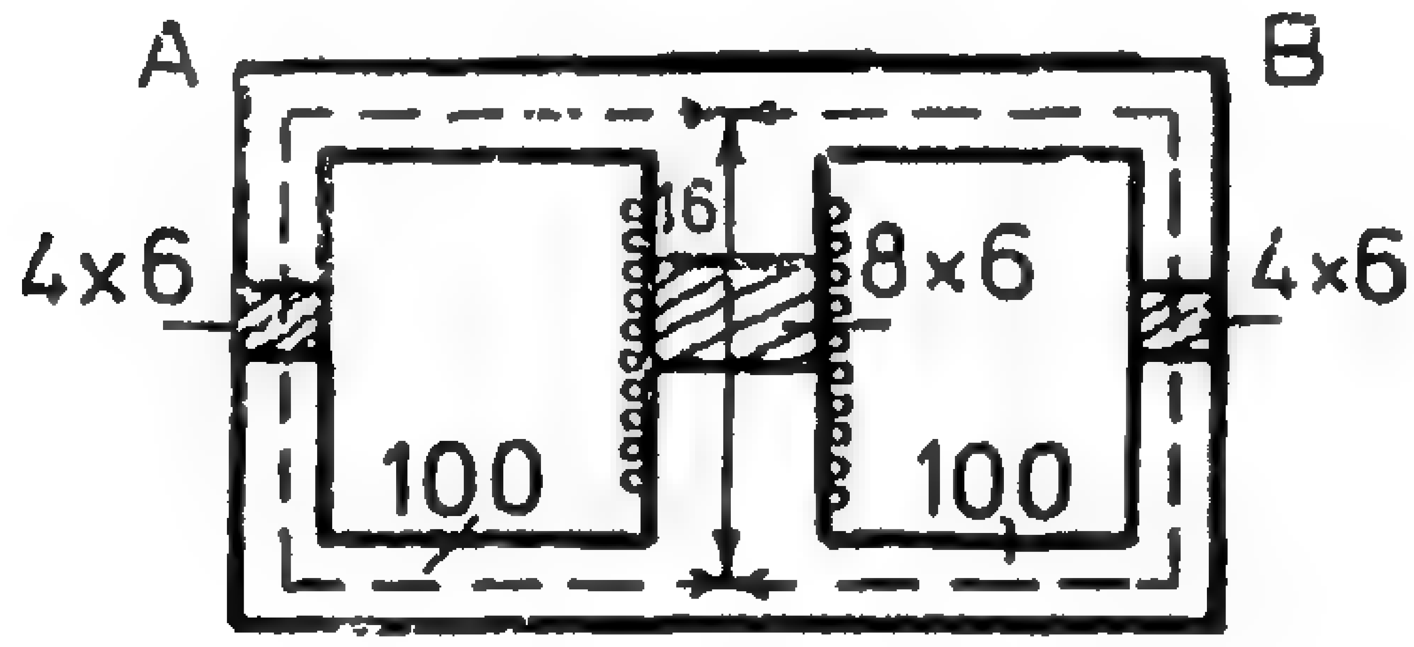


Fig. 8.8.

Branches A and B of the core are in parallel with the central leg C. Hence we have that

$$NI = H_c L_c + H_B L_B \qquad (1)$$

$$H_B L_B = H_A L_A$$

The flux in the branches A and B is

$$\phi_A = \phi_B = 1 \times 24 \times 10^{-4}$$

so that

$$\phi_c = \phi_A + \phi_B = 48\times10^{-4} \quad Wb$$

and

$$B_c = \phi_c / S_c = 1 \ tesla$$

From the magnetization curve we find that,

$$H_A = H_B = H_c = 210 \qquad AT/m$$

and from equation (1)

$$100I = 210\times1.16$$

This gives

$$I = 2.436 \ A.$$

11. *To what value must the current in the central coil of the core shown in Fig. 8.8 be increased if an air gap of 1 mm is introduced in one of the outer legs. Allow for fringing at the air gap but neglect leakage flux and assume $B_g = 1.0$ tesla.*

The effect of fringing is to make the flux density in the air gap less than that in the ferromagnetic core, that is, the effective cross-section of the gap is actually greater than that of the core at the air gap. If a and b are the cross-sectional dimensions of the parallel faces of an air gap the effective area is given by the empirical formula,

$$S_g = (a+g).(b+g)$$

In the present problem $a = 4$, $b = 6$, and $g = 0.1$ so that $S_g = 25$ *sq cm*. Thus with the air gap in branch A,

$$\phi_A = 25\times10^{-4}\times1 = 25\times10^{-4} \ Wb$$

Thus $\quad B_A = \phi_A / S_A = 1.04$

and the magnetizing force is $H_A = 240$. In the air gap we have that,

$$H_g = B_g / \mu_o = 8\times10^5$$

Also we have

$$H_B L_B = H_A L_A + H_g L_g$$

$$= 240 \times 1 + 8 \times 10^{5} \times 10^{-3} = 1040$$

and $\quad H_B = 1040 \ , \ B_B = 1.35 \ tesla \ ,$ and $\phi_B = 32.4 \times 10^{-4}$

Hence

$$\phi_c = \phi_A + \phi_B = 57.4 \times 10^{-4}$$

and $\quad B_c = \phi_c \ / \ S_c = 1.2$

The corresponding value of H_c is 400 AT/m. We also have that,

$$VI = H_B L_B + H_c L_c$$

$$= 1040 + 400 \times 16 \times 10^{-2}$$

$$= 1104$$

and $\quad I = 11.04 \ A.$

12. *A long solenoid of N closely wound turns and length l has a closely fitting core of permeability μ; the core is made up of two halves separated by an air gap of length d at the center of the coil (Fig. 8.9). If the coil carries a current I and all end effects are neglected, show that the magnetic flux density in the air gap is $\mu NI \ / \ [l + d(\mu_r - 1)]$.*

Since end effects are negligible we have that,

$$NI = H\,(l - d) + H_g \, l_g$$

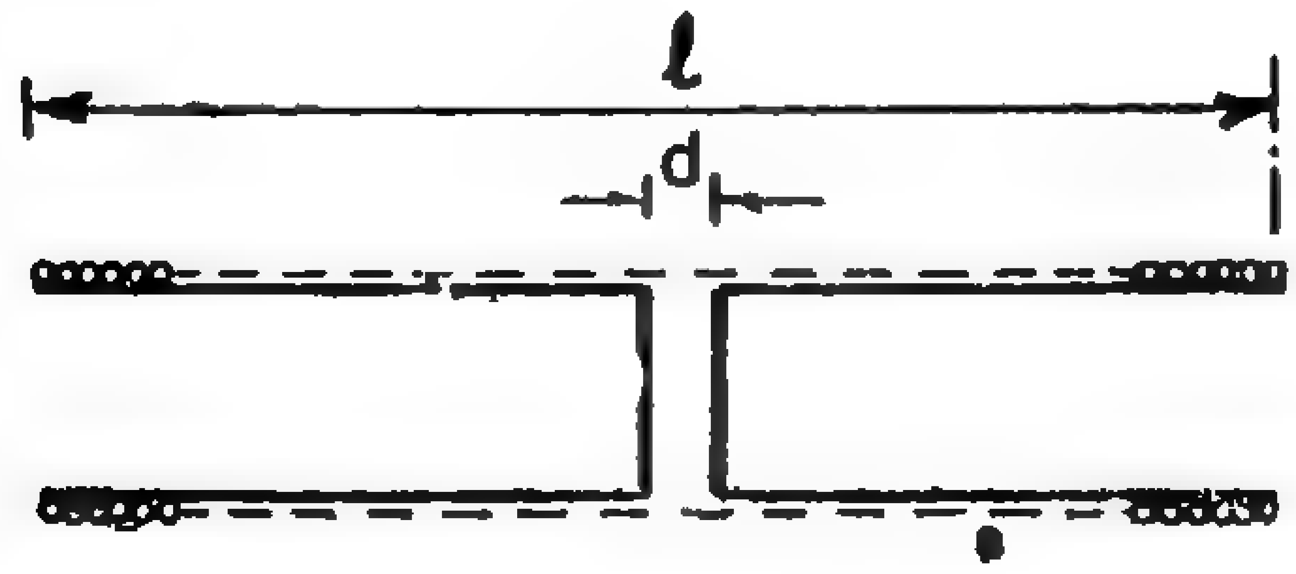


Fig. 8.9.

and since $B_g = B$, then $\mu_0 H_g = \mu H = \mu_0 \mu_r H$, and

$$NI = H_g \left[l/\mu_r - d\,(1/\mu_r - 1) \right]$$

so that

$$B_g = \mu_0 H_g = \mu NI \,/\, [l + d\,(\mu_r - 1)]$$

13. *The magnetic circuit shown in Fig. 8.10 is made of USS steel and has a uniform cross-section throughout. Find the current in the 400-turn winding to produce a flux of 9×10^{-4} Wb in the right-hand leg. Assume a stacking factor of 0.9.*

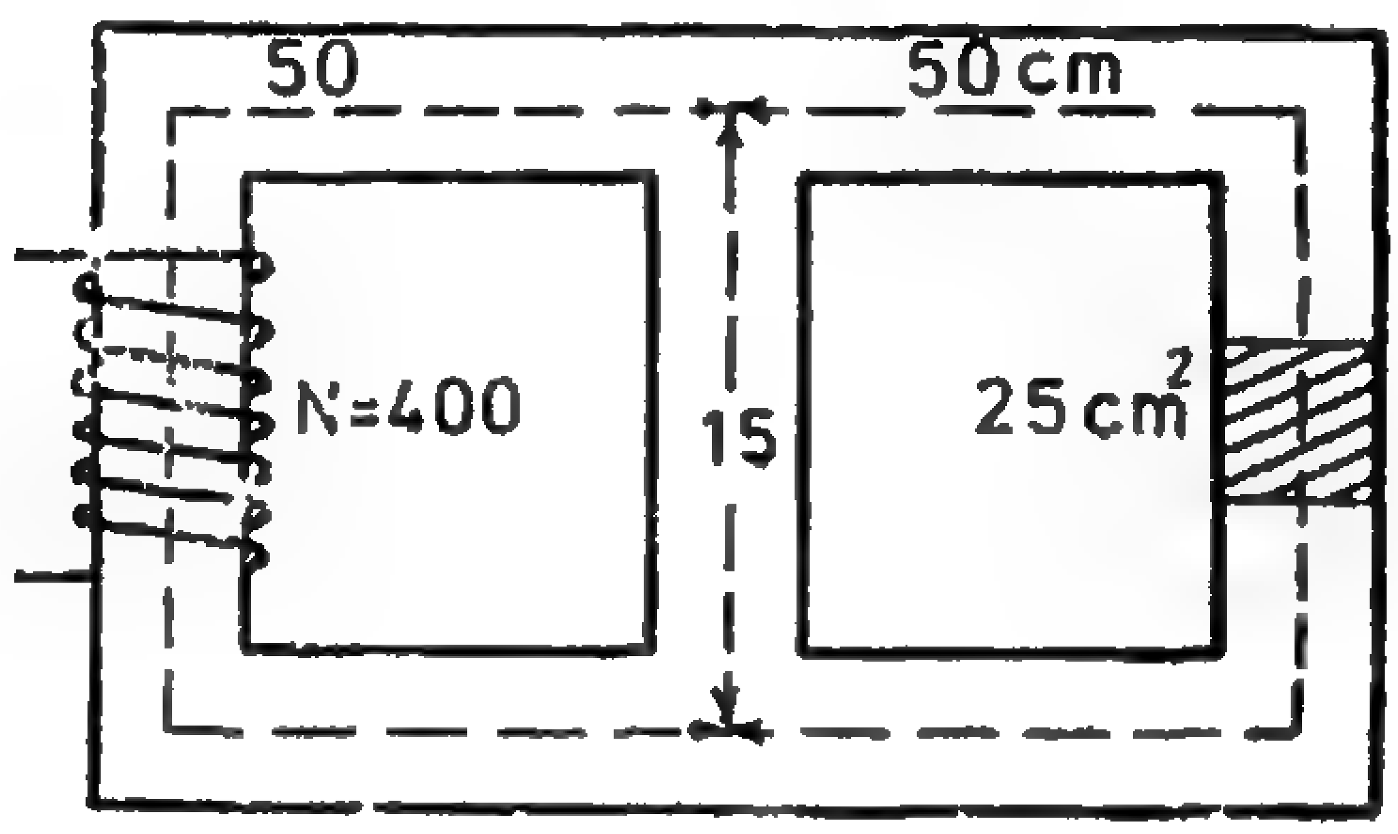


Fig. 8.10.

In order to minimize the eddy current losses, the core is made up of thin laminations with each lamination insulated from the others by a thin layer of insulating varnish applied over its surfaces. This makes the effective cross sectional area of the laminated core less than the area obtained from its dimensions. The ratio of this effective area to the actual area is called the stacking factor and varies between 0.75 and 0.95.

In the present case the effective cross-sectional area is $0.9 \times 25 = 22.5$ *sq cm*. The equations for the magnetic circuit are

$$NI = H_1 l_1 + H_2 l_2$$

$$H_2 l_2 = H_3 l_3$$

$$\phi_1 = \phi_2 + \phi_3$$

Now $B_3 = 9 \times 10^{-4} / 22.5 \times 10^{-4} = 0.4$ *tesla.*

The corresponding value of H_3 from the magnetization curve is 59 AT/m, so that $H_3 l_3 = 29.5$. Hence

$$H_2 = 29.5 / 0.15 = 197 \quad AT/m$$

The corresponding value of B_2 is 0.98 *tesla*. Since the cross-sectional area is uniform,

$$B_1 = B_2 + B_3 = 1.38$$

and $H_1 = 1300$

Thus $NI = 1300 \times 0.5 + 29.5 = 679.5$

Hence $I = 1.70$

14. *The magnetic circuit shown in Fig. 8.11a is made of USS steel laminations with a stacking factor 0.9. Find the flux in the air gap, if the mmf of the exciting winding is 2000 AT. Neglect leakage flux but allow for fringing at the air gap.*

The circuit equations are

$$NI = Hl + H_g l_g$$

$$\phi_g = \phi \text{ or } B_g S_g = BS$$

It is required to find the value of H and H_g which satisfy the mmf equation. This equation may be written as

$$B = (\mu_0 S_g / S l_g)(NI - Hl) \qquad (1)$$

But the relationship between B and H must also satisfy the magnetization curve for sheet steel,

$$B = f(H) \qquad (2)$$

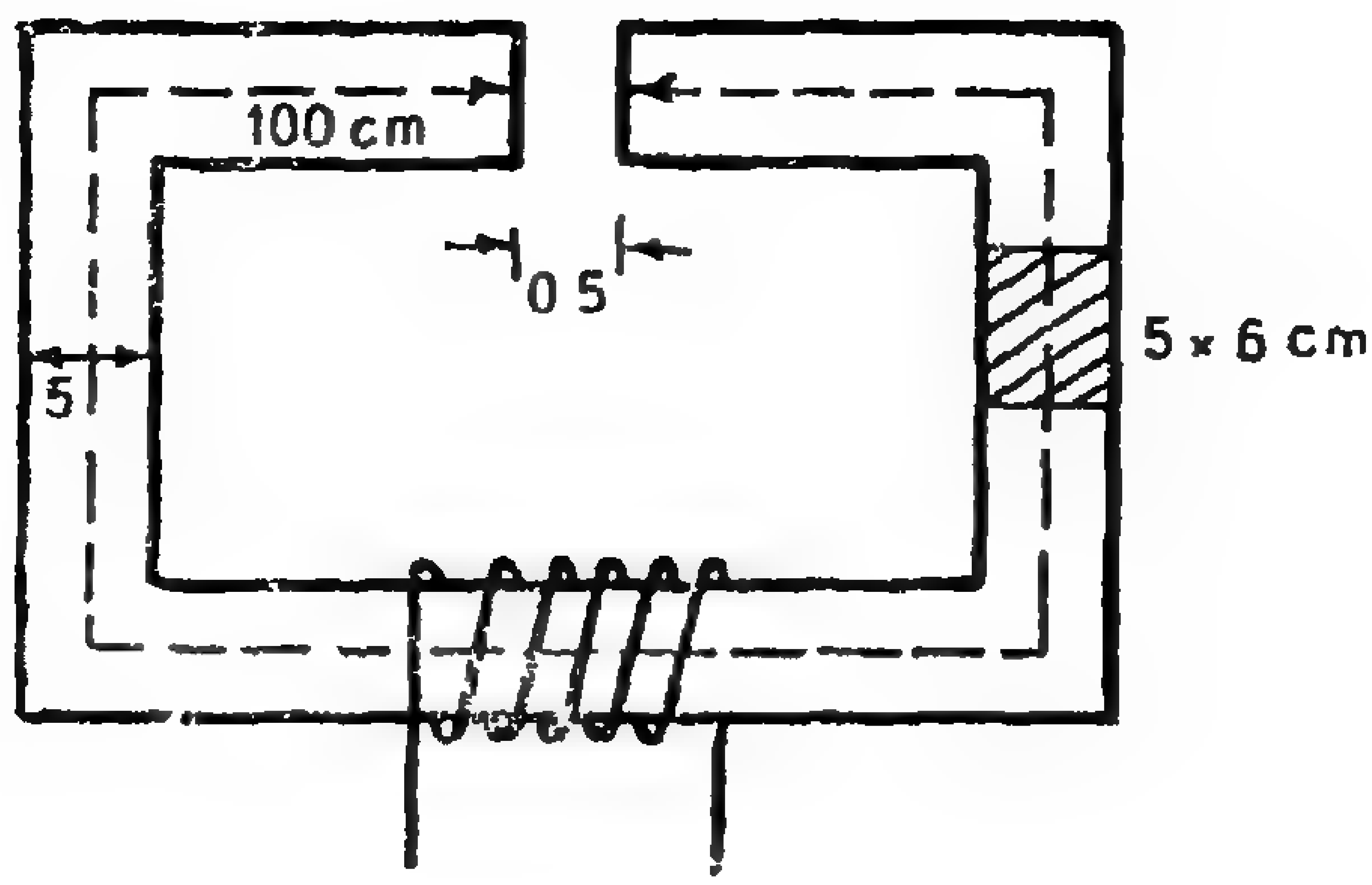


Fig. 8.11 a

The value of B which satisfies equations (1) and (2) is given by the point of intersection of these two curves. Equation (1) represents a straight line which intersects the B and H axes at the points $\mu_0 NI S_g / S l_g$ and NI / l respectively (Fig. 8.11.b). Allowing for fringing the value of S_g is,

$$S_g = 5.5 \times 6.5 = 35.75 \ sq. \ cm.$$

From the numerical data given in the problem we find that

$$OA = 2000 \ AT/m$$

$$OD = 0.60 \ Wb/m^2$$

At the point of inersection $B = 0.58$ and hence $\phi = 0.00173 \ Wb$

15. *A long solenoid has a cross-sectional area S. If a cylindrical magnetic core of permeability* μ *and cross-section* $S'(S' < S)$ *is introduced coaxially into the solenoid, show that the inductance of the solenoid is increased by* $1 + (\mu_r - 1)(S'/S)$.

Without the core we have that,

$$L = N\phi/I \ , \qquad \phi = \mu_0 NIS/l$$

With the core the total flux inside the solenoid is

$$\phi' = \mu NIS'/l + \mu_0 NI(S-S')/l$$

$$L' = N\phi'/I$$

Thus $\quad L'/L = \phi'/\phi = 1 + (\mu_r - 1)(S'/S)$

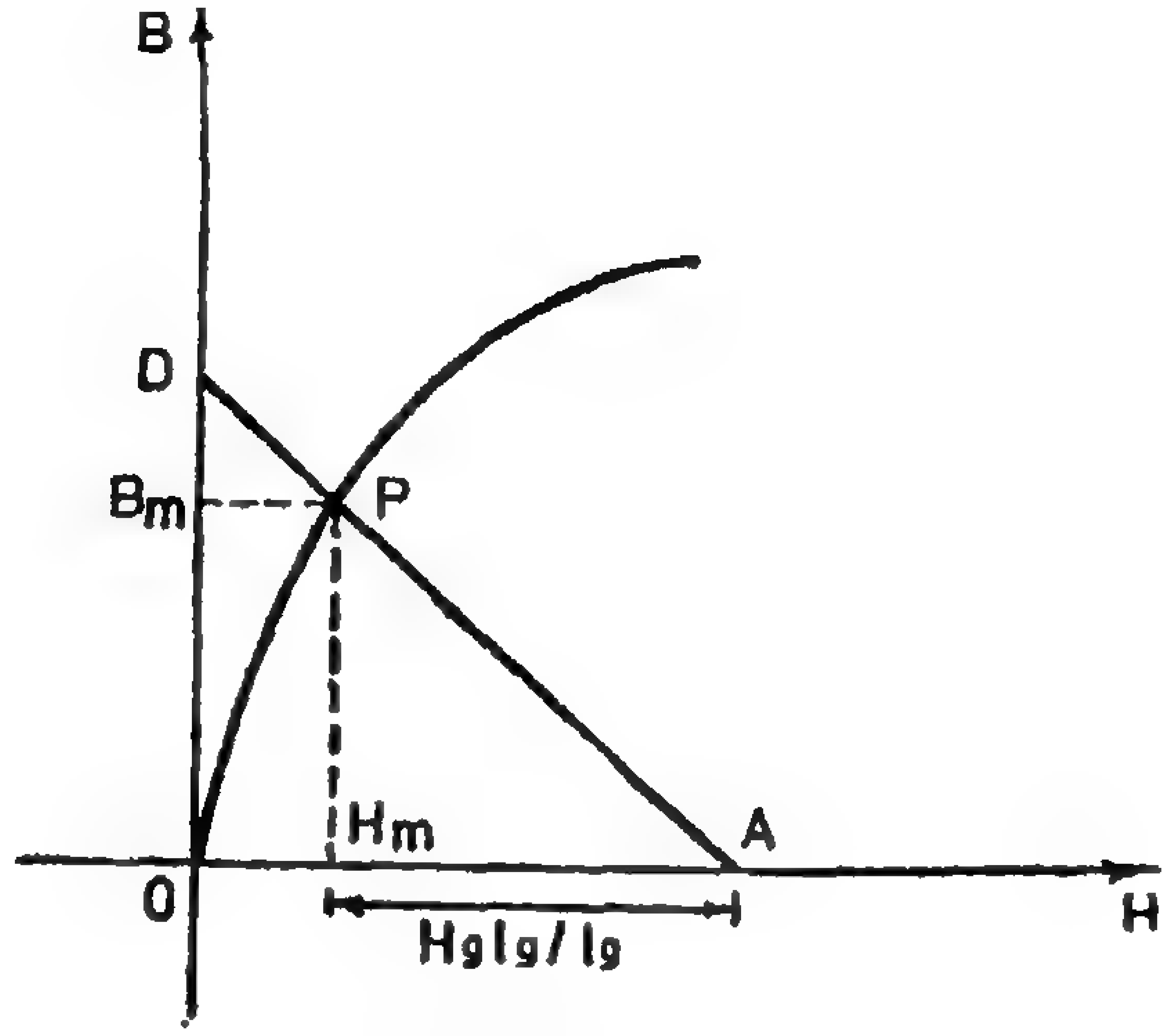


Fig. 8.11. b

16. *An iron ring with a rectangular cross-section of width d, internal and external diameters a and b respectively. A radial air gap which subtends an angle θ radians at the center has been cut out of the ring (Fig. 8.12). If the coil of N turns is uniformly wound on the ring show that, neglecting end effects, the inductance of the system is given by*

$$L = \frac{\mu N^2 d}{2\pi + \theta(\mu_r - 1)} \log(b/a)$$

Since the gap and the magnetic core are in series,

$$NI = H_m L_m + H_g L_g$$

where $H_g = \mu_r H_m$. For a path of radius r inside the core we have that

$$NI = H_m [2\pi r - r\theta + r\theta \mu_r]$$

Also $B_m = \mu H_m = \mu NI / r [2\pi - \theta + \theta \mu_r]$

The flux through an element of area $d\,dr$ is

$$d\phi = B_m \, d \, dr$$

and

$$\phi = \{ \mu NId / [2\pi + (\mu_r - 1)\theta] \} \int_a^b dr/r$$

$$= \frac{\mu N I d}{2\pi + \theta(\mu_r - 1)} \log (b/a)$$

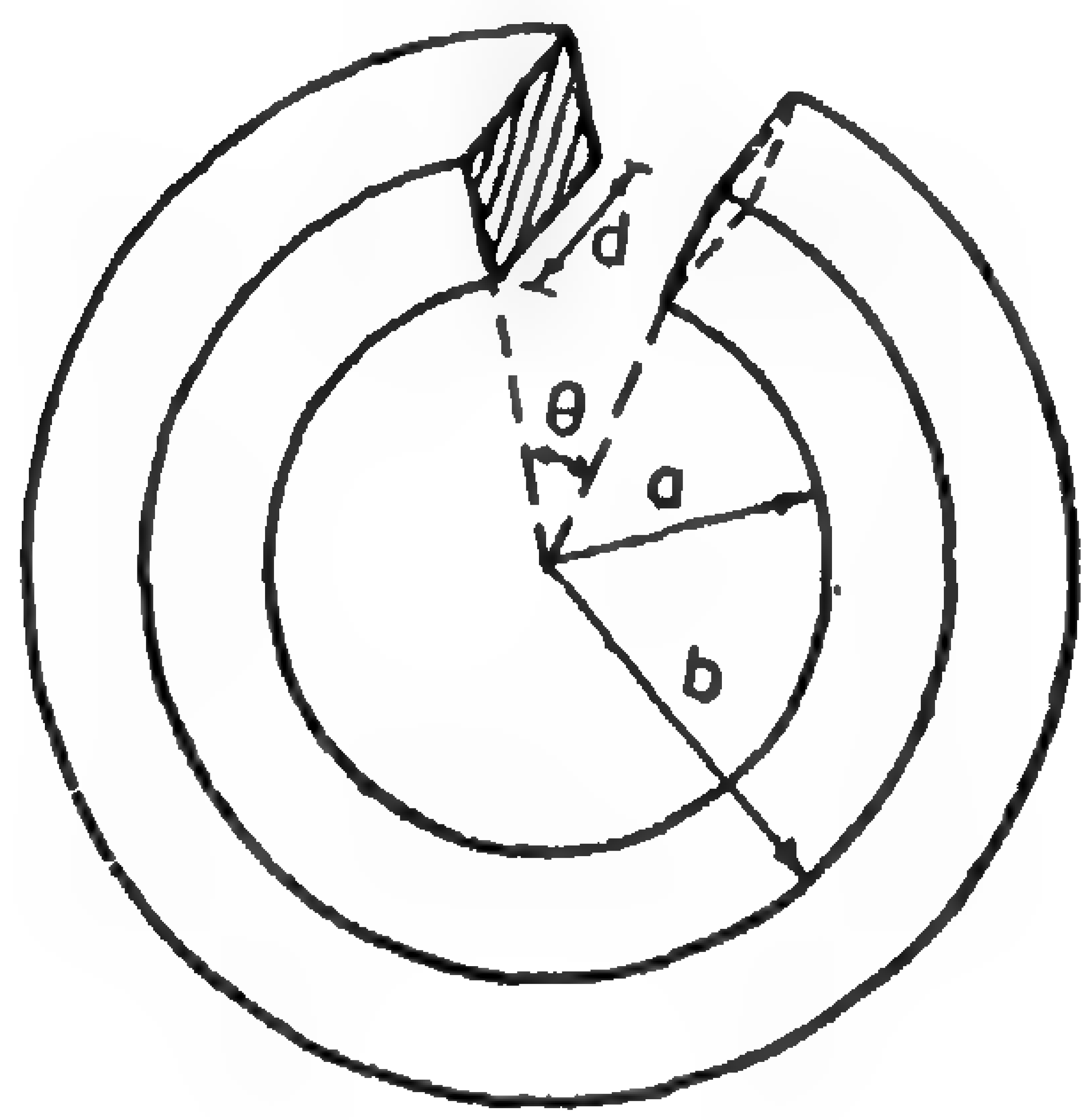


Fig. 8.12.

The expression for the inductance follows immediately from $L = N\phi/I$.

17. *The demagnetization curve of a permanent magnet material is given by $B = (B_o/H_o) H + B_o$. A permanent magnet made from this material is to have a fixed volume; it is to be fitted with pole shoes of*

negligible reluctance to produce flux in an air gap of length l_g and area S_g. Show that the flux density in the gap will be a maximum if the ratio of the length l of the magnet to its area S is $B_o l_g / \mu_o S_g H_o$. Neglect leakage and fringing.

Neglecting any leakage flux we may write that

$$\phi = \phi_g$$

$$BS = B_g S_g \tag{1}$$

and since the flux does not link any current,

$$Hl + H_g l_g = 0$$

or $$Hl = -H_g l_g \tag{}$$

The expression for H in terms of B is obtained from the given equation of the demagnetization curve

$$H = [(B / B_o) - 1] H_o$$

Hence

$$[(B/B_o) - 1] H_o l = -B_g l_g / \mu_o$$

Substitiuting for B from equation (2) gives,

$$B_g [(S_g H_o/B_o) l/S + l_g / \mu_o] = H_o l$$

Let $l / S = x$, and since the magnet has a fixed volume $l S = K$. Setting $a = S_g H_o/B_o$ and $b = l_g/\mu_o$ we may write,

$$B_g = H_o K / (al + b S)$$

Now we have that

$$dB_g / dx = (\partial B_g / \partial l) (dl / dx) + (\partial B_g / \partial S) (dS / dx)$$

$dl / dx = S$, $dS/dx = -S^2/l$, and for maximum B_g, $dB_g/dx = 0$. Thus

$$H_o K a S / (a l + b S)^2 - H_o Kb S^2/l (al+bS)^2 = 0$$

$$a - Sb/l = 0$$

Hence $l/S = b/a = B_o l_g / \mu_o S_g H_o$

It will now be shown that for a given air gap volume and flux density, the above ratio of l/S is that which corresponds to the minimum volume of the magnetic material. From equations (1) and (2) we have that

$$V / V_g = - H_g B_g / HB \tag{3}$$

where $V = S l$ and $V_g = S_g l_g$ are the volumes of the magnetic material and the air gap respectively. Equation (3) shows that the volume V will be minimum when the product (BH) is a maximum. From the given equation of the demagnetization curve we have that,

$$(BH) = (B_o/H_o) H^2 + B_o H$$

$$d(BH)/dH = 2 (B_o/H_o) H + B_o$$

and for maximum (BH),

$$H = - \tfrac{1}{2} H_o \text{ and } B = \tfrac{1}{2} B_o$$

Equations (1) and (2) may be written as

$$\tfrac{1}{2} B_o S = B_g S_g$$

$$\tfrac{1}{2} H_o l = H_g l_g$$

and it follows that

$$l / S = B_o l_g / \mu_o S_g H_o$$

18. *It is required to produce a flux density of 0.5 tesla in an air gap 1 mm long and 1 cm² in area. For this purpose a permanent magnet is fitted with soft iron pole shoes of negligible reluctance. Find the optimum dimensions of the magnet if its demagnetization curve is given by*

$$B = 0.9 \, [H/19900 + 1]$$

From the demagnetization curve we have that

$$B_o = 0.9 \quad , \text{ and } H_o = 19900$$

Mak'ng use of the results of the previous problem we have that,

$$B = \tfrac{1}{2} B_o = 0.45 \text{ , anl } H = -\tfrac{1}{2} H_o = -9950$$

From equation (') of problem 8.17

$$S = B_g S_g / B$$

$$= 0.5 / 0.45 = 1.11 \quad sq.\ cm.$$

and from equation (2),

$$l = H_g l_g / H$$

$$= 0.5 / 4\pi \times 10^{-7} \times 9950 = 40 \quad mm.$$

8 - 9 مسائل إضافية

SUPPLEMENTARY PROBLEMS

1. A magnetic dipole at the origin of a coordinate system produces at the point (0, 0, 5) a magnetic flux density $B_x = B_y = 0, B_z = 0.1$ μ tesla. Find the magnitude and direction of **B** at the point (4, 0, 0). Determine the equation of that magnetic line of force which passes through the point (4, 0, 0) and plot it.

2. Show that the energy of a small magnet of moment **m** in the presence of a long straight wire carrying current **I** is $\mathbf{m} . \mathbf{I} \times \mathbf{r}/2\pi r^2$, where **r** is a radial vector from the wire to **m**. Find the force and torque on the magnet.

3. Two small magnets of moment m and mass M are suspended at their centers by light strings. If they are a great distance d apart, show that the magnets will attract one another and that in equilibrium the strings supporting them will be displaced from the vertical by a small angle

$$\theta = 3\ m^2/2\pi\ \mu_o\ d^4\ Mg$$

4. Show that the magnetic scalar potential due to a long thin bar uniformly magnetized in the direction of its length reduces to $V_m = -(m/4\pi\ \mu_0)\ (1/r_1 - 1/r_2)$ at distances great compared with its cross sectional dimensions, where m is the moment per unit length, and r_1, and r_2 are the distance from the ends of the bar to the point at which the potential is V_m.

5. Show that to terms of the order of l^2/r^2 the potential due to the magnet of the previous problem near its axis is $V_m = (-m/4\pi\ \mu_o\ r^2)\ [1+(l/2r)^2]$, where l is the length of the magnet. Calculate the force between two collinear magnets to this approximation.

6. A cast iron ring with a mean radius of 15 cm and cross section of 1.2 sq cm has been diametrically bridged by a rod of the same material and 0.8 sq cm in cross section. On one of the halves of the ring there is a 160 turns wire. If the flux in the rod is 4×10^{-5} weber find the current in the wire.

7. An electromagnet consists of a spherical iron shell of inner radius 10 cm and outer radius 10.5 cm. Two solid cylinders are placed along a diameter with a 5 mm. gap between them at the center. The diameters of the cylinders are 7 cm and the permeability of the iron is assumed constant and equal to 500. A 200 hundred turns coil is wound around each cylinder. Find the current in the coil to produce a flux of 0.4 μWb in the air gap. Neglect leakage.

8. A ring of transformer steel has an external diameter of 15 cm and a square cross section 4×4 cm. The ring carries a 500 turns coil which carries a current of 0.5 A. How large is the error made in the calculation of the magnetic flux, if use is made of the mean magnetic path corresponding to the mean diameter of the ring.

9. A number of annular discs of transformer steel, the inner diameter of which is 38 cm and the external 42 cm, are stacked on top of each other to form a hollow cylinder 4 cm high. The stack is kept in place by two rings of cast iron one on each side of it. They have the same annular form as the discs, but are each 8 mm thick. Around this core 2000 turns of wire are wound. Find the current for which the flux in the core is 1790 μWb.

10. An iron toroid of mean diameter 30 cm has been welded together from a tube of internal diameter 2 cm and external diameter 2.5 cm. On the tube a probe winding of 20 turns is wound, and directly on top of this a magnetizing winding of 2000 turns is wound. When

the direction of a current in the magnetizing winding is reversed, a fluxmeter indicates a change in the flux of 0.016 Wb in the probe winding. Determine the current in the winding by making use of the magnetization curve for transformer steel. (The flux in the air inside the tube should not be neglected. Solve graphically).

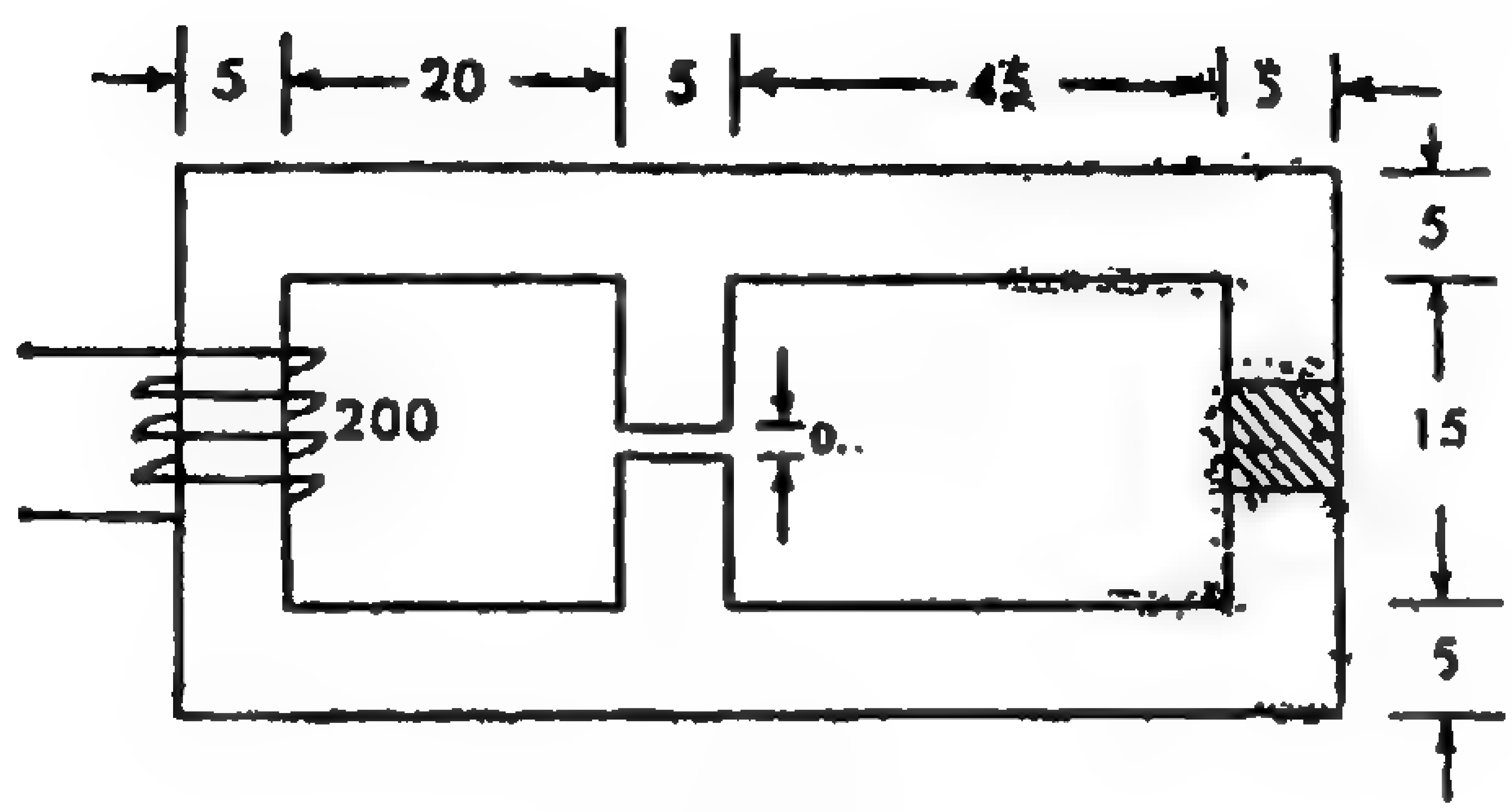


Fig. 8.13.

11. The magnetic core shown in Fig. 8.13 is made of transformer steel laminations. The stacking factor is 0.85. The flux in the air gap is 0.6 μ Wb. Calculate the mmf and current in the exciting winding neglect leakage and fringing effects.

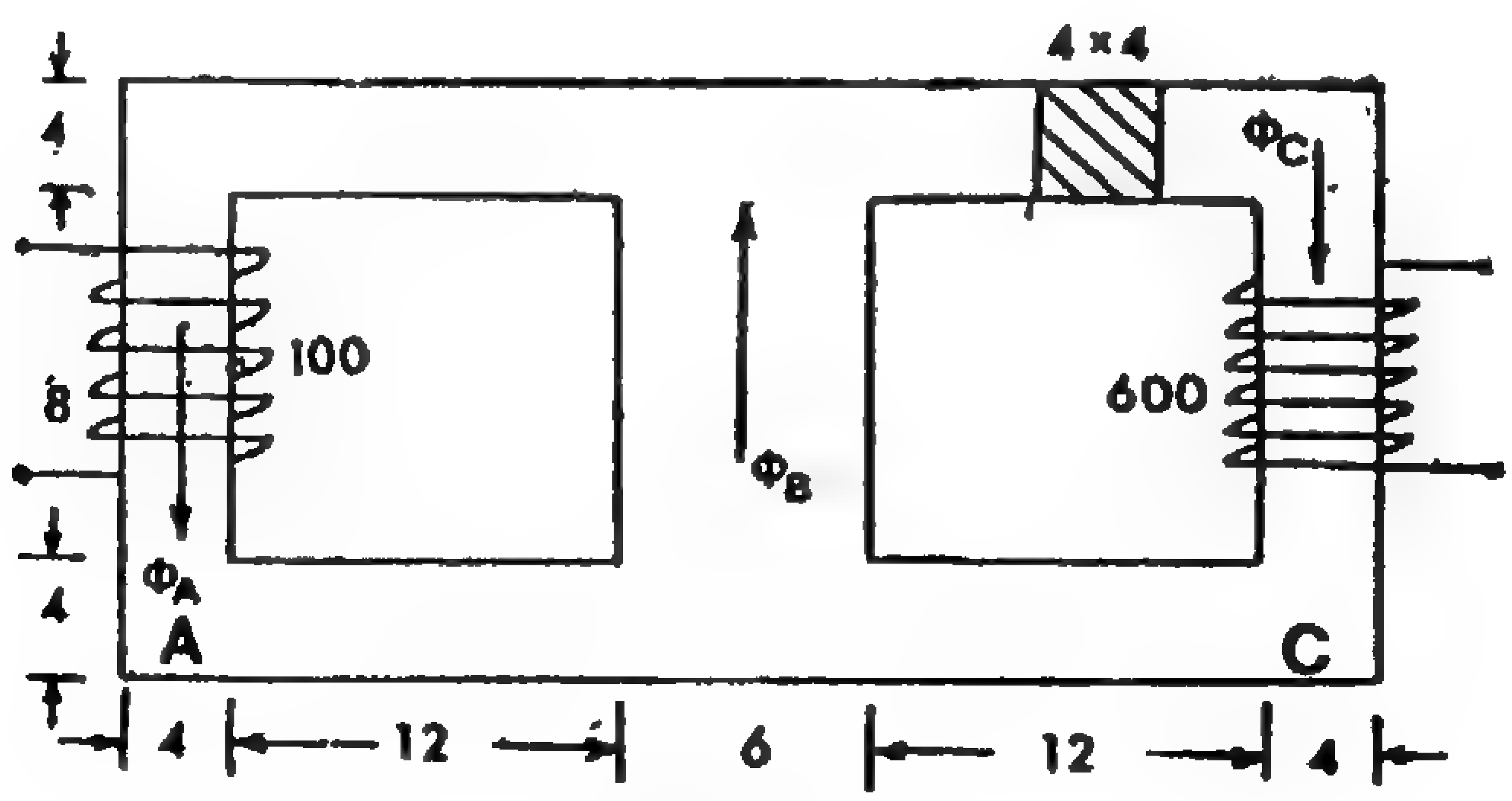


Fig. 8.14.

12. The magnetic core shown in Fig. 8.14. is made of transformer steel laminations. The stacking factor is 0.9. The fluxes in the three legs are $\phi_A = 0.4\ \mu$ Wb $\phi_B = 0.6\ \mu$ Wb, and $\phi_c = 0.2\ \mu$ Wb in the directions shown. Find the magnititude and direction of the current in each coil.

13. In the previous problem, if the flux in legs A and B is $0.4\ \mu$ Wb in the counter clockwise direction and the flux in leg C is zero, find the magnitude and direction of the currents in the two windings.

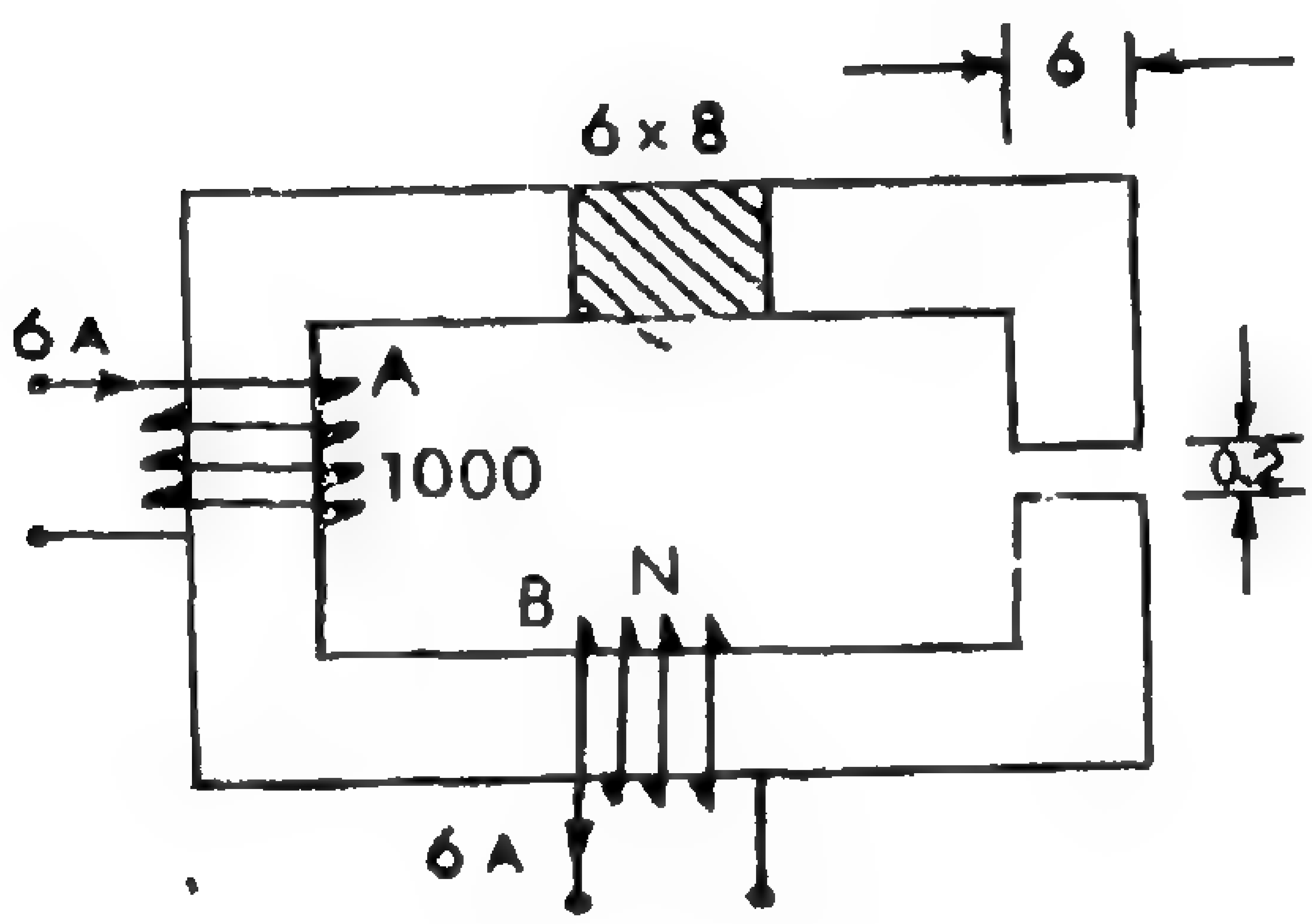


Fig. 8.15.

14. The magnetic core shown in Fig. 8.15 is made of transformer steel laminations with a stacking factor of 0.9. The mean length of magnetic path is 0.75 meter in the steel. The flux in the air gap is 4 mWb. Find the number of turns of coil B. All dimensions shown are in cm. Neglect leakage flux but allow for fringing.

15. In the cast steel core shown in Fig. 8.16 all the dimensions are in cm. The coil in leg A carries a current of 0.5 A. Find the current in coil B in the direction shown to make the flux in the centeral leg zero.

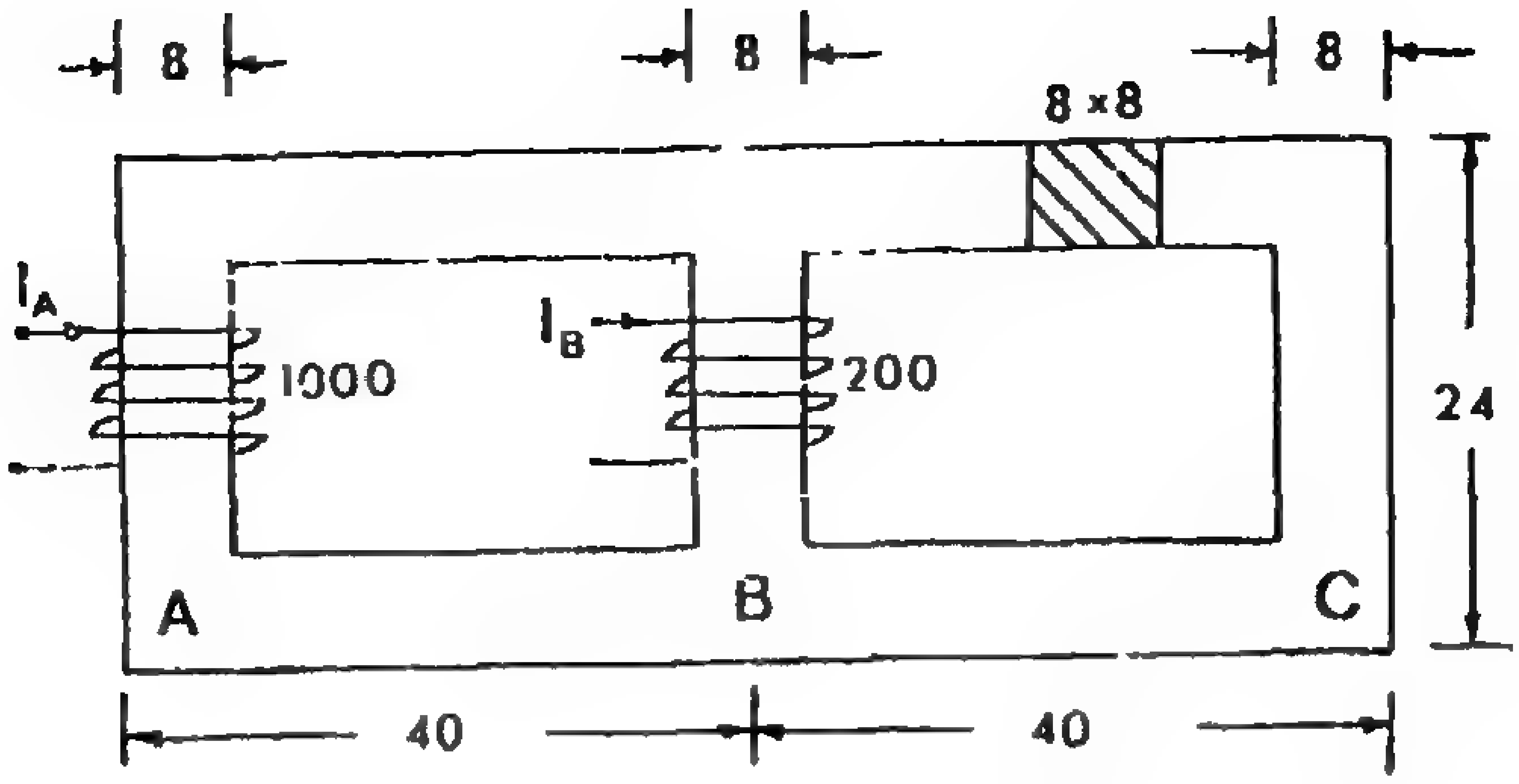


Fig. 8.16.

16. In the transformer steel core shown in Fig. 8.17 the flux in branch A is 100 μWb. Find the fluxes in branches B and C and the mmf required in the exciting coil.

17. For the magnetic circuit of Fig. 8.17 the flux in leg B is 1600 μWb. Find the fluxes in legs A and C and the mmf of the winding.

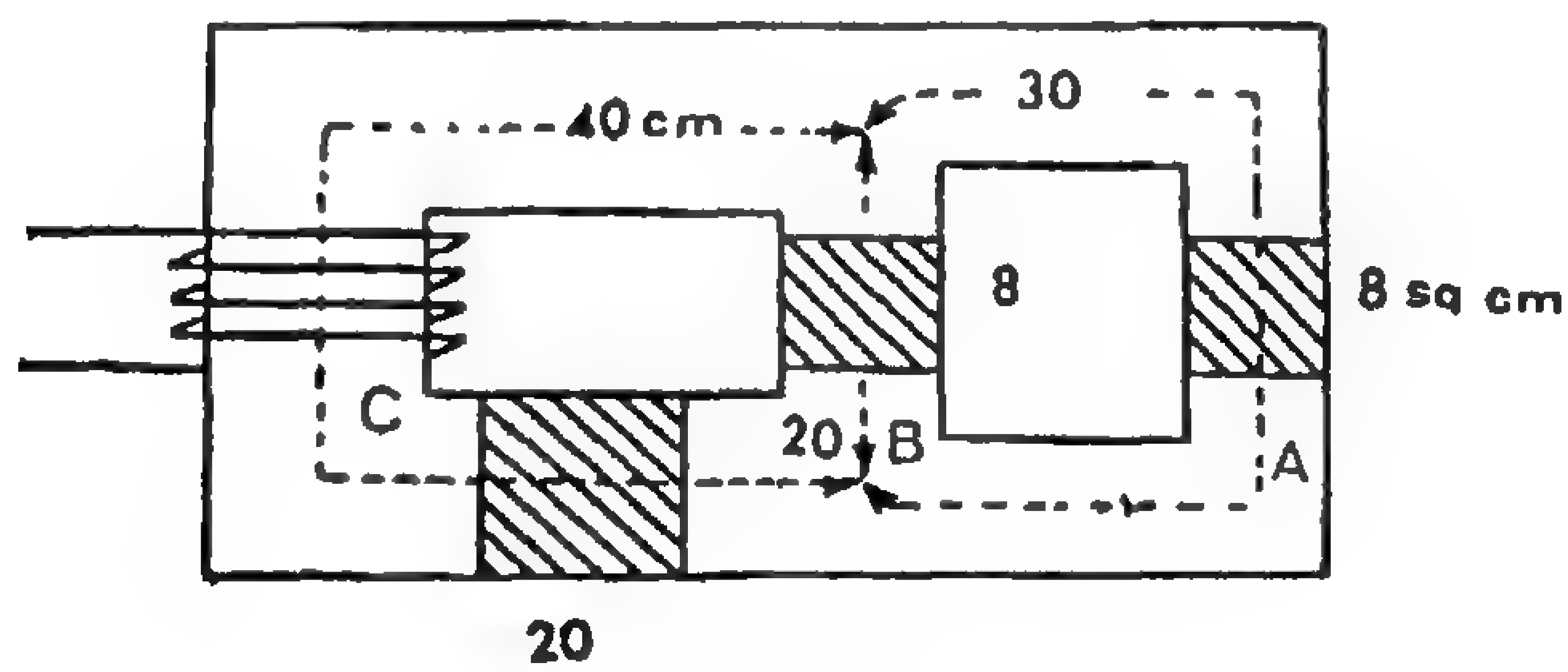


Fig. 8.17.

18. For the magnetic circuit of Fig. 8.17 if there are 1200 ampere turns in the exciting winding, find the flux in each branch.

19. A sample of ferromagnetic material is in the shape of a complete toroidal ring and forms the core of a toroidal inductor. Application of a current H per unit length round the core produces a magnetic flux density B in the core. The demagnetization curve is given by the equation

$$B = B_s \left\{ \coth \left[(H+H_1) / H_2 \right] - H_2 / (H+H_1) \right\}$$

where B_s, H_1, and H_2 are constants. Calculate (*a*) the magnetic flux density in the core when not current is applied round it, (*b*) the current per unit length that must be applied to reduce the magnetic flux density in the core to zero, (*c*) the value of $dB/\mu_0 dH$ when $H = 0$, and (*d*) the value of $dB/\mu_0 \, dH$ when $B = 0$. Evaluate these quantities if $B_s = 1$ tesla, $H_1 = 50000$ A/m, and $H_2 = 10000$ A/m.

20. For the demagnetization curve described in the previous problem find a numerical method for locating the point for which the product $-BH$ is maximum. Find the values of B and H for which $-BH$ is maximum when $B_s = 1$ tesla, $H_1 = 50000$ A/m, and $H_2 = 10000$ A/m.

21. A magnetic circuit of uniform cross sectional area S and average length l is wound with an N-turn coil. Leakage of flux from the circuit may be neglected but the relation between B and H for the ferromagnetic material used is

$$B = B_s \left[\coth (3\mu H/B_s) - B_s/3\mu H \right)$$

Where B_s is the saturation flux density. Calculate the magnetic flux ϕ threading the electric circuit when there is a steady current I round it, and evaluate $d\phi / dI$ when $I = 0$. Sketch the relation between ϕ and I when $\mu = 1000\ \mu_0$, and $B = 1.0$ tesla.

22. A magnet in air has the form of a circular cylinder of length L and radius a. The magnet is permanently and uniformly magnetized parallel to its axis, so that the magnetic flux emanates from one flat end with a uniform distribution M per unit area and returns to the other flat end with the same uniform distribution. Calculate B and H at a point on the axis at a distance z from the mid-point of the magnet.

23. A permanent magnet is made of tungsten steel in the form of a horse shoe and has a magnetic path length of 24 cm and a cross section of 1.5 sq cm. It is furnished with pole pieces of soft iron which provide for an air gap 0.1 cm long and 0.8 sq cm in area. How large is the flux density in the air gap if the magnetization is assumed to follow the curve for tungsten steel. Assume a leakage coefficient of 1.4.

B	0.0	0.2	0.4	0.6	0.8	1.0
H_1	—5000	—4800	—4500	—4000	—3150	—1300
H_2	+5000	+5100	+5400	+5700	+6000	+6600
B	1.1	1.2	1.4	1.5	1.6	
H_1	0.0	+2300	+10000	+17000	+29500	
H_2	+7000	+8000	+12500	+19000	+29500	

24. In a magnetic system in which the dimensions of the air gap are fixed and where leakage due to the pole shoes is small, it is necessary to produce a magnetic flux of 0.14 mWb in the air gap. The reluctance of the pole shoes, air gap and armature is 8 mega A/Wb. Find the dimensions of a permanent magnet which will supply the necessary magnetic flux with the minimum amount of material. The magnetic steel that is available has the maximum product $-BH$ when $B = 0.7$ tesla and $H = 16000$ A/m.

25. In order to provide the necessary magnetic flux in an electric instrument use is made of a permanent magnet which has a mean magnetic path of 15 cm and a cross section of 1.5 sq cm. The magnetization curve of the material is approximately given by

$$B = 1 + H \times 10^{-4} \text{ in the second quadrant.}$$

The magnet is provided with an armature and pole shoes of a material whose relative permeability is 10000. The total magnetic path in the armature and the pole shoes is 5 cm and cross section 2 sq cm. The two air gaps which exist between the pole shoes and the armature have each a length of 2 mm. Determine the flux in the air gap neglecting leakage.

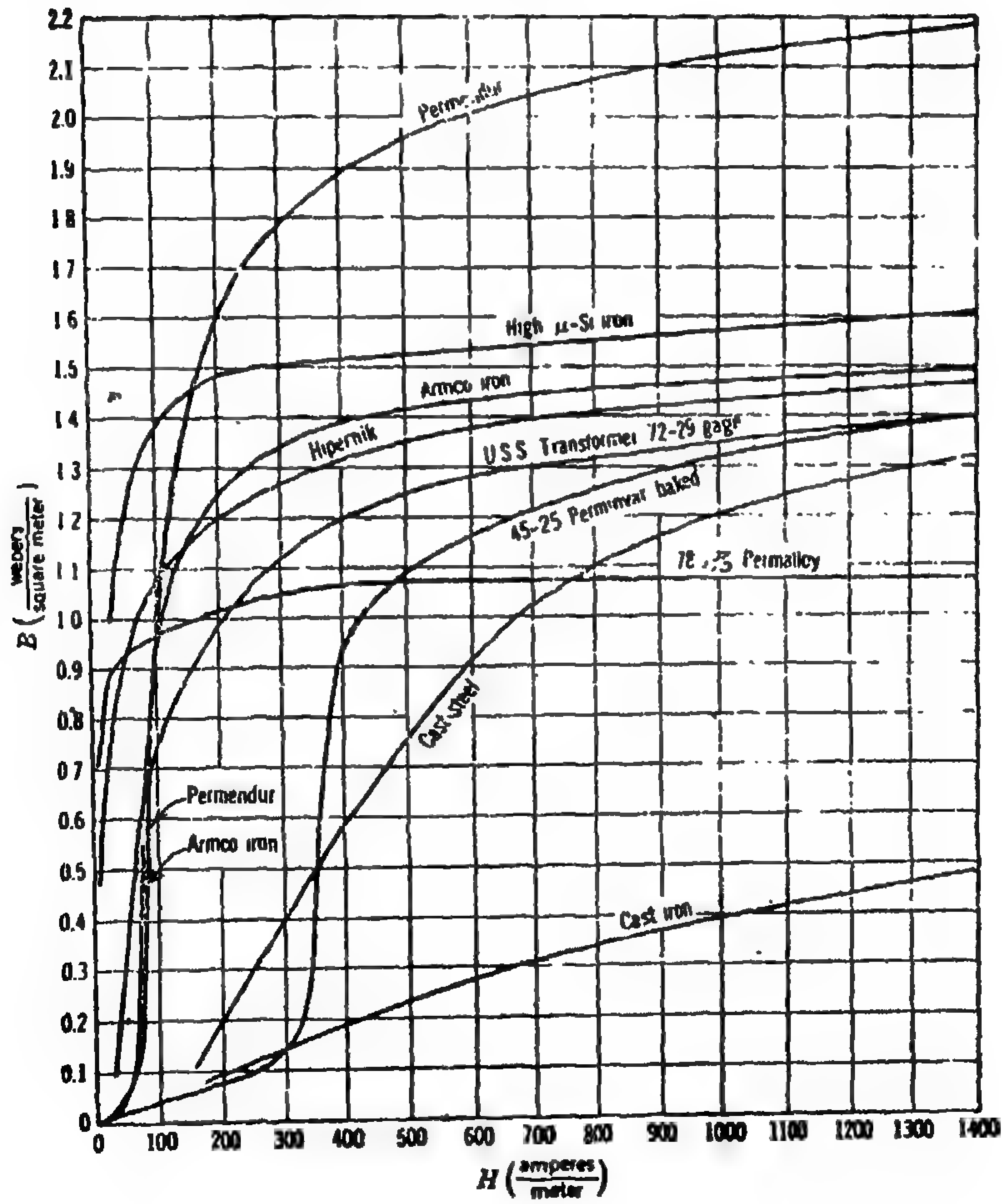


Fig. 8.18.

الفصل التاسع

المسائل الحدية في المجالات الكهربية والمغناطيسية الساكنة

BOUNDARY-VALUE PROBLEMS IN THE ELECTROSTATIC AND MAGNETOSTATIC FIELDS

1-9 مقدمة :

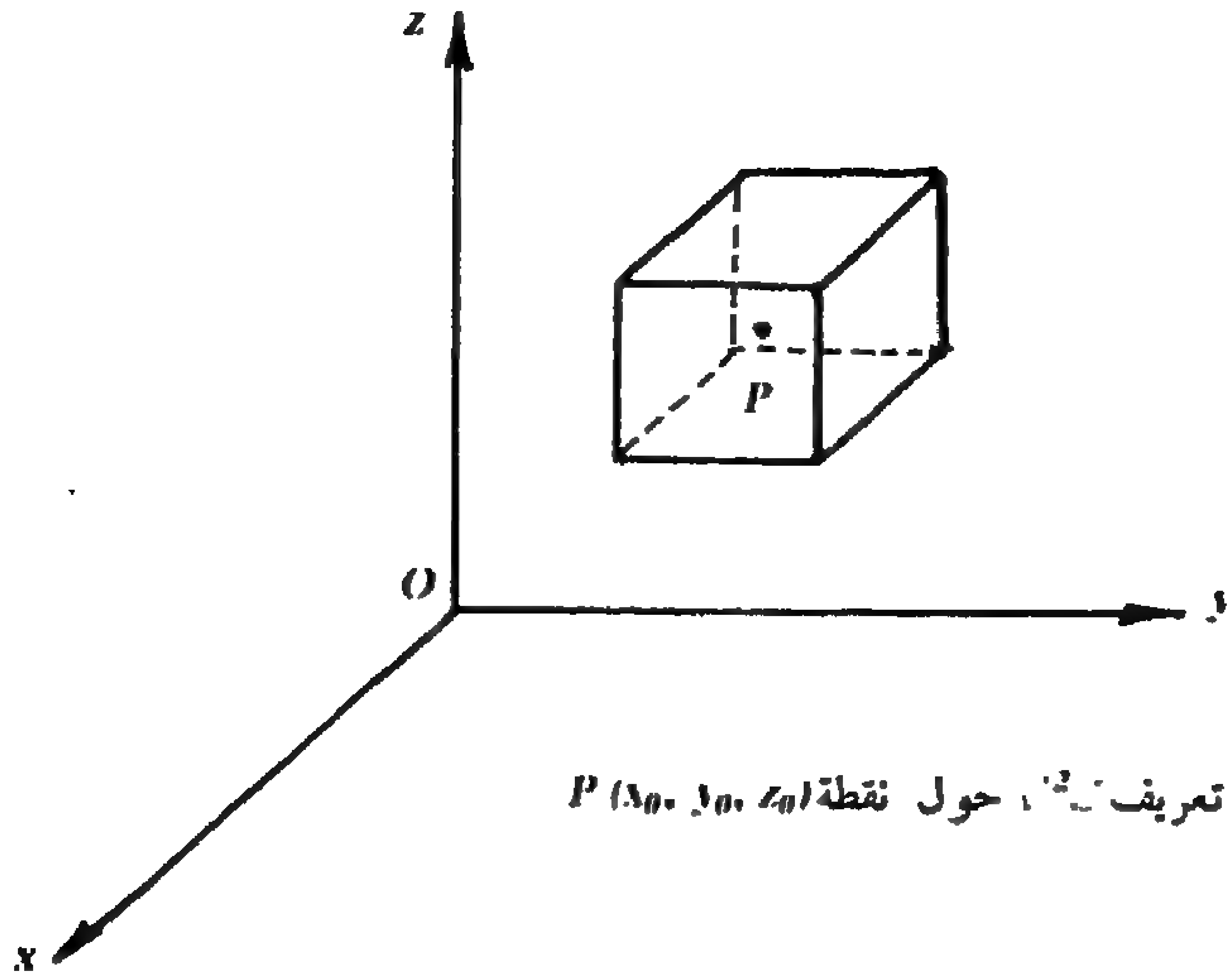
في الأوساط الكهربية ذات الخواص الموحدة تكون المجالات الكهربية والمغناطيسية الساكنة وشبه الساكنة (quasi - static) غير دوّارة وحلزونية وبالتالي تحقق معادلة لابلاس $\nabla^2 \varnothing = o$ حيث $\varnothing$ هي دالة قياسية ومستمرة ومحدودة ولها تفاضلات جزئية حتى الدرجة الثانية بالنسبة لمتغيراتها المستقلة .

2-9 المعنى الطبيعي للدالة $\nabla^2 \varnothing$ عند نقطة :

إفرض أن $\varnothing$ هي دالة قياسية من الاحداثيات الكارتيزية (x,y,z) وأنها مستمرة ومحدودة ولها تفاضلات جزئية من الدرجة الثانية . أفرض كذلك أن قيمة الدالة عند نقطة $P\ (x_o, y_o, z_o)$ هي $\varnothing_P$. شكل (a-9) يبين نقطة P كمركز في مكعب صغير طول ضلعه 2δ . يمكن حساب متوسط قيمة الدالة $\varnothing$ داخل هذا المكعب من المعادلة .

$$< \varnothing > = \frac{1}{8\delta^3} \int\int\int_{cube} \varnothing\ (x,y,z)\ dx\ dy\ dz \qquad (1\text{-}9)$$

ويمكن إيجاد مفكوك تيلور للدالة $\varnothing$ حول نقطة P على الصورة .



شكل (9 - a) تعريف ∇^2 حول نقطة $P(x_0, y_0, z_0)$

$$\varnothing \cong \varnothing_p + x\left(\frac{\partial \varnothing}{\partial x}\right)_p + y\left(\frac{\partial \varnothing}{\partial y}\right)_p + z\left(\frac{\partial \varnothing}{\partial z}\right)_p$$

$$+ \frac{1}{2}x^2\left(\frac{\partial^2 \varnothing}{\partial x_2}\right)_p + \frac{1}{2}y^2\left(\frac{\partial^2 \varnothing}{\partial y_2}\right)_p + \frac{1}{2}z^2\left(\frac{\partial^2 \varnothing}{\partial z^2}\right)_p$$

$$+ yz\left(\frac{\partial^2 \varnothing}{\partial y \partial z}\right)_p + xz\left(\frac{\partial^2 \varnothing}{\partial x \partial z}\right)_p + yz\left(\frac{\partial^2 \varnothing}{\partial y \partial z}\right)_p$$

بالتعويض عن قيمة $\varnothing$ من *(2-9)* في المعادلة *(1-9)* نحصل على :

$$< \varnothing > = \varnothing_p + \frac{\delta^2}{24}(\nabla^2 \varnothing)_p$$

$$< \varnothing > - \varnothing_p = \frac{\delta^2}{24}(\nabla^2 \varnothing)_p \qquad (3\text{-}9)$$

أي أن $\nabla^2 \varnothing$ عند نقطة P تناسب مع الفرق بين قيمة الدالة $\varnothing$ عند نقطة P ومتوسط قيمتها حول هذه النقطة . وبناء على ذلك يمكن تفسير المعنى الطبيعي لمعادلة لابلاس ومعادلة الانتشار وكذلك المعادلة الموجبة .

(a) معادلة لابلاس — *Laplaces equation*

$$\nabla^2 \varnothing = o$$

تدل هذه المعادلة على أن قيمة الدالة $\varnothing$ عند نقطة ما تساوي متوسط قيمتها حول هذه النقطة .

(b) معادلة الانتشار — *diffusion equation*

$$\nabla^2 \varnothing = K \frac{\partial \varnothing}{\partial t}$$

تدل هذه المعادلة على أن قيمة الدالة $\varnothing$ عند نقطة ما تختلف عن متوسط قيمتها حول هذه النقطة . ويتلاشى هذا الفرق مع مرور الزمن .

(c) المعادلة الموجبة — Wave Equation

$$\nabla^2 \varnothing = \frac{1}{c^2} \frac{\partial^2 \varnothing}{\partial t^2}$$

وتدل هذه المعادلة على أن قيمة الدالة عند نقطة ما تختلف عن متوسط قيمتها حول هذه النقطة وان هناك قوى تعمل على إستعادة التوازن بينهما

3-9 نظرية القيمة القصوى : (Maximum Value Theorem)

هذه النظرية هي نتيجة مباشرة من المعنى الطبيعي لمعادلة لابلاس وهي تنص على أن دالة الجهد لا يمكن أن تأخذ قيمة قصوى أو صغرى عند أي نقطة في مجال هذه الدالة . ولاثبات ذلك تفرض أن الدالة $\varnothing$ لها قيمة قصوى عند نقطة ما P . متوسط قيمة الدالة $\varnothing$ حول نقطة P يكون حسب هذا الفرض أصغر من قيمة الدالة عند P وهذا يتعارض مع معادلة لابلاس وبذلك لا يمكن أن يكون للذالة $\varnothing$ قيمة قصوى عند أي نقطة .

4-9 الدوال التوافقية : (Harmonic Functions)

لايجاد دالة الجهد $\varnothing$ عند أي نقطة يجب حل معادلة لابلاس في المجال

التي تقع داخله هذه النقطة وذلك بعد تطبيق الشروط الحدية والتأكد من أن الدالة $\varnothing$ تنعدم في اللانهاية إذا كان الحيز المطلوب فيه الحل يصل إلى اللانهاية . وتسمى أي دالة تحقق معادلة لابلاس بالدالة التوافقية . وتتحقق معادلة لابلاس في النظرية الكهرومغناطيسية في الحالات الآتية :

(a) الجهد الكهروستاتيكي (The Electrostatic Potential)

في المناطق التي لا يوجد بها شحنات كهربية تحقق دالة الجهد الكهربي $\varnothing = V$ معادلة لابلاس وتكون السطوح المتساوية الجهد وخطوط القوى الكهربية مجموعتين متعامدتين .

(b) الجهد المغناطيسي الاستاتيكي (The Magnetostatic Potential)

في المناطق التي ليس بها تيار في الوسط المغناطيسي الساكن تحقق دالة الجهد المغناطيسي $\varnothing_H$ معادلة لابلاس وتكون السطوح المتساوية الجهد $\varnothing_H = const.$ وخطوط القوى المغناطيسية $= const.$ مجموعتين متعامدتين .

(c) التيار الثابت في وسط موصل موحد الخواص : (Steady Currents in Isotropic Conducting Media)

في وسط موصل موحد الخواص وبه تيار ثابت تحقق دالة الجهد الكهربي $\varnothing$ معادلة لابلاس عند أي نقطة . وتكون خطوط إنسياب التيار $J = const.$ والخطوط المتساوية الجهد $\varnothing = const.$ مجموعتين متعامدتين .

9-5 الشروط الحدية : (Boundary Conditions)

معادلة لابلاس هي معادلة تفاضلية جزئية . خطية من الدرجة الثانية وهي لا تعبر عن ظاهرة طبيعية معينة بحد ذاتها ولكنها تمثل قانون عام يحكم ظواهر فيزيائية كثيرة . وعند اعتبار مسألة معينة فإن حل معادلة لابلاس لا يكون محدد إلا إذا تحققت بعض الشروط التي تلم بظروف المسألة نفسها .

ويقال عن ظاهرة لها دالة قياسية معينة $\varnothing$ تحقق معادلة لابلاس ولها شروط حدية إنها مصاغة بطريقة مناسبة إذا تحققت الشروط التالية :

(a) الحل الذي نحصل عليه يكون وحيد وذلك لأن الطبيعة تعطي قيمة واحدة لمجموعة من الشروط الحدية المقاسة عملياً فمثلاً في وسط كهربي يحمل شحنات معينة وله قيم حدية مقاسة يكون للجهد الكهربي عند أي نقطة قيمة واحدة فقط مهما تعددت الطرق التي نتبعها للحصول على هذه الدالة .

(b) الحل الذي نحصل عليه يكون مستقر بمعنى أنه إذا حدث تغيير صغير في أحد الشروط الحدية فإن التغيير المصاحب له في قيمة الدالة $\varnothing$ يكون صغيراً . وهذا راجع إلى أن هناك أخطاء صغيرة تحدث عند قياس الشروط الحدية عملياً وهذه الاخطاء الصغيرة لا يجب أن تعطي فروق كبيرة في دالة الجهد المقاسة عند مقارنتها بدالة الجهد . التي نحصل عليها عند حل معادلة لابلاس .

وهناك ثلاثة أنواع من الشروط الحدية ينتج عنها ثلاثة أنواع من مسائل الحدود المعينة (Boundary-value Problems) وهي كالآتي :

النوع الأول : الدالة القياسية $\varnothing$ تحقق معادلة لابلاس $\nabla^2 \varnothing = o$ داخل حجم معين v حيث تكون قيمة $\varnothing$ معلومة عند أي نقطة على السطح S الذي يحدد الحجم v . وتسمى هذه المسألة بمسألة دريشلت (Dirichlet's Problem) .

النوع الثاني : الدالة القياسية $\varnothing$ تحقق معادلة لابلاس $\nabla^2 \varnothing = o$ عند أي نقطة داخل حجم معين v والتفاضل الجزئي العمودي على السطح $\frac{\partial \varnothing}{\partial n}$ معلوم عند أي نقطة على السطح S الذي يحدد الحجم v وتسمى هذه المسألة بمسألة نيومان (Neumann's Problem) .

النوع الثالث : الدالة القياسية $\varnothing$ تحقق معادلة لابلاس $\nabla^2 \varnothing = o$ عند أي نقطة داخل الحجم v والمعلوم هو $\frac{\partial \varnothing}{\partial n}$ على جزء من السطح S المحدد للحجم v وكذلك معلوم $\varnothing$ على الجزء الباقي من S وتسمى المسألة في هذه الحالة بمسألة الحدود المتاخمة المختلطة (Mixed-boundary value Problem) . ويلاحظ أن حل معادلة لابلاس في منطقة ما يجب أن يحقق الشروط الحدية عندما يكون هناك اكثر من مادة داخل الحجم v الذي تحل داخله معادلة لابلاس . وهذه الشروط هي :

(a) المجال الكهروستاتيكي على سطح موصل :

$$V = constant \tag{4-9}$$

$$\frac{\partial V}{\partial n} = -\frac{\sigma}{\epsilon} \tag{5-9}$$

حيث   هي كثافة الشحن السطحي على الموصل

وعند السطح الفاصل بين وسطين عازلين .

$$V_1 = V_2 \tag{6-9}$$

$$\epsilon_1 \frac{\partial V_1}{\partial n} = \epsilon_2 \frac{\partial V_2}{\partial n} \tag{7-9}$$

(b) في المجالات المغناطيسية الساكنة :

$$\varnothing_1 = \varnothing_2 \tag{8-9}$$

$$\mu_1 \frac{\partial \varnothing_1}{\partial n} = \mu_2 \frac{\partial \varnothing_2}{\partial n} \tag{9-9}$$

حيث المجال المغناطيسي H معطى بالعلاقة :

$$H = -\nabla \varnothing \tag{10-9}$$

(c) في التيارات الكهربية الثابتة في موصل :

على الحد الفاصل بين وسطين موصلين :

$$V_1 = V_2 \tag{11-9}$$

$$\gamma_1 \frac{\partial V_1}{\partial n} = \gamma_2 \frac{\partial V_2}{\partial n} \tag{12-9}$$

وعلى السطح الفاصل بين موصل وعازل يكون التيار الكهربي دائماً في إتجاه المماس أي أن :

$$\frac{\partial V}{\partial n} = 0 \tag{13-9}$$

6-9 الطرق المتبعة لحل معادلة لابلاس :

هناك طرق كثيرة كل معادلة لابلاس وذلك مع الشروط الحدية المناسبة وهي :

(a) الطرق الرياضية التحليلية حيث يعطي الحل على صورة دالة رياضية محددة .

(b) الطرق العددية التقريبية مثل طريقة العزوم (Method of Moments) وطريقة الإسترخاء (Relaxation Method) .

(c) الطرق البيانية :

(d) الطرق المقارنة (Analogue Methods) مثل طريقة الخزان الالكتروليتي (Electrolytic Tank) وطريقة الورق المقاوم (Resistance. Paper Method) .

وتعتمد طريقة الحل على الشكل الهندسي لحدود الحجم v المراد إيجاد حل معادلة لابلاس داخله وكذلك على الشروط الحدية للمسألة. وأخيراً

على درجة الدقة المطلوبة في الحل . الطرق الرياضية التحليلية المتبعة هي طريقة الصور، طريقة التعاكس، طريقة فصل المتغيرات، طريقة التحليل المركب، وطريقة التحويل التشابهي . وتستعمل طريقتي التحليل المركب والتحويل التشابهي فقط في المسائل ذات البعدين أما الطرق الاخرى فهي تستخدم في الحالات ذات البعدين أو الثلاثة أبعاد وذلك بشرط أن يكون الحجم v ذات حدود بسيطة مثل المستويات والخطوط المستقيمة . وتحتاج طريقة فصل المتغيرات لمعرفة معادلات حدود المساحة المطلوب فيها الحل . وإذا كانت حدود الحجم v أو السطح S المراد إيجاد الحل داخله أو عليه تعطي بمعادلات لا يمكن حلها تحليلياً فيمكن أن نستخدم الطرق (b) أو (c) أو (d) . وتعتمد طريقة الحل على طبيعة المسألة المراد حلها فمثلاً إذا كان المطلوب تعيين الحدود المثلى لمنطقة معينة تتحقق داخلها معادلة لابلاس فإن الطريقة الرياضية قد تكون مكلفة وبالتالي يفضل إستخدام طريقة المقارنة .

7-9 وحدانية الحل (Uniqueness of Solution)

عند حل معادلة لابلاس داخل حجم معين v لايجاد دالة الجهد V عند أي نقطة داخلة مع الأخذ في الاعتبار بالشروط الحدية التي تنص عليها المسألة فإننا نحصل على حل معين V_1 لدالة الجهد . والآن لنا أن نتساءل كيف نعرف أنه لا توجد حلول أخرى لنفس المسألة وبنفس الشروط الحدية المعطاة . ويمكننا إثبات أنه إذا تواجد حل لمسألة تحقق معادلة لابلاس فإن هذا الحل يكون وحيد وبالتالي يكون المجال المصاحب لهذا الحل وحيد .

إفرض أن الحجم v يحدده سطح مغلق S وأن الدالة $\varnothing$ هي دالة قياسية مستمرة ومحدودة وذات تفاضلات جزئية ثنائية داخل v وعلى سطحه S . بتطبيق نظرية جاوس على الدالة المتجهة .

$$F = \varnothing \nabla \varnothing$$

نجد أن :

$$\oint_S \emptyset \nabla \emptyset . dS = \int_v (\emptyset \nabla^2 \emptyset + \nabla \emptyset . \nabla \emptyset)\, dv \quad (14\text{-}9)$$

أفرض أن هناك دالتي جهد V_1 وV_2 كلاهما تحقق المعادلة (14-9) . فإذا كانت الشحنة الكهربية داخل v هي ρ فإن .

$$\nabla^2 V_1 = -\rho/\epsilon_0$$

$$\nabla^2 V_2 = -\rho/\epsilon_0$$

فإذا كانت :

$$\emptyset = V_1 - V_2$$

فإن :

$$\nabla^2 \emptyset = \nabla^2 V_1 - \nabla^2 V_2 = 0$$

أي أن $\emptyset$ تحقق معادلة لابلاس وبالتعويض في المعادلة (14-9) نجد أن :

$$\oint_S \emptyset \nabla \emptyset . dS = \oint_S \emptyset \frac{\partial \emptyset}{\partial n}\, dS = \int (\nabla \emptyset)^2\, dv$$

فإذا إنعدم التكامل السطحي نتيجة للشروط الحدية عليه فإن :

$$\int_v (\nabla \emptyset)^2\, dv = 0$$

ومنها يكون $\nabla \emptyset = 0$ أي أن العلاقة الأتية تتحقق .

$$\emptyset = V_1 - V_2 = \text{ثابت}$$

عند أي نقطة داخل v ومن هذه العلاقة يتضح أن المجال الكهربي الناتج عن V_1 هو نفسه الناتج عن V_2 وأن الجهد معين مع امكانية اضافة حد ثابت لدالته .

ويكون التكامل السطحي منعدماً في الحالات الآتية :

(1) انعدام الدالة $\varnothing$ على السطح S وفي هذه الحالة يكون $V = V_2 =$ V_1 ويكون الحد الثابت الذي ذكرناه منعدماً أي أن المجال موحد ودالة الجهد V موحدة كذلك . وتكون المسألة هي مسألة دريشلت .

(2) انعدام الدالة $\frac{\partial \varnothing}{\partial n}$ على السطح S وفي هذه الحالة يكون :

$$\frac{\partial V_1}{\partial n} = \frac{\partial V_2}{\partial n} = \frac{\partial V}{\partial n}$$

وهذا الشرط يتحقق إذا كانت الدالة $\frac{\partial V}{\partial n}$ موصوفة عند كل نقط السطح S. وتصبح المسألة هي مسألة نيومان .

(3) تكون الدالة $\varnothing$ معطاة بالعلاقة :

$$\varnothing = V_1 - V_2 = constant$$

عند أي نقطة على السطح S وهذا واضح من العلاقة .

$$\oint \varnothing \frac{\partial \varnothing}{\partial n} dS = \varnothing \oint \frac{\partial \varnothing}{\partial n} dS = \varnothing \int_{v} \nabla^2 \varnothing \, ds = 0$$

وبناء على ما سبق فإن مفهوم وحدوية الحل يتحقق إما بالعلاقة

$$(A) \quad V_1 - V_2 = constant \ on \ S$$

أو بالعلاقة :

$$(B) \quad \frac{\partial V_1}{\partial n} = \frac{\partial V_2}{\partial n}$$

على السطح S

فإذا تحققت العلاقة (A) وكانت a و b هما نقطتان على السطح S فإن

$$V_{1a} - V_{2a} = V_{1b} - V_{2b} = const.$$

$$V_{1a} - V_{1b} = V_{2a} - V_{2b} = const.$$

وهذا لا يمكن أن يحدث إلا إذا كان المجال الكهربي المماس موصوف عند أي نقطة على S. وإذا تحققت العلاقة (B) فإن المركبة العمودية للمجال الكهربي يجب أن تكون موصوفة على السطح S.

9 - 8 مسائل محلولة

SOLVED PROBLEMS

(LAPLACE'S EQUATION)

1. *A system consists of an arbitrary number N of conductors each insulated from the others and the whole system is in electrostatic equilibrium. Show that there is only one way in which the charge on each conductor can distribute itself and that there can be only one value for the potential (and hence field) at each point in space. (Uniqueness Theorem).*

Assume that a certain charge distribution $\sigma_1, \sigma_2, \ldots, \sigma_N$ on the system of N conductors will produce at any point in space a potential V. Let us assume further that there exists another charge distribution $\sigma_1', \sigma_2', \ldots, \sigma_N'$ which will produce the same potential. Then,

$$V = (1/4\pi\epsilon_0) \; \Sigma \int_{S_i} \sigma_i \, dS_i / r$$

$$= (1/4\pi\epsilon_0) \; \Sigma \int_{S_i} \sigma_i' \, dS_i / r$$

Suppose that the conductors have charge distribution $\sigma_i'' = \sigma_i - \sigma_i'$ $i = 1, 2, \ldots, N$. Then the resulting potential is,

$$V'' = (1/4\pi\epsilon_0) \; \Sigma \int_{S_i} (\sigma_i - \sigma_i') \, dS_i / r$$

$$= (1/4\pi\epsilon_0) \; [\Sigma \int_{S_i} \sigma_i \, dS_i / r - \Sigma \int_{S_i} \sigma_i' \, dS_i / r]$$

$$= 0$$

It follows that the field in turn must be zero everywhere. Thus on the surface of any conductor the normal field is zero,

$$E_n'' = (\sigma_i - \sigma_i') / \epsilon_0 = 0$$

Thus we have that, $\sigma_i = \sigma_i'$.

Therefore for given potentials of each of the conductors, the resulting field strength and surface charge distribution have unique values.

2. *The potential in the plane $z = 0$ is given by*

$$V = V_0 , \ -a/2 < x < a/2$$

$$= -V_0 , \ a/2 < x < 3a/2 \quad \text{(Fig. 10.1)}$$

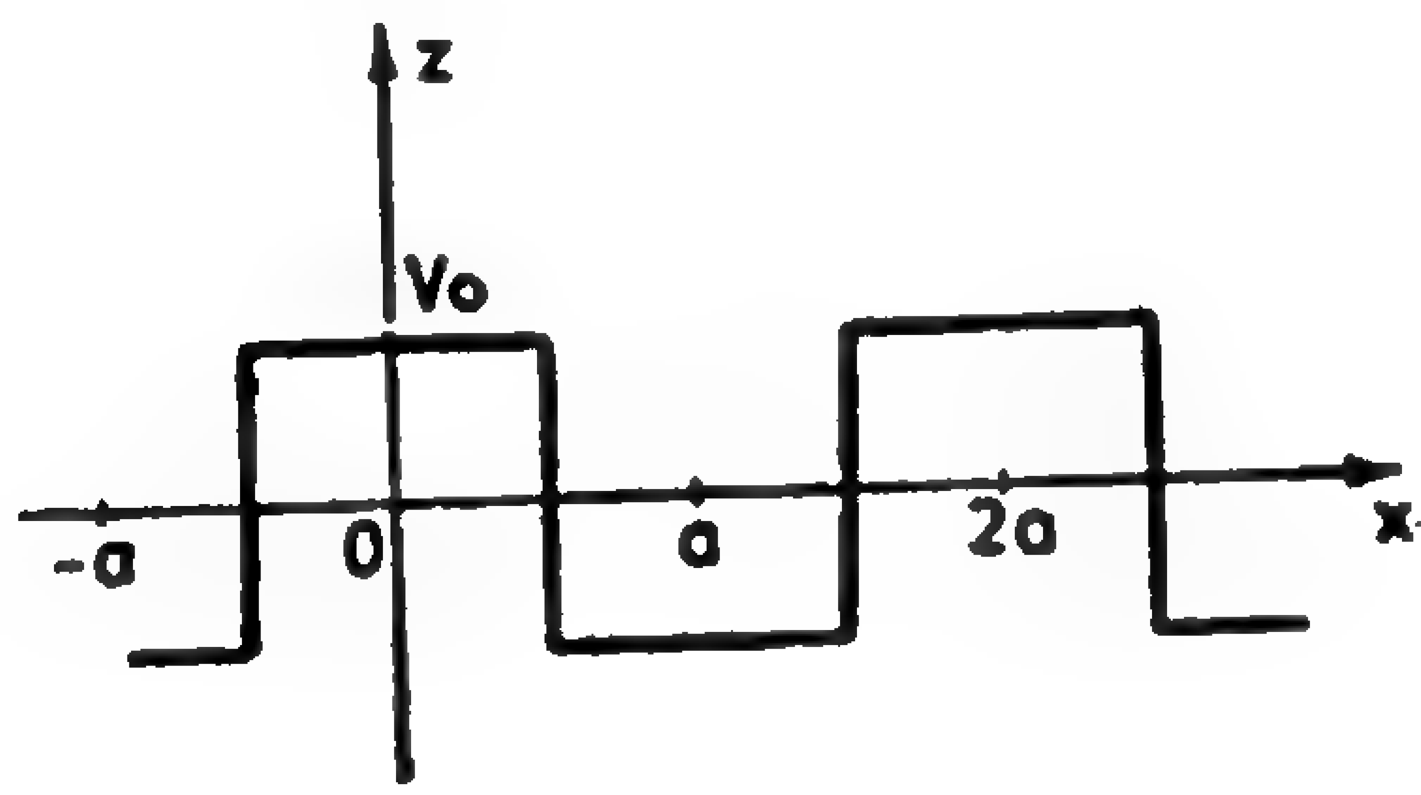


Fig. 10.1.

and is independent of y. Find an expression for the potential at all points.

The potential at any point satisfies Laplace's equation $\nabla^2 V = 0$. A suitable solution is,

$$V = \sum_{n=0}^{\infty} (A_n \sin a_n x + B_n \cos a_n x) \, [C_n \exp(a_n z) + D_n \exp(-a_n z)] \qquad (1)$$

In order to satisfy the boundary conditions when z is very large, $C_n = 0$ for $z > 0$ and $D_n = 0$ for $z < 0$. At $z = 0$ the boundary condition can be satisfied by expanding the given potential distribution in the xy—plane using Fourier series, which gives

$$V = \sum_{n \text{ odd}} (4V_0 / n\pi)(-1)^{\frac{1}{2}(n-1)} \cos(n\pi x/a) \tag{2}$$

Equations (1) and (3) are identical at $z = 0$. This gives,

$A_n = 0$, $B_n = (4V_0/n\pi) \; (-1)^{\frac{1}{2}(n-1)}$ n odd, and $\alpha_n = n\pi/a$

Therefore the required solution is ($z \geq 0$),

$$V = \sum_{n \text{ odd}} (4V_0/ n\pi)(-1)^{\frac{1}{2}(n-1)} \cos(n\,\pi x/a) \exp(-n\pi z/a)$$

and, ($z \leq 0$)

$$V = \sum_{n \text{ odd}} (4V_0/n\pi)(-1)^{\frac{1}{2}(n-1)} \cos(n\pi x/a) \exp(n\pi z/a)$$

3. *The potential is such that we have a sawtooth $V = V_0\, x/a$ as shown in Fig. 10.2 in the xy—plane, and the plane at $z = b$ is kept at zero potential. Determine the potential at all points between the planes $z = 0, b$.*

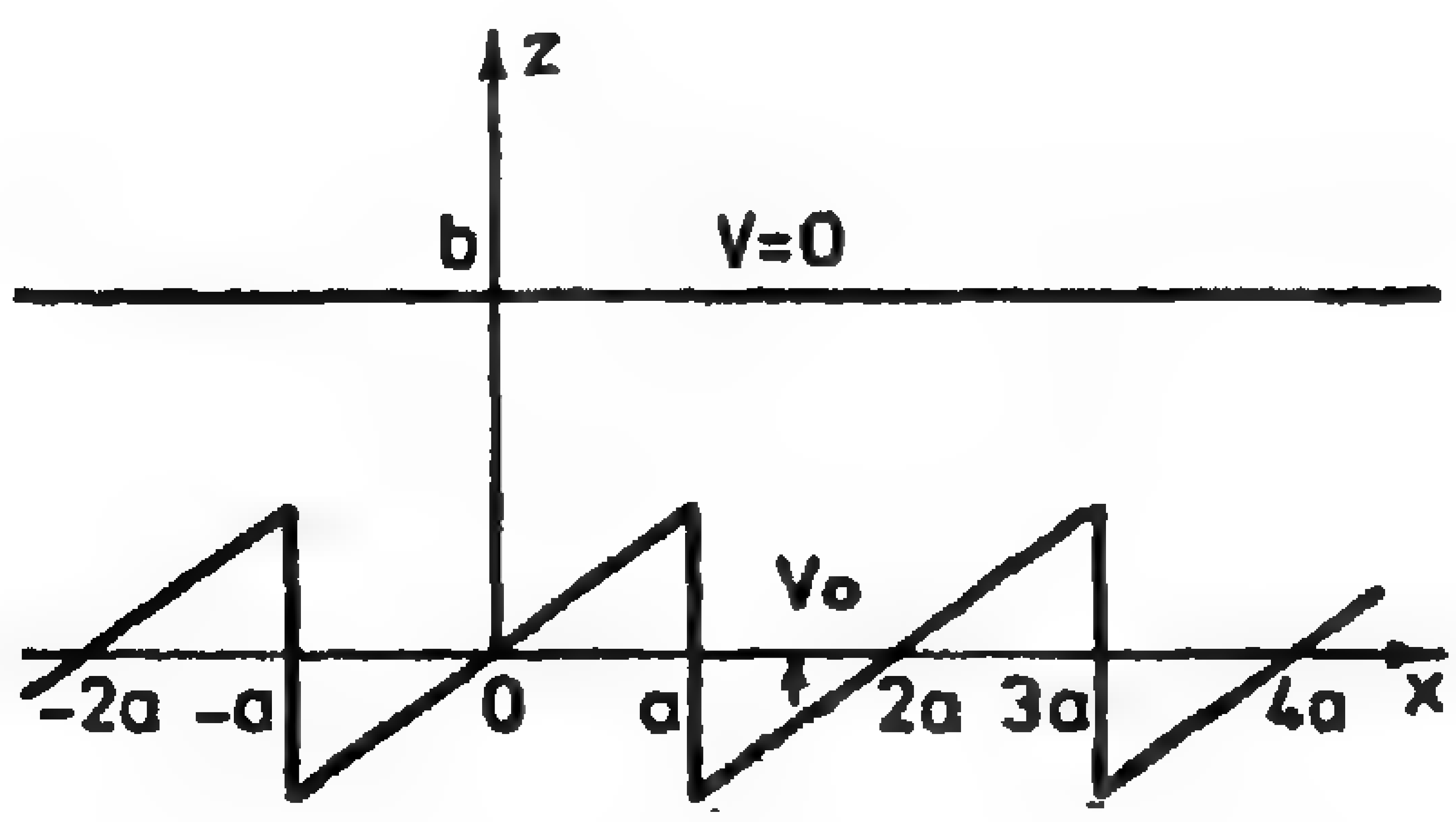


Fig. 10.2.

A suitable solution for Laplace's equation in the region $o \leq z \leq c$ is (Appendix II),

$$V = \sum_{n=o}^{\infty} (A_n \sin \alpha_n x + B_n \cos \alpha_n x)\ [\exp\ (\alpha_n z) + D_n \exp(-\alpha_n z)] \qquad (1)$$

where C_n is absorbed in A_n and B_n.

The potential in the xy— plane given by the sawtooth when expanded in a Fourier series gives

$$V = \sum_{1}^{\infty} (2V_o/n\,\pi)\ (-1)^{n+1} \sin\ (n\pi\, x/a) \qquad (2)$$

This suggests that $B_n = 0$ and $\alpha_n = n \cdot \pi / a$

The coefficient D_n can be obtained from the boundary condition at $z = b$, *i.e.* $V = 0$. This gives, $D_n = -\exp(2\alpha_n\ b)$ so that (1) becomes,

$$V = \sum_{1}^{\infty} -2\,A_n \exp\ (\alpha_n b)\ \sin\ (\alpha_n x)\ \sinh\ [\alpha_n (b-z)] \qquad (3)$$

The coeficient A_n is determined by equating (2) and (3) at $z = 0$,

$$A_n = -\ (V_o/n\pi)\ \exp\ (-n\,\pi\, b/a)$$

Thus we have that,

$$V = \sum_{1}^{\infty} (2V_o/n\pi)\,(-1)^{n+1} \sinh\ [n\pi(b-z)/a]\ \sin\,(n\pi x/a)\ / \sinh\ (n\pi\, b/a)$$

4. *The potential distribution in the plane $z = c$ is given by,*

$$V = V_o \qquad -\tfrac{1}{2}a \leq x < \tfrac{1}{2}a$$

$$= -V_o \qquad \tfrac{1}{2}a \leq x < 3a/2$$

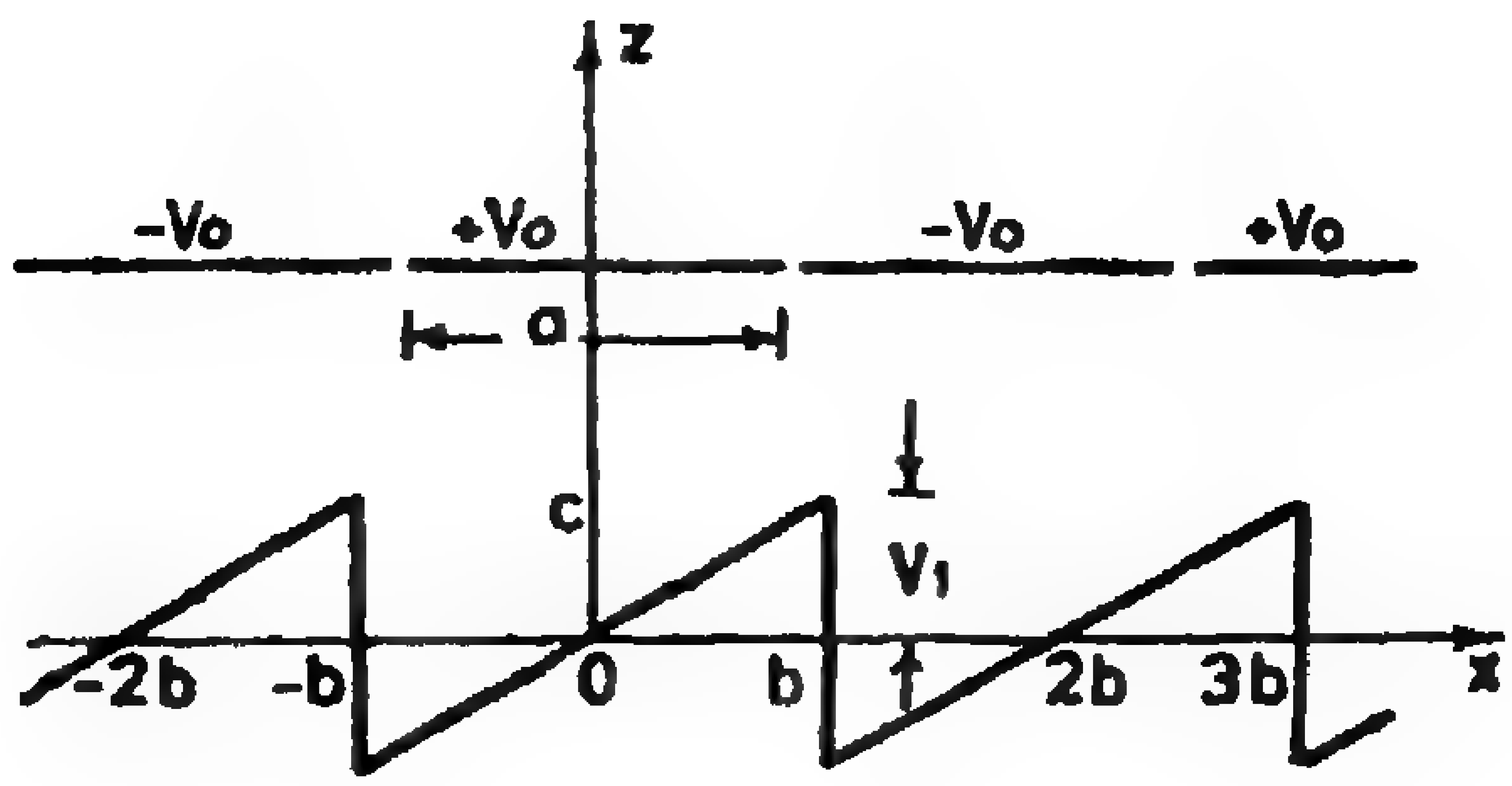


Fig. 10.3.

while in the $z = 0$ *plane it is given by the sawtooth* $V = V_1 x / \iota$ *alon g the x—direction Fig. 10.3 Find an expression for the potentio at all points in the region* $0 \leq z \leq c$.

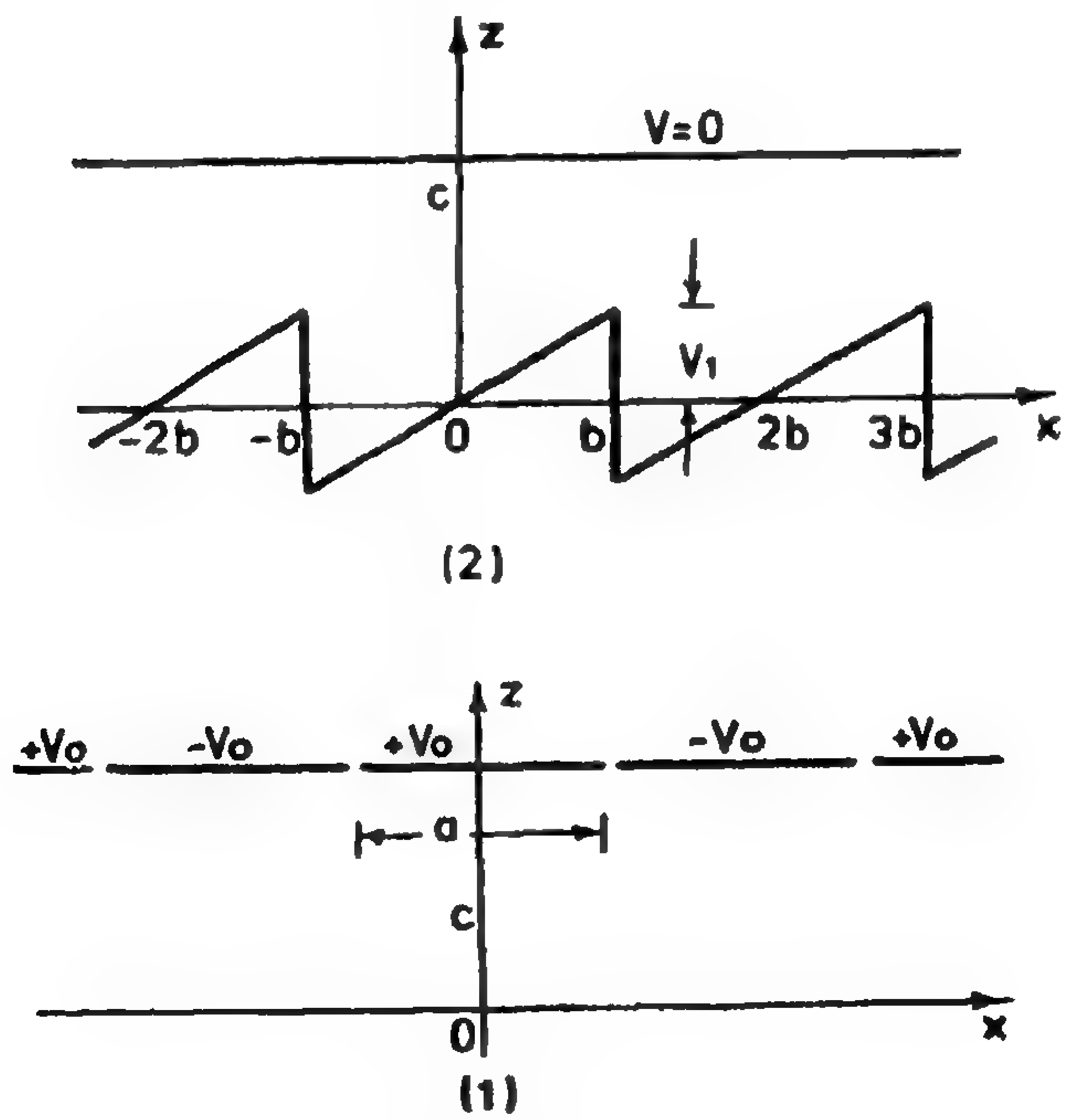


Fig. 10.4.

The problem can be solved using the method of superposition. It is the sum of two problems, Fig. 10.4. (1) and (2). Problem (1) satisfies the boundary condition at $z = c$ and does not contribute to the potential at $z = 0$. Problem (2) satisfies the boundary condition at $z = 0$ and does not contribute to the potential at $z = c$. The sum of these two problems satisfies Laplace's equation and at the same time satisfies the boundary conditions at $z = 0, c$. Therefore the sum of the solutions of the two problems gives the required solution.

The solution of problem (1) is,

$$V_1 = \sum_{n\ odd} (4V_0/n\pi)(-1)^{\frac{1}{2}(n-1)} \sinh(n\pi z/a) \cos(n\pi x/a) \,/$$

$$\sinh(n\pi c/a)$$

and that of problem (2) is (see previous problem),

$$V_2 = \sum_1^{\infty} (2V_1/n\pi)(-1)^{n+1} \sinh[n\pi(c-z)/b]$$

$$\sin(n\pi x/b) \,/\, \sinh(n\pi c/b)$$

The whole solution of the problem is thus,

$$V = V_1 + V_2$$

5. *Find an expression for the potential everywhere within the rectangle shown in Fig. 10.5, given the potential along the four boundaries (Fig. 10.6.).*

The problem is solved in three parts in a way similar to that adopted in the previous problem.

First assume that we have the potential indicated along the boundary (1) and zero at all others. The solution of Laplace's equation is,

$$V_1(x,y) = \sum_1^{\infty} [A_n \sin(a_n x) + B_n \cos(a_n x)] \, [\exp(a_n y) +$$

$$C_n \exp(-a_n y)] \qquad (1)$$

The boundary conditions $V = 0$, $x = 0$, $x = a$, $y = b$ reduce (1) to

$$V_1(x,y) = \Sigma \; A_n \sin(n\pi\, x/a) \sinh[n\,\pi\,(b-y)/a] \div \sinh(n\pi\, b/a)$$

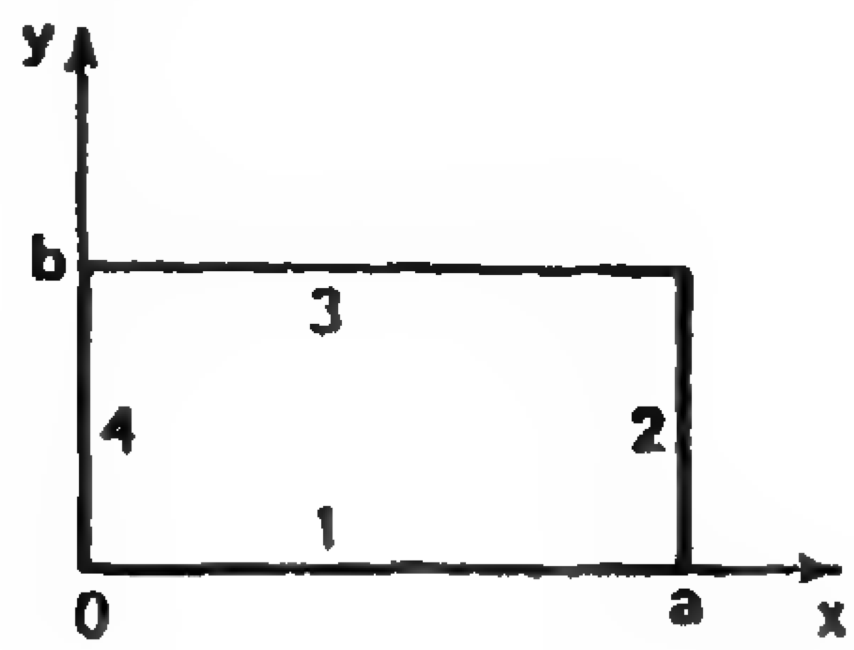


Fig. 10.5.

From the given boundary condition,

$$V_1(x,o) = V_1 \sin(\pi x/a) \quad , \text{ and from (1)},$$

$$= \sum_{1}^{\infty} A_n \sin(n\pi\, x/a)$$

Thus, $\quad A_1 = V_1$, $A_n = 0$, $n>1$.

The solution for the first part becomes

$$V_1(x,y) = V_1 \sin(\pi x/a) \sinh[\pi(b-y)/a] \,/\, \sinh(\pi b/a) \qquad (2)$$

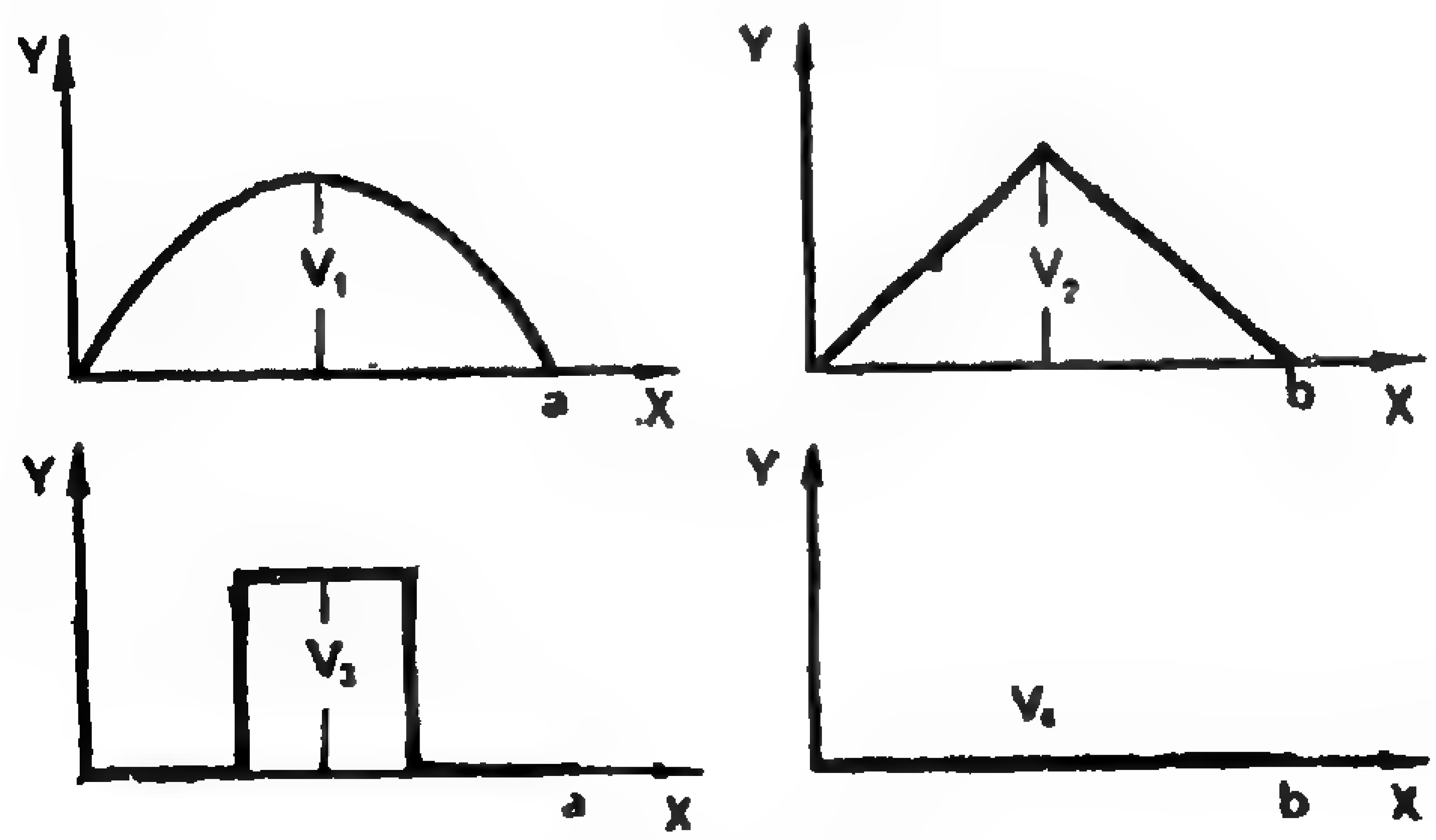


Fig. 10.6.

For the second part consider the potential along the boundary (2) as specified and zero on all other boundaries. General solution of Laplace's equation in this case is,

$$V_2(x,y) = \sum_{1}^{\infty} B_n \sin(n\pi y/b) \sinh(n\pi x/b) \tag{3}$$

Or in terms of slightly different coefficients,

$$V_2(x,y) = \sum_{1}^{\infty} B_n' \sin(n\pi y/b) \sinh(n\pi x/b) / \sinh(n\pi a/b)$$

where, $$B_n' = B_n \sinh(n\pi a/b) \tag{4}$$

$$V_2(a,y) = \sum_{1}^{\infty} B_n' \sin(n\pi y/b) \tag{5}$$

The coefficients B_n' are the Fourier coefficients which occur in the series representation of the specified potential distribution. The coefficients are given by,

$$B_n' = (2/b) \int_0^b V_2(a,y) \sin(n\pi y/b)\, dy \tag{6}$$

From symmetry only odd sine terms are present in the series, hence it is sufficient to integrate over half the total interval and multiply by two.

$$B_n' = (4/b) \int_0^{b/2} (2V_2/b)\, y \sin(n\pi y/b)\, dy$$

This gives,

$$B_1' = 8V_2/\pi^2, \quad B_3' = -8V_2/\pi^2 3^2, \quad B_5' = 8V_2/\pi^2 5^2, \ \ldots$$

The general solution of part (2) is thus,

$$V_2(x,y) = \sum_{n\ odd} (-1)^{\frac{1}{2}(n-1)} (8V_2/\pi^2 n^2) \sin(n\pi y/b) \sinh(n\pi x/b) / \sinh(n\pi a/b) \tag{7}$$

For part (3)

$$V_3(x,y) = \sum_{1}^{\infty} C_n \sin(n\pi x/a) \sinh(n\pi y/a) / \sinh(n\pi b/a)$$

or $$V_3(x,b) = \sum_{1}^{\infty} C_n \sin(n\pi x/a)$$

The coefficients C_n are obtained in the same manner as above and again from the symmetry only odd sine terms are present.

$$C_n = (4/a) \int_{a/3}^{a/2} V_3 \sin(n\pi x/a)\, dx$$

This gives,

$$C_1 = 2V_3/\pi\,, \quad C_3 = -4V_3/3\pi\,, \quad C_5 = 2V_3/5\pi, \ldots$$

$$V_3(x,y) = (2V_3/\pi) \sin(\pi x/a) \sinh(\pi y/a) / \sinh(\pi b/a)$$
$$- (4V_3/\pi) \sin(3\pi x/a) \sinh(3\pi y/a) / \sinh(3\pi b/a)$$
$$+ \ldots$$

The general solution to the complete problem is thus,

$$V(x,y) = V_1(x,y) + V_2(x,y) + V_3(x,y)$$

6. *Find the potential at any point in the space above an infinite checkerboard in the xy-plane shown in Fig. 10.7. The black squares are at potential V_0 while the white squares are at potential $-V_0$.*

A solution of Laplace's equation suitable for this problem is,

$$V(x,y,z) = \sum_{n,m} (A_n \cos k_n x + B_n \sin k_n x)(C_m \cos h_m y + D_m \sin h_m y) \exp(-C_{mn} z) \qquad (1)$$

In the xy-plane the potential distribution can be obtained in terms of a double Fourier series. Since the potential function in the xy-plane

is odd in both x and y as can be noticed from Fig .10.7, a suitable double Fourier series is

$$V(x,y,o) = \sum_{n,m} E_{mn} \sin(n\pi x/a) \sin(m\pi y/b) \tag{2}$$

with

$$E_{mn} = (4/ab) \int_0^a \int_0^b V \sin(n\pi x/a) \sin(m\pi y/b)\, dx\, dy$$

Substituting for the value of V in the range $(0, 0)$, (a,b), $V = V_o$ we get

$$E_{mn} = 16V_o \ \pi^2 mn \qquad \text{for } n,m \text{ odd}$$

$$= 0 \qquad \text{for } n \text{ or } m \text{ even}$$

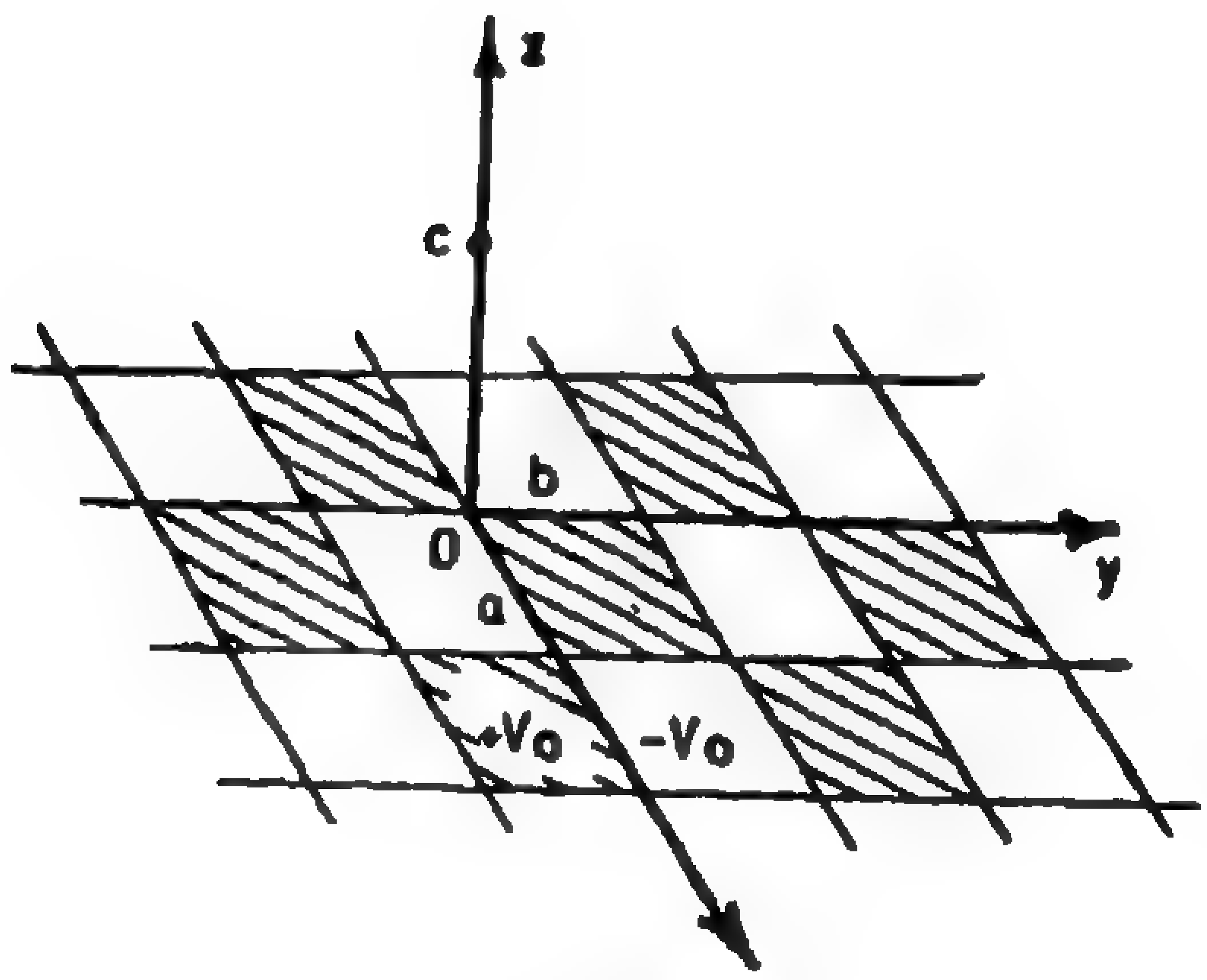


Fig. 10.7.

Thus (2) becomes

$$V(x,y,o) = (16V_o/\pi^2) \sum_{m,\, n \text{ odd}} (1/nm) \sin(n\pi x/a) \sin(m\pi y/b) \tag{3}$$

Comparing equation (1) with (3) at $z = 0$ gives

$$A_n = 0 \ , C_m = 0 \ , D_m B_n = 16V_o/\pi\, nm$$

$$k_n = n\pi/a \ , h_m = m\pi/b \ , \text{and}$$

$$C_{mn} = [\, (n\pi/a)^2 + (m\pi/b)^2]^{1/2}$$

and m, n are both odd.

Therefore above the plane $z = 0$ the potential is

$$V(x,y,z) = (16V_o/\pi^2) \sum_{m,\, n\ odd} (1/mn) \sin (n\pi x/a) \sin (m\pi y/b)$$

$$\exp \{ -[(n\pi/a)^2 + (m\pi/b)^2]^{1/2} z \} \qquad (4)$$

7. *In the previous example if in addition to the checkerboard we have an infinite plane at $z = c$ kept at zero potential. Find an expression for the potential in the region $o \leq z \leq c$.*

In this problem the boundary condition is different from that of the previous one in that; the potential is zero at $z = c$, and the solution is confined in a finite region $o \leq z \leq c$. Thus the solution in the z-direction will differ . Let the solution in the z-direction be

$$E_{mn} \exp (C_{mn} z) + F_{mn} \exp (-C_{mn}\, z)$$

This must be zero at $z = c$, so that we get

$$F_{mn} = - \exp (2\, C_{mn}\, c)$$

and E_{mn} is absorbed in A_n, B_n of equation (1) in the previous problem. Hence the solution in the z-direction is

$$\exp (C_{mn}\, z) - \exp [C_{mn}\, (2c-z)]$$

$$= 2 \exp (C_{mn} c) \sinh [C_{mn}\, (c-z)]$$

Thus the pote...l at any point in $o \leq z \leq c$ is,

$$V(x,y,z) = \sum_{m,n} (A_n \cos k_n x + B_n \sin k_n x)(C_m \cos h_m y + D_m \sin h_m y)\, 2 \exp (C_{mn} c) \sinh [C_{mn}(c-z)] \quad (1)$$

Applying the boundary condition at $z = o$ we get the same function given by equation (3) of the previous problem so that after equating with (1) at $z = o$ we get,

$$A_n = 0, C_m = 0, \; k_n = n\pi/a \;, \; h_m = m\pi/b, \; m, n \text{ both odd}$$

and

$$D_m B_n = [8 \; V_o/(\pi^2 m \; n \; \sinh C_{mn} c)] \exp (-C_{mn} \, c)$$

From (1) the potential at any point in the region $o \leq z \leq c$ is,

$$V(x,y,z) = (16 V_o/\pi^2) \; \Sigma \; [1/m \, n \; \sinh (C_{mn} c)] \; \sin (n\pi x/a) \sin (m\pi y/b) \sinh [C_{mn} (c-z)]$$

where,

$$C_{mn} = [(n\pi/a)^2 + (m\pi/b)^2]^{1/2}$$

8. *Find the potential at any point in the space determined by* $0 < x < a$, $o < y < b$, $o < z < c$ *with the boundary conditions*

(*i*) $V = o$ *at* $x = o, a, y = o, b,$ *and* $z = o$.

(*ii*) $V = V_o \, xy$ *at* $z = c$, *and* $o \leq x \leq a$, $o \leq y \leq b$.

A general solution of Laplace's equation suitable for this problem is,

$$V(x,y,z) = \sum_{m,n} (A_n \sin k_n x + B_n \cos k_n x)(C_m \sin h_m y + D_m \cos h_m y)(E_{mn} \cosh C_{mn} z + F_{mn} \sinh C_{mn} z) \quad (1)$$

To satisfy the boundary condition (i) we must have,

$$B_n = D_m = E_{mn} = 0,$$

$$k_n = n\pi/a,\ h_m = m\pi/b, \quad \text{and}$$

$$C_{mn} = (k_n^2 + h_m^2)^{1/2}$$

$$= [(n\pi/a)^2 + (m\pi/b)^2]^{1/2} \tag{2}$$

so that (1) becomes,

$$V(x,y,z) = \sum_{m,n} A_{mn} \sin (n\pi x/a) \sin (m\pi y/b) \sinh (C_{mn} z) \tag{3}$$

The constants A_{mn}, $m,n = 1, 2, \ldots, \infty$ can be determined from the second boundary condition. At $z = c$ the potential given by (2) must be equal to $V_0 xy$, so that we must expand this function using a double Fourier series. Since the function $V_0 xy$ is odd in both x and y a suitable double Fourier series is,

$$V(x,y,c) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} H_{mn} \sin (n\pi x/a) \sin (m\pi y/b) \tag{4}$$

where

$$H_{mn} = 4/ab \int_0^a \int_0^b V \sin (n\pi x/a) \sin (m\pi y/b)\, dx\, dy \tag{5}$$

Equating (3) and (4) at $z = c$ we get,

$$A_{mn} = H_{mn} \operatorname{cosech} (C_{mn} c) \quad (6$$

The coefficient H_{mn} can be determined by substituting $V = V_0 xy$ in (5). Now we have,

$$\int_0^a x \sin (n\pi x/a)\, dx = (-1)^{n+1} (a^2/n\pi)$$

$$\int_0^b y \sin (m\pi y/b)\, dy = (-1)^{m+1} (b^2/m\pi)$$

Then,

$$H_{mn} = (-1)^{m+n}\, 4a\, b\, V_0/(\pi^2\, mn) \qquad (7)$$

Substituting from (6) and (7) in (3) we get,

$$V(x,y,z) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (4abV_0 / \pi^2 mn)\, (-1)^{m+n} \operatorname{cosech}\, (C_{mn}\, c)$$

$$\sin (n\pi x/a)\, \sin\, (m\pi y/b)\, \sinh (C_{mn}\, z)$$

where C_{mn} is given by (2).

9. *An earthed conducting cylinder of radius a and length 2L is closed at both ends by two plates of radius a and potential V_0. (Fig. 10.8). Determine the potential at all points inside the cylinder.*

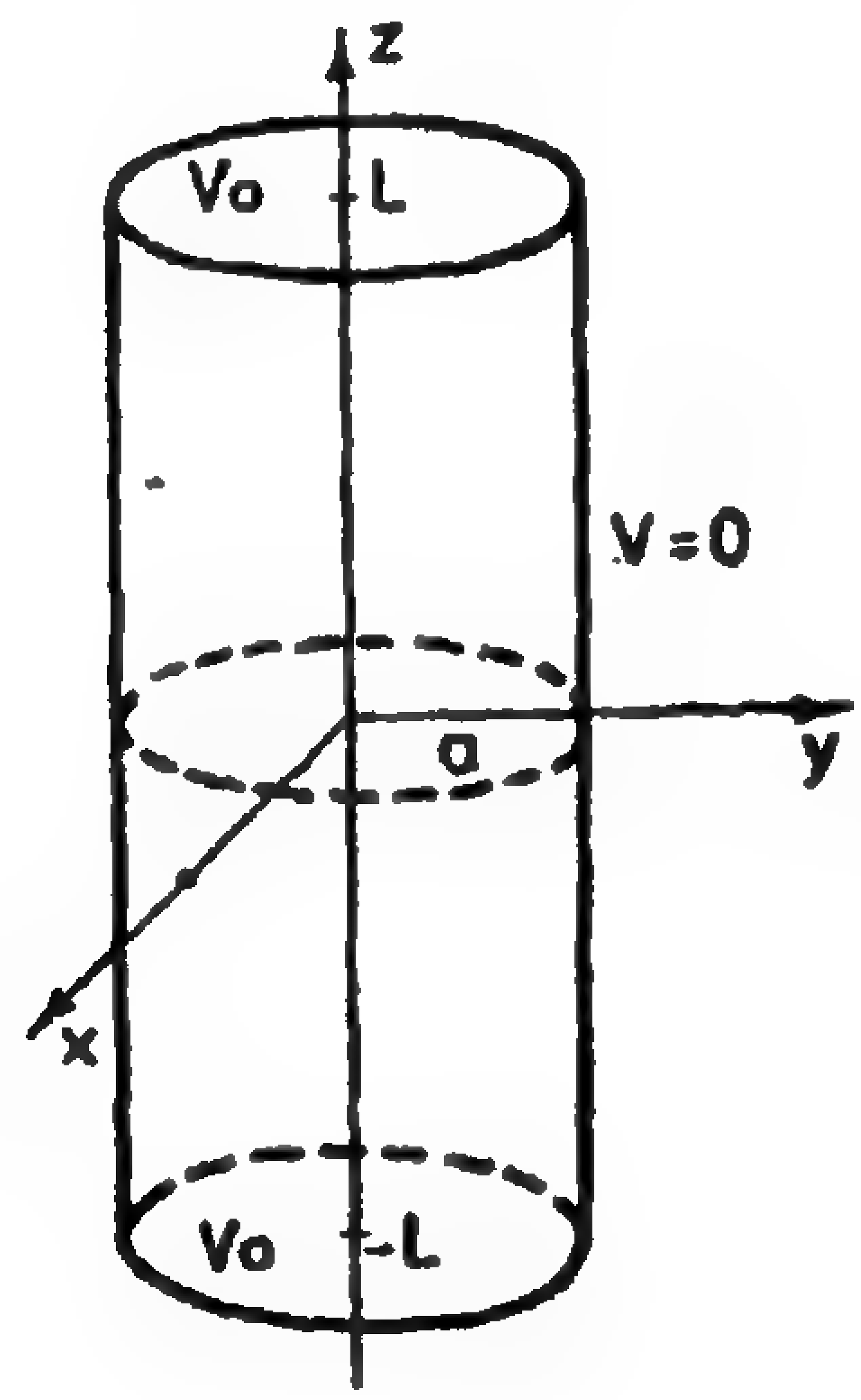


Fig. 10.8.

A solution of Laplace's equation in cylindrical coordinates suitable for this problem is,

$$V = [A' J_p(kr) + B' Y_p(kr)] \, (C \sin p\theta + D \cos p\theta)$$
$$(E \cosh kz + F \sinh kz) \qquad (1)$$

Since the potential on the axis does not go to infinity the solution doe not contain $Y_p(kr)$ since it tends to infinity as r tends to zero. Also we have symmetry about the z-axis, thus $p=o$, and the solution (1) becomes,

$$V = A J_o(kr) \, (\cosh kz + B \sinh kz) \qquad (2)$$

Actually the general solution is a collection of solutions as(2) but with different k,p, that is

$$V = \sum_{m=1}^{\infty} A_m J_o(k_m r) \, (\cosh k_m z + B_m \sinh k_m z) \qquad (3)$$

The boundary conditions at $z = \pm L$ are satisfied if we take $B_m = 0$.

$$V = \sum_{m=1}^{\infty} A_m \, J_o(k_m r) \cosh k_m z$$

at $z = L$, $V = V_o$ so that we have,

$$V = V_o = \sum_{m=1}^{\infty} A_m J_o(k_m r) \cosh(k_m L) \qquad r<a \quad (5)\text{–}a$$

$$V = 0 \qquad r \geq a \quad (5)\text{–}b$$

(5) can be easily satisfied if $J_o(k_m a) = 0$, for $m = 1,2,3,\ldots,\infty$. That is $k_m a$, $m = 1,2,3,\ldots \infty$ are the zeros of $J_o(x)$. The coefficients A_n can be determined by multiplying both sides of (5)–a by $r \, J_o(k_n r)$ and integrating between o and a.

$$V_o \int_0^a r J_o(k_n r) \, dr = \sum_{m=1}^{\infty} A_m \cosh(k_m L) \int_0^a r \, J_o(k_n r)$$
$$J_o(k_m r) \, dr \qquad (6)$$

By using the orthogonal property of Bessel functions; that is if α and β are two roots of $J_o(x)$ we have that,

$$\int_0^1 x J_o(\alpha x)\, J_o(\beta x)\, dx = \begin{cases} o & \alpha \neq \beta \\ \tfrac{1}{2} J_1^2(\alpha) & \alpha = \beta \end{cases}$$

Thus

$$\int_0^a r J_o(k_n r)\, J_o(k_m r)\, dr = \int_0^1 (r/a) J_o(k_n ar/a)\, J_o(k_m ar/a) d(r/a)$$

$$= \begin{cases} o & n \neq m \\ \tfrac{1}{2} J_1^2(k_n a) & n = m \end{cases}$$

and (6) gives

$$A_n = .2V_o\, a \,/\, [k_n \cosh (k_n L)\, J_1 (k_n\, a)]$$

and the potential at any point inside the cylinder is,

$$V = \sum_{m=1}^{\infty} 2V_o\, a\, J_o(k_n r) \cosh (k_m z) \,/\, [k_m \cosh (k_m L)\, J_1 (k_m\, a)]$$

where $k_m\, a$, $m=1,2\ \ldots$ are the roots of $J_o(x)$.

10. *Find an expression for the potential on the interior of the section shown in Fig. 10.9. The potential along the outer circular surface has the distribution indicated in Fig. 10.10 and the potential on all other surfaces is zero.*

The potential at any point on the interior of the section satisfies Laplace's equation, $\nabla^2 V = 0$. A suitable solution in polar coordinates is,

$$V = \Sigma (A_n r^n + B_n r^{-n})\ [\cos (n\theta) + C_n \sin (n\theta)] \qquad (1)$$

Since we have symmetry about the y-axis, $C_n = 0$. Also at $\theta = \pm\pi/2$ the potential is zero, so that the even order coefficients vanish, $A_n = 0$, $B_n = 0$, n even. Then,

$$V = \sum_{n\ odd} (A_n r^n + B_n / r^n) \cos(n\theta) \tag{2}$$

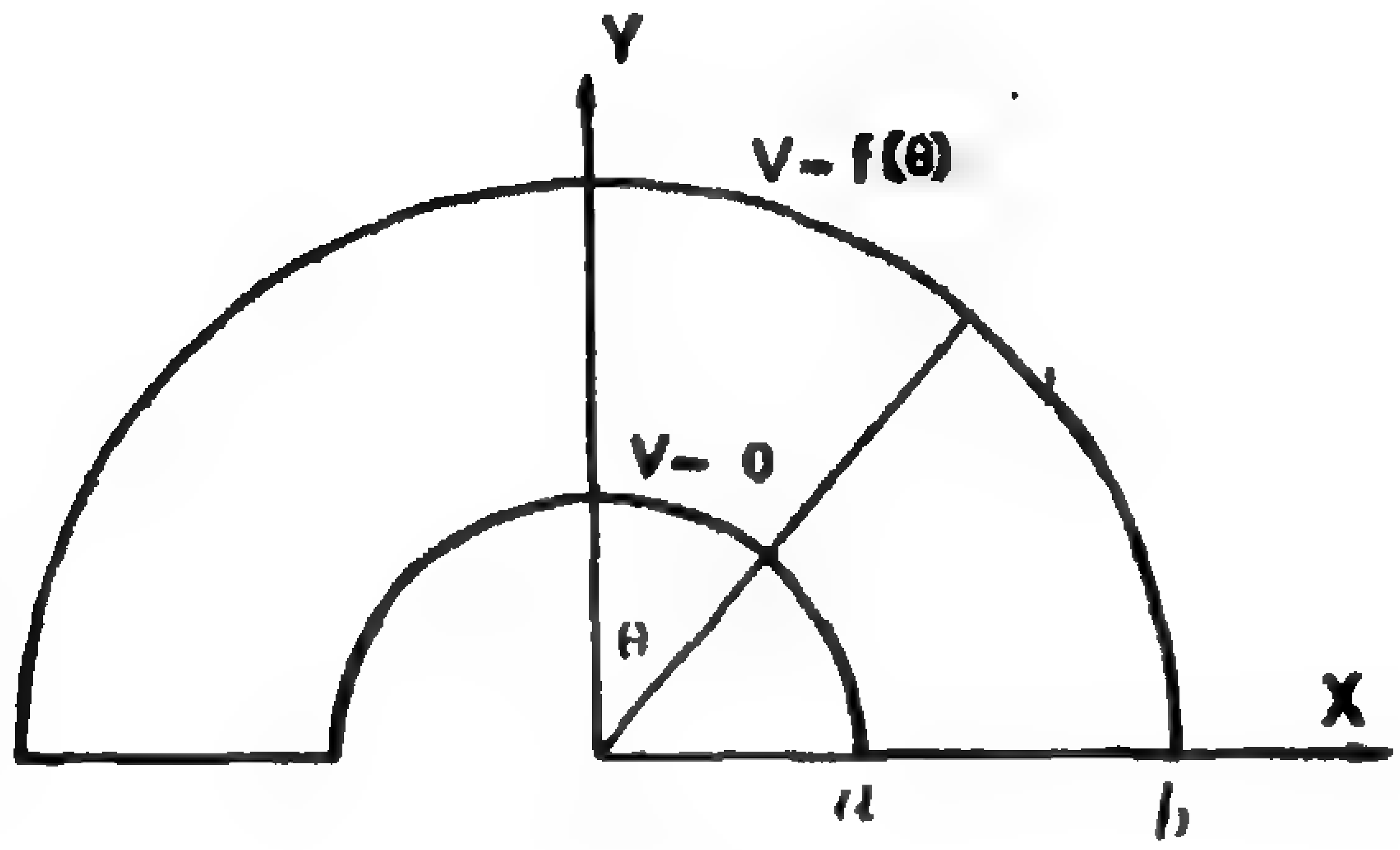


Fig. 10 9.

With the aid of Fig. 10.9, the boundary condition at $r = b$ is,

$$V = V_o (1 - 6\theta/\pi) , 0 < \theta < \pi/6$$

$$= V_o (1 + 6\theta/\pi) , -\pi/6 < \theta < 0$$

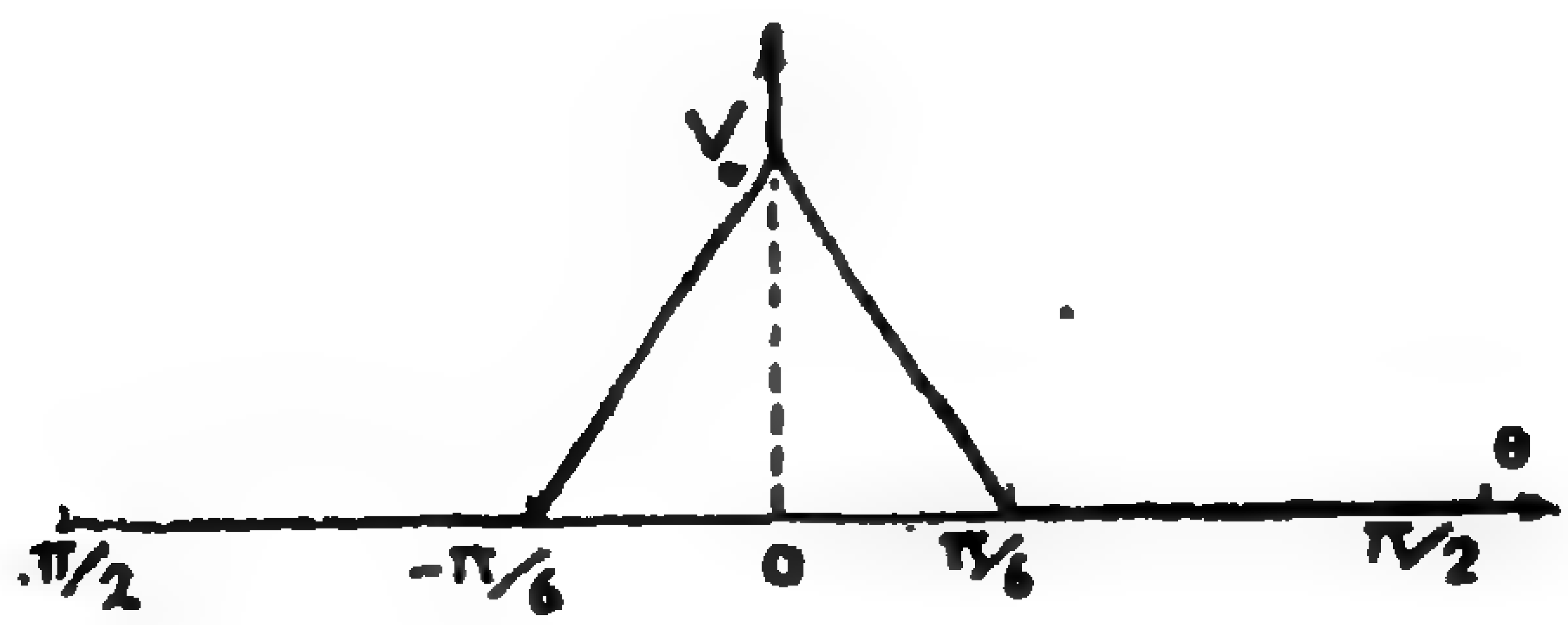


Fig. 10.10.

This can be expanded in a Fourier series of odd orders only,

$$V = \sum_{n\ odd} a_n \cos(n\theta) \tag{3}$$

where,

$$a_n = \frac{24 V_0}{n^2 \pi^2}\left[1 - \cos\frac{n\pi}{6}\right]$$

Now at $r = a$, $V = 0$ for all θ, thus (2) gives,

$$B_n = -a^{2n} A_n$$

Then,

$$V = \sum_{n\ odd} A_n (r^n - a^{2n}/r^n) \cos(n\theta) \tag{4}$$

At $r = b$ (3) and (4) must be identical. This gives,

$$A_n = a_n \,/\, (b^n - a^{2n}/b^n)$$

Thus the potential at any point on the interior of the given section is,

$$V = \sum_{n\ odd} a_n \left[(r^n - a^{2n}/r^n) \,/\, (b^n - a^{2n}/b^n)\right] \cos(n\theta) \tag{5}$$

11. *Find the potential distribution between a four-segment commutator of radius a and a grounded concentric cylinder of radius b, $b>a$. Alternate segments of the commutator are at potentials $+V_0$ and $-V_0$.*

Since we have no variation in the z-direction, the solution of Laplace's equation in cylindrical coordinates is (Appendix II).,

$$V = \sum_{n=0}^{\infty} (A_n r^n + B_n r^{-n})(D_n \cos n\theta + C_n \sin n\theta) \tag{1}$$

when $r = a$, the potential is given by (Fig. 10.11,12).

$$V = V_0 \quad 0<\theta<\pi/2 \quad , \quad V = -V_0 \quad \pi/2 <\theta<\pi$$

$$V = V_0 \quad \pi<\theta<3\pi/2 \quad , \quad V = -V_0 \quad 3\pi/2\leq\theta<2\pi$$

This can be expanded in a Fourier series with period π it contains only sine terms,

$$V = \sum_{n\ odd}^{\infty} (4V_0/\pi n) \sin (2n\theta) \tag{2}$$

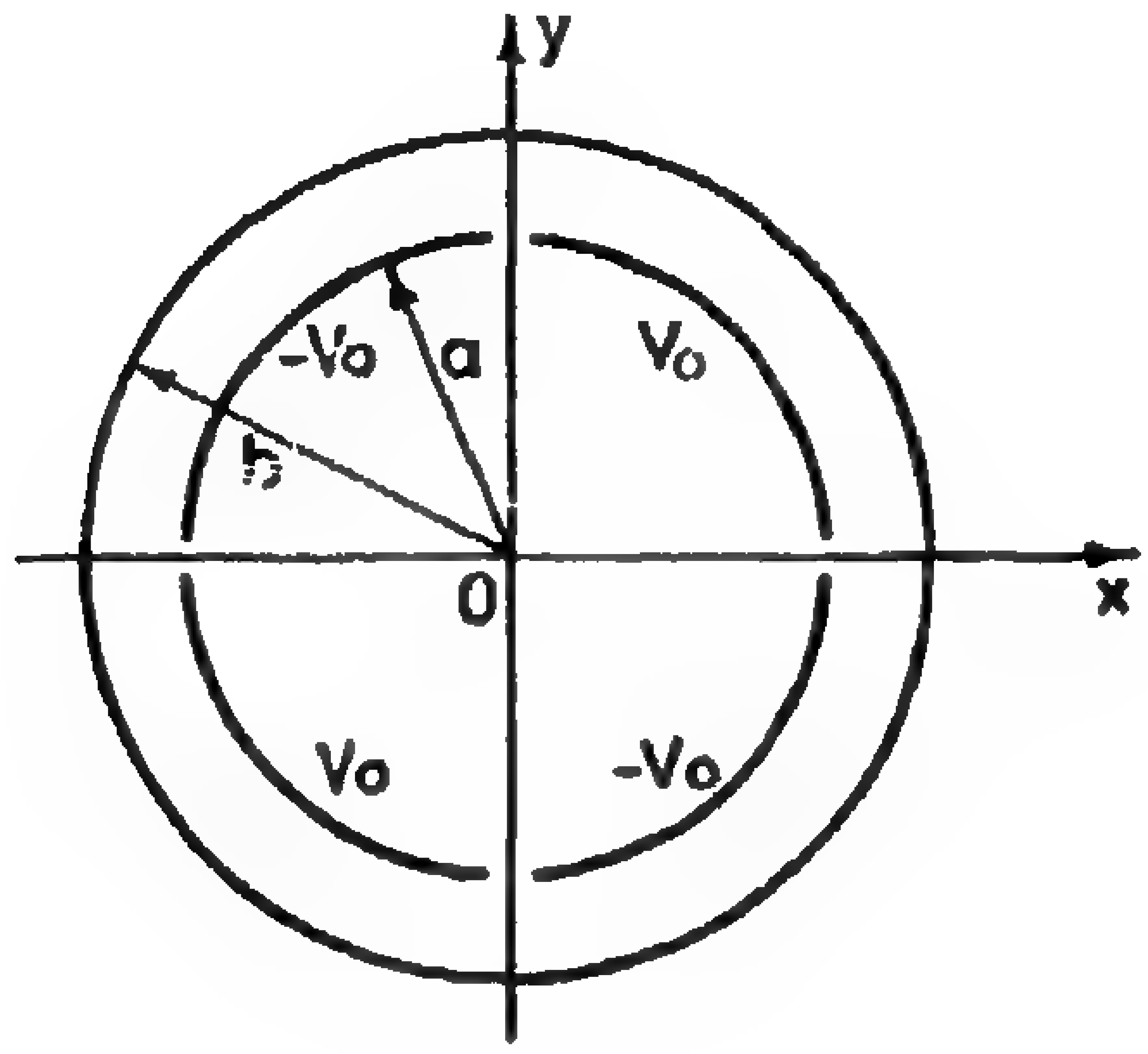


Fig. 10.11.

This suggests a modification of (1) to,

$$V = \sum_{n\ odd} (A_{2n}\, r^{2n} + B_{2n}/r^{2n}) \sin (2n\theta) \qquad a \leq r \leq b \tag{3}$$

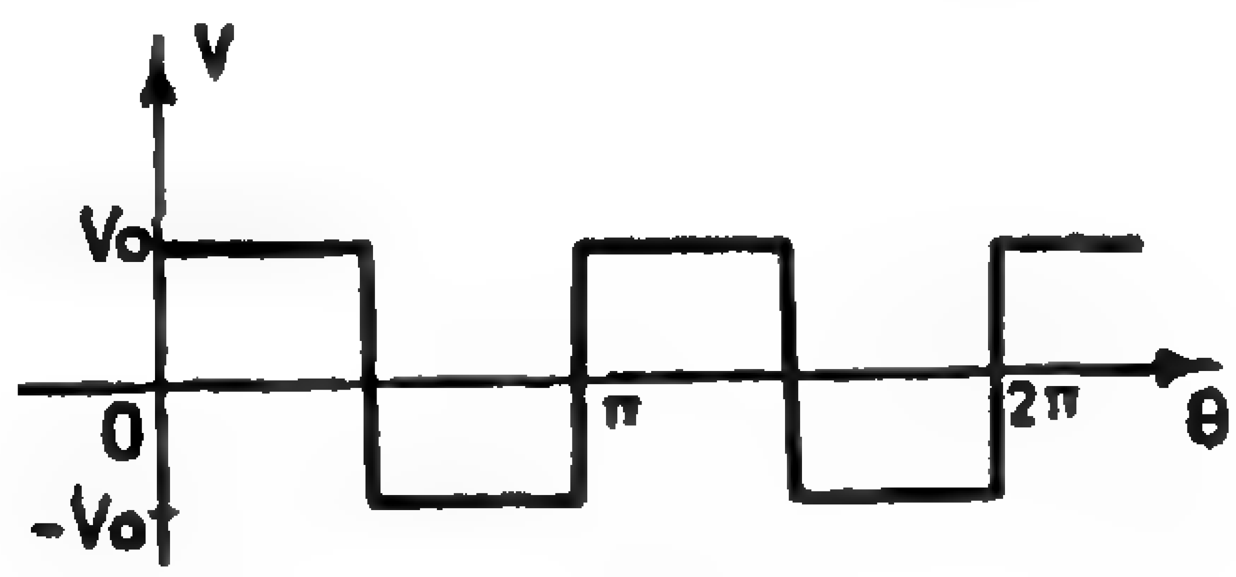


Fig. 10.12.

At $r = b$, $V = 0$ so that (3) gives,

$$A_{2n}\, b^{2n} + B_{2n}\, b^{-2n} = 0$$

i.e. $\qquad B_{2n} = -A_{2n}\, b^{4n}$

Thus (3) becomes,

$$V = \sum_{n \text{ odd}} A_{2n} (r^{2n} - b^{4n} r^{-2n}) \sin (2n\,\theta) \tag{4}$$

At $r = a$, (2) and (4) are identical, hence

$$A_{2n} = 4V_0\, a^{2n}/n\pi\,(a^{4n} - b^{4n})$$

Substituting for A_{2n} in (4) we get the required result.

12. *A grounded conducting cylinder of radius a is perpendicular to a uniform electric field E_0. Find the potential at all points.*

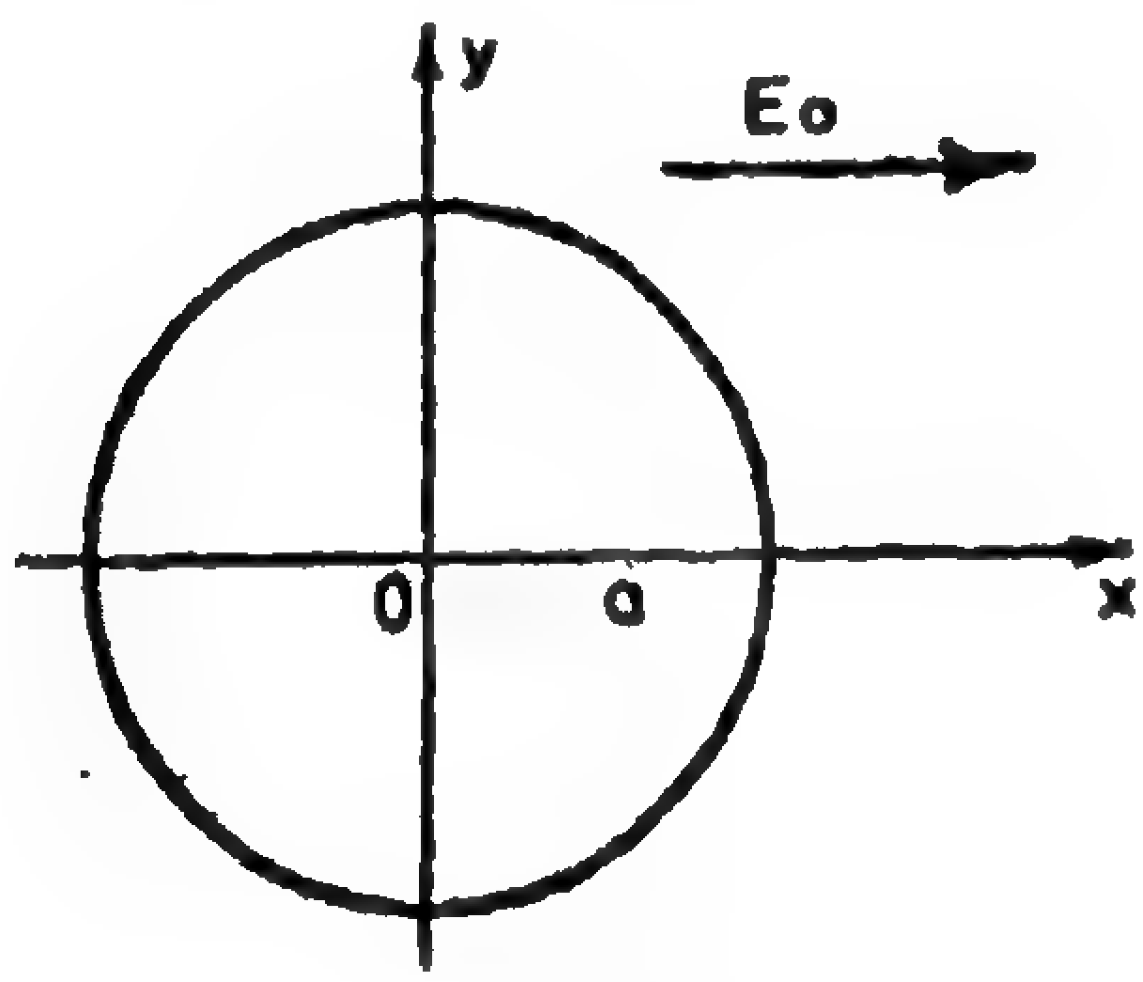


Fig. 10.13.

As shown in Fig. 10.13 we have no z-dependence, the solution of Laplace's equation in this case is (Appendix II)

$$V = \sum_n (A_n r^n + B_n r^{-n}) (C_n \cos n\,\theta + D_n \sin n\,\theta) \tag{1}$$

Since we have symmetry with respect to the x-axis, *i.e.* the potential at (r,θ) is the same as that at $(r,-\theta)$, we must have $D_n = 0$, and (1) becomes,

$$V = \sum (A_n r^n + B_n/r^n) \cos n\theta \tag{2}$$

where C_n is absorbed in A_n and B_n. At large values of r the electric field is E_o so that the potential at large r is,

$$V = -E_o x = -E_o\, r \cos\theta \tag{3}$$

Comparing (2) and (3) at large r we get,

$$A_1 = -E_o\,,\; A_n = 0,\; n = 0,2,3,4,\ldots$$

Thus (2) becomes

$$V = -E_o r \cos\theta + \sum_{n=o}^{\infty} B_n r^{-n} \cos n\theta \tag{4}$$

Also $V = 0$ when $r = a$ so that (4) gives,

$$0 = -E_o\, a \cos\theta + \sum_{n=o}^{\infty} B_n\, a^{-n} \cos n\theta$$

This gives $B_o = 0$, $B_1 = a^2 E_o$, $B_n = 0$, $n = 2, 3, \ldots$

Thus the required potential is

$$V = -E_o\,(r - a^2/r) \cos\theta\,, \qquad r \geq a \tag{5}$$

13. *An infinitely long cylinder of radius a and magnetic permeability μ is placed with its axis perpendicular to a, magnetic field* $\mathbf{H}_0$; *find the magnetic field produced by the induced magnetization in the magnetic cylinder.*

Let the axis of the cylinder coincide with the z-axis and the field $\mathbf{H}_o$ be along the x-axis. The induced magnetic scalar potential, which is only due to the induced magnetization in the cylinder, satisfies Laplace's equation. A suitable solution is,

$$V_1 \text{ induced} = \sum_{1}^{\infty} r^n A_n \cos(n\theta) \qquad r \leq a \tag{1}$$

$$V_2 \text{ induced} = \sum_{1}^{\infty} r^{-n} B_n \cos(n\theta) \qquad r \geq a \tag{2}$$

Note that the $n=o$ term has a constant term which adds nothing to the field while the logarithmic term gives a singularity at $r=o$. The total external magnetic potential is,

$$V_2 = V_2 \textit{ induced} + V \textit{ external} \qquad (3)$$

$$V \textit{ external} = -H_o\, r \cos\theta \qquad (4)$$

which corresponds to a uniform field along the x-axis. The constants A_n and B_n of (1) and (2) can be determined from the boundary conditions at $r = a$.

(i) The induced scalar potential is continuous.

$$V_1 \textit{ induced} = V_2 \textit{ induced}$$

This gives,

$$B_n = a^{2n} A_n \qquad (5)$$

(ii) The normal component of **B** must be continuous at $r=a$.

$$\mu\, \partial V_1/\partial r = \mu_o\, \partial V_2/\partial r$$

that is,

$$\mu \,\Sigma\; n\, a^{n-1} A_n \cos(n\theta) - \mu\, H_o \cos\theta$$

$$= \mu_o \,\Sigma\; (-n) a^{-n-1} B_n \cos(n\theta) - \mu_o H_o \cos\theta$$

This gives,

$$\mu\, n\, a^{n-1} A_n = -\mu_o\, n\, a^{-n-1} B_n\,, \qquad n \neq 1 \qquad (6)$$

$$\mu\, A_1 = (\mu - \mu_o) H_o - \mu_o B_1 a^{-2} \qquad (7)$$

Equations (5)—(7) give,

$$A_n = B_n = 0\,,\; n \neq 1$$

$$A_1 = (\mu-\mu_o)\, H_o \,/\, (\mu+\mu_o)$$

$$B_1 = (\mu-\mu_o)\, a^2 H_o \,/\, (\mu+\mu_o)$$

so that we have

$$V_1 = -2\mu_o H_o r \cos\theta / (\mu+\mu_o) \qquad r \leq a$$

$$V_2 = -H_o r \cos\theta + (\mu-\mu_o) a^2 H_o \cos\theta / r (\mu+\mu_o) \qquad r \geq a$$

The magnetic induction field **B** is,

$$\mathbf{B} = 2\mu_o \mu H_o / (\mu+\mu_o) \ \mathbf{a}_x \qquad r \leq a$$

$$= \mu_o H_o \mathbf{a}_x + \mu_o (\mu-\mu_o) H_o a^2/r^2 (\mu+\mu_o)$$

$$[\cos\phi \ \mathbf{a}_r + \sin\phi \ \mathbf{a}_\theta] \qquad r \geq a$$

Inside the cylinder the induced magnetization can be obtained from the relation,

$$\mathbf{B} = \mu_o (\mathbf{M}+\mathbf{H})$$

$$\mathbf{M} = \mathbf{B}/\mu_o - \mathbf{H} = (\mu-\mu_o)\mathbf{B}/\mu\mu_o$$

$$= 2 [(\mu-\mu_o) H_o / (\mu+\mu_o)] \ \mathbf{a}_x$$

14. *An infinite line charge of strength λ is placed at distance b parallel to the axis of an infinitely long conducting cylinder of radius a. If the cylinder is earthed show that the potential at any point outside the cylinder is,*

$$\Psi = +(\lambda/2\pi\epsilon) [A \log r + B + \log R + \Sigma (a^{2n}/nb^n r^n) \cos(n\theta)]$$

where A and B are constants, and R is the distance between the line charge and the point at which the potential is considered. Show that the above expression can be written in the form,

$$\Psi = +(\lambda/2\pi\epsilon) [\log (r/b) + \log R - \log R']$$

where R′ is the distance from the inverse of the given line charge.

The potential at point $P(r, \theta)$, Fig. 10.14, due to the line charge only is,

$$\Psi_1 = \int (\lambda/2\pi \epsilon R)\, dR = (\lambda/2\pi \epsilon)\,(\log R + C) \qquad (1)$$

The potential at point P due to the induced charges on the cylinder satisfies Laplace's equation $\nabla^2 \Psi_2 = 0$. A suitable solution in cylindrical coordinates is, (Appendix II)

$$\Psi_2 = (\lambda/2\pi \epsilon) \sum_{1}^{\infty} A_n r^{-n} \cos(n\theta) + A' \log r + B' \qquad (2)$$

Hence the total potential is,

$$\Psi = (\lambda/2\pi \epsilon)\left[A \log r + B + \log R + \sum_{1}^{\infty} A_n r^{-n} \cos(n\theta)\right] \quad (3)$$

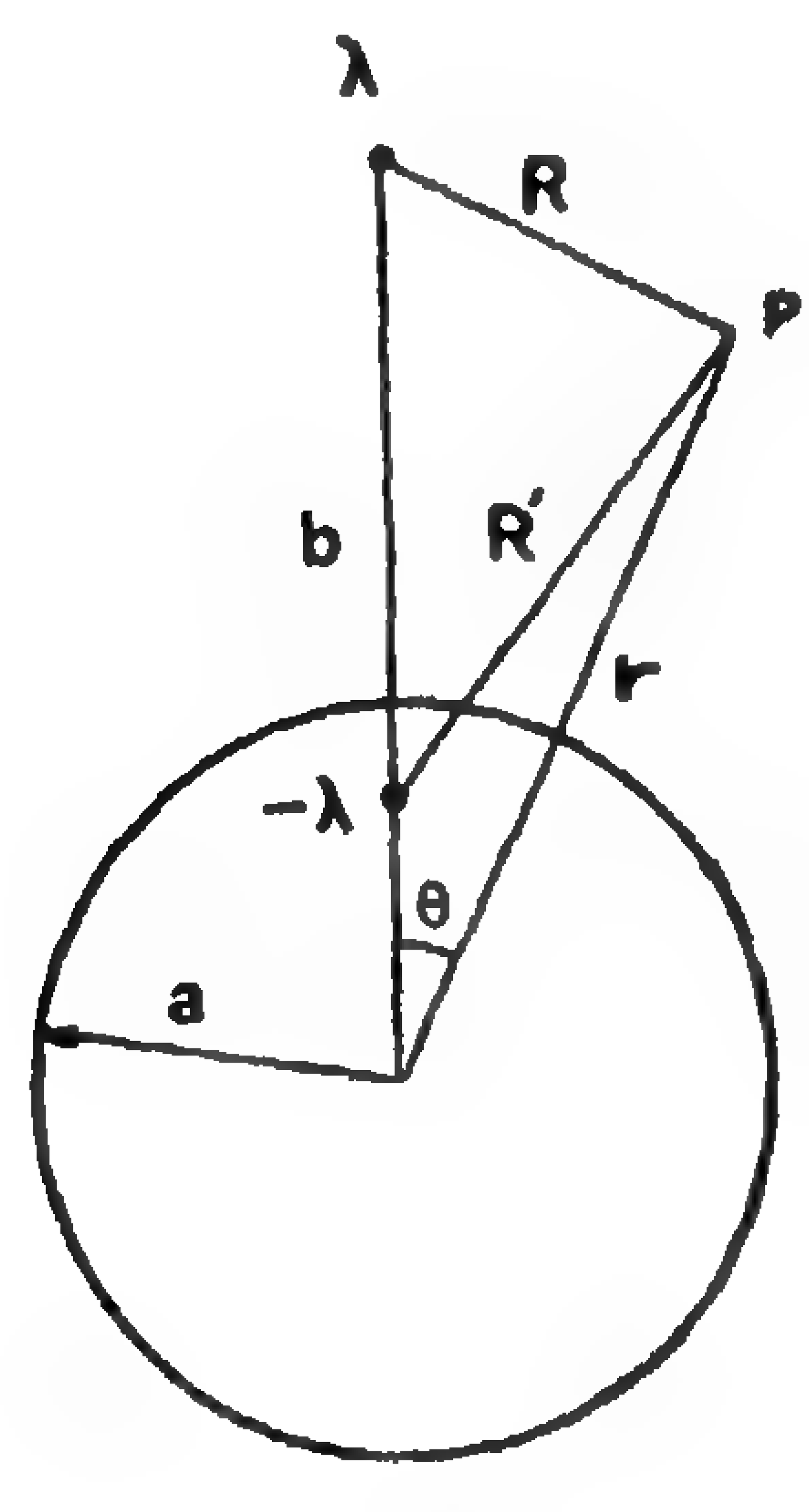


Fig. 10.14.

From Fig. 10.14.

$$R = (r^2 + b^2 - 2\, r\, b \cos\theta)^{1/2}$$

$$= b\,[1 + r^2/b^2 - 2\,(r/b) \cos\theta]^{1/2}$$

$$= b\,[1 - \exp(j\theta)\,(r/b)]^{1/2}\,[1 - \exp(-j\theta)\,(r/b)]^{1/2}$$

so that we have,

$$\log R = \log b + \tfrac{1}{2} \log [1 - \exp(j\theta)(r/b)]$$
$$+ \tfrac{1}{2} \log [1 - \exp(-j\theta)(r/b)]$$

Expanding the last two log terms we get,

$$\log R = \log b - \sum_{1}^{\infty} [(r/b)^n/n] \cos(n\theta) \quad , \quad r \leq b \tag{4}$$

The constant A_n can be determined from the boundary condition at $r = a$, namely, $\Psi = 0$ at $r = a$. From (3) and (4) we have,

$$A \log a + B + \log b - \sum^{\infty} [(a/b)^n/n)] \cos(n\theta) + \sum^{\infty} A_n a^{-n}$$
$$\cos(n\theta) = 0$$

$$A \log a + B + \log b = 0$$

$$A_n a^{-n} = a^n/n\, b^n$$

Thus we have, $B = -\log b - A \log a$, $A_n = a^{2n}/(n\, b^n)$. Substituting in (3) we get

$$\Psi = (\lambda/2\pi\epsilon) \left\{ A \log r + B + \log R + \sum_{1}^{\infty} [a^{2n}/(n\, b^n r^n)] \right.$$
$$\left. \cos(n\theta) \right\} \tag{5}$$

From Fig. 10.13.

$$R' = [r^2 + a^4/b^2 - 2\,(ra^2/b) \cos\theta]^{1/2}$$
$$= r\,[1 + a^4/b^2r^2 - 2\,(a^2/br) \cos\theta]^{1/2}$$
$$= r\,[1 - \exp(j\theta)(a^2/rb)]^{1/2}\, [1 - \exp(-j\theta)(a^2/rb)]^{1/2}$$

$$\log R' = \log r - \sum_{1}^{\infty} [a^{2n}/(n\, r^n b^n)] \cos(n\theta)$$

Thus (5) reduces to,

$$\Psi = (\lambda/2\pi\epsilon)\,[(A+1)\,log\ r+B+log\ R-log\ R']$$

Substituting for B,

$$\Psi = (\lambda/2\pi\epsilon)\,[log\ (r/b) + log\ R-log\ R' + A\ log\ (r/a)]$$

Taking $A = 0$ we arrive at the required result.

15. *A dielectric sphere of radius a and relative permittivity* ϵ_r *is placed in a uniform field* E_0, Fig. 10.15. *Find the potential at any point.*

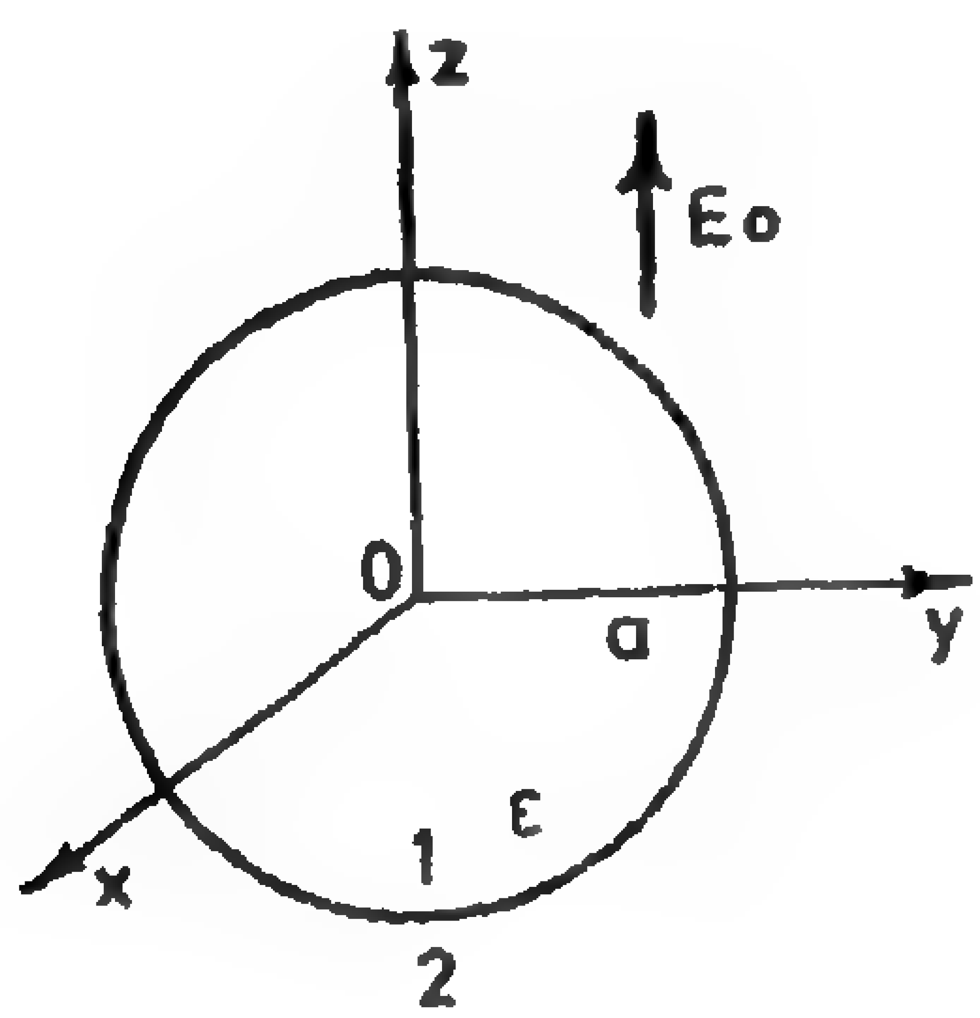


Fig. 10.15.

Suitable solutions to Laplace's equation inside and outside the sphere are,

$$V_1 = \Sigma\ (A_n\, r^n + B_n\, r^{-n-1})\ P_n\,(cos\ \theta) \tag{1}$$

$$V_2 = \Sigma\ (C_n\, r^n + D_n\, r^{-n-1})\ P_n\,(cos\ \theta) \tag{2}$$

In region 1 the potential is finite at origin, $B_n = 0$, while in 2 the potential at large distance r is,

$$V_2 = -E_0\, z = -E_0\, r\ cos\ \theta \tag{3}$$
$$= \Sigma\ C_n\, r^n\ P_n\,(cos\ \theta)$$

This gives, $C_1 = -E_o$, $C_n = 0$, $n \neq 1$. Thus (1) and (2) reduce to,

$$V_1 = \Sigma \; A_n \, r^n \, P_n \, (\cos \theta) \tag{4}$$

$$V_2 = -E_o \, r \cos \theta + \Sigma \, D_n \, r^{-n-1} \, P_n \, (\cos \theta) \tag{5}$$

The coefficients A_n and D_n are obtained from the boundary conditions at $r = a$, namely,

(i) $V_1 = V_2$, and (ii) $\epsilon_r \, \partial V_1/\partial r = \partial V_2/\partial r$

The first boundary condition gives,

$$A_o = D_o/a \quad , \quad A_1 = -E_o + C_1/a^3 \tag{6}$$

$$A_n = a^{-2n-1} \; C_n \; , \quad n>1 \tag{7}$$

The second boundary condition gives,

$$\epsilon_r A_1 = -E_o - 2 \, C_1/a^3 \tag{8}$$

$$\epsilon_r A_n = -[(n+1)/n] \, a^{-2n-1} \, C_n \, , \quad n>1 \tag{9}$$

Equations (7) and (9) are satisfied simultaneously if $A_n = C_n = 0$, $n>1$, $A_o = C_o = 0$. Also (6) and (8) give,

$$C_1 = (\epsilon_r - 1) \; E_o \; a^3/(\epsilon_r + 2) \tag{10}$$

$$A_1 = -3 \; \epsilon_r \; E_o/(\epsilon_r + 2) \tag{11}$$

Thus we have that,

$$V_1 = -[3\epsilon_r/(\epsilon_r+2)] \; E_o \; r \cos \theta \; , \qquad r < a \tag{12}$$

$$V_2 = -E_o \; r \cos \theta + [(\epsilon_r - 1)/(\epsilon_r + 2)] \; E_o \, a^3 \, r^{-2} \cos \theta \tag{13}$$

16. *In the previous problem calculate the density of bound charges on the sphere and the net force acting on it due to its existence in the electric field* E_o.

The polarized dielectric can be replaced by a charge distribution of volume density $\rho = \nabla .\mathbf{P}$ and surface density $\sigma = \mathbf{P}.\mathbf{n}$ over the surface of the dielectric sphere, where,

$$\mathbf{P} = \epsilon_0 \epsilon_r E_1 - \epsilon_0 E_1 = \epsilon_0 (\epsilon_r - 1) E_1$$

$$= -\epsilon_0 (\epsilon_0 - 1) \nabla V_1$$

Thus we have that,

$$\nabla .\mathbf{P} = -\epsilon_0 (\epsilon_r - 1) \nabla^2 V_1 = 0$$

thus no volume distribution of charge exists. Also

$$\sigma_b = \mathbf{P}.\mathbf{n} = -\epsilon_0 (\epsilon_r - 1) \nabla V_1 . \mathbf{n}$$

But we have that

$$\nabla V_1 = -[3\epsilon_r E_0 / (\epsilon_r + 2)] \; [\cos\theta \, \mathbf{a}_r - (\sin\theta / r) \, \mathbf{a}_\theta]$$

Now we have, $\mathbf{a}_r.\mathbf{n} = 1$, and $\mathbf{a}_\theta.\mathbf{n} = 0$,

$$\sigma_b = [\epsilon_0 \epsilon_r (\epsilon_r - 1) E_0 / (\epsilon_r + 2)] \cos\theta$$

The net force acting on the dielectric sphere is,

$$F = \int (\sigma_b{}^2 / 2\epsilon_0) \cos\theta \, d S$$

$$= (2\pi a^2 / 2 \epsilon_0) \int_0^\pi \sigma_b{}^2 \cos\theta \sin\theta \, d\theta$$

Substituting for σ_b we get, $F = 0$ *i.e.* the net force acting on the sphere in the z-direction is zero.

17. *A conducting sphere of radius a and potential V_0 is set in a uniform field E_0. Find an expression for the force acting on this sphere.*

As in the previous problem the uniform field E_0 is represented by potential $V = -E_0 \cdot r \cos\theta$, Fig. 10.15. The solution of Laplace's

equation outside the sphere is thus,

$$V = -E_o\, r \cos\theta + \Sigma\; B_n\, r^{-n-1}\, P_n(\cos\theta) \qquad (1)$$

At $r = a$, $V = V_o$ so that we get,

$$B_o = a\, V_o\,, \quad B_1 = a^3 E_o\,, \quad \text{and } B_n = 0, \qquad n > 1$$

Thus (1) reduces to,

$$V = V_o\, a/r - E_o\,(r - a^3/r^2)\cos \qquad (2)$$

The surface charge density at any point on the sphere is,

$$\sigma = -\epsilon_o\, \partial V/\partial r \quad \text{at } r = a$$

$$= \epsilon_o\, V_o/a + 3\;\epsilon_o E_o \cos\theta \qquad (3)$$

The force acting on the conducting sphere is,

$$F = \int_0^{\pi} (\sigma^2/2\epsilon_o)\, 2\pi\, a^2 \sin\theta \cos\theta\, d\theta$$

in the z-direction. Substituting for σ from (3) and integrating we get,

$$\mathbf{F} = 4\;\epsilon_o a V_o E_o\, \mathbf{a}_z$$

18. *An earthed metal sphere of radius a is placed in an electrostatic field that was* $V = \sum_0^{\infty} A_n\, r^n\, P_n$ *until the sphere was introduced with center at origin. Show how the field is affected by the sphere. Also find an expression for the force acting on the sphere. If the initial field was* $\mathbf{E} = 2\, a^2\, E_o\, \mathbf{a}_z$, *deduce the value of this force.*

After introducing the conducting sphere the field is distorted but still satisfies Laplace's equation. Thus an adequate solution to this equation is

$$V = \sum_0^{\infty} (A'_n\, r^n + B'_n\, r^{-n-1})\, P_n(\cos\theta) \qquad r \geq a \qquad (1)$$

The coefficients A'_n and B_n' are obtained from the boundary conditions, namely,

$$\text{(i)} \quad V = o \ \text{at} \ r = a, \qquad \text{(ii)} \ V = \sum_{0}^{\infty} A_n r^n P_n (\cos\theta) \ \text{at large } r$$

Thus we get, $B_n' = -a^{2n+1} A_n'$ and $A_n' = A_n$ for all n.

$$V = \sum_{0}^{\infty} A_n (r^n - a^{2n+1} r^{-n-1}) P_n (\cos\theta) \tag{2}$$

Equation (2) gives the potential after introducing the metallic sphere. To find the force on the conducting sphere we must first find an expression for the surface charge density on it.

$$\sigma = -\epsilon_0 (\partial V/\partial r) \ \text{at} \ r = a$$

$$= -\epsilon_0 \sum_{0}^{\infty} (2n+1) a^{n-1} A_n P_n (\cos\theta) \tag{3}$$

The force per unit area at the point $A(r, \theta)$ on the sphere is $\sigma^2/2\epsilon_0$. Thus the total force on the sphere is,

$$F_t = (½ \epsilon_0) \int \int \sigma^2 \, dS$$

$$= (\pi a^2 \epsilon_0) \int_0^{\pi} [\sum_{0}^{\infty} (2n+1) a^{n-1} A_n P_n]^2 \cos\theta \sin\theta \, d\theta$$

$$= (\pi a^2 \epsilon_0) \int_{-1}^{1} [\sum_{0}^{\infty} (2n+1) a^{n-1} A_n P_n]^2 P_1 \, d\mu$$

$$= \pi a^2 \epsilon_0 \int \Big\{ \sum_{0}^{\infty} (2n+1)^2 a^{2n-2} A_n^2 P_n^2$$

$$+ 2 \sum_{0}^{\infty} \sum_{0}^{\infty} (2n+1)(2m+1) a^{n+m-2} P_n P_m \Big\} P_1 \, d\mu$$

Using the three term recurrence formula and the orthogonal property of the Legendre polynomials, namely,

$$(2n+1)\, P_1 P_n = n\, P_{n-1} + (n+1)\, P_{n+1}$$

$$\int_{-1}^{1} P_n P_m \, d\mu = \begin{cases} 0 & n \neq m \\ 2/(2n+1) & n = m \end{cases}$$

$$(2n+1) \int_{-1}^{1} P_n^2 P_1 \, d\mu = n \int_{-1}^{1} P_n P_{n-1} \, d\mu + (n+1) \int_{-1}^{1} P_{n+1} P_n d\mu$$
$$= 0$$

$$(2n+1) \int_{-1}^{1} P_1 P_n P_m \, d\mu = (n+1) \int_{-1}^{1} P_{n+1} P_m d\mu + n \int_{-1}^{1} P_{n-1} P_m \, d\mu$$

If $m = n-1$ or $n+1$, integrals of the form

$$(2n+1) \int_{-1}^{1} P_1 P_n P_{n-1} d\mu$$

will not be zeros. Thus

$$F_t = 2\pi a^2 \epsilon_0 \sum_{0}^{\infty} \int_{-1}^{1} (2n+1)(2n-1)\, a^{2n-3}\, P_n P_{n-1}\, P_1 \, d\mu$$

$$= 4\pi \epsilon_0 \sum_{1}^{\infty} n a^{2n-1} \; A_n A_{n-1} \tag{4}$$

From the given electric field,

$$\begin{aligned} dV &= \nabla V . d\mathbf{r} \\ &= -\mathbf{E} . d\mathbf{r} \\ &= -2a^2 \, E_0 \, dz \end{aligned}$$

$$V = -2a^2 E_0 z = -2a^3 E_0 r \cos\theta = -2a^3 E_0 r P_1(\cos\theta)$$

Comparing with the given potential we have that,

$$A_1 = -2a^2 E_0 \quad , \quad A_n = 0 \quad \text{for } n \neq 1$$

Substituting in the force expression we conclude that the force on the sphere is zero.

19. *A metal sphere is placed in an electrostatic field that was uniform until the sphere was introduced. How is the field distorted by the sphere ?*

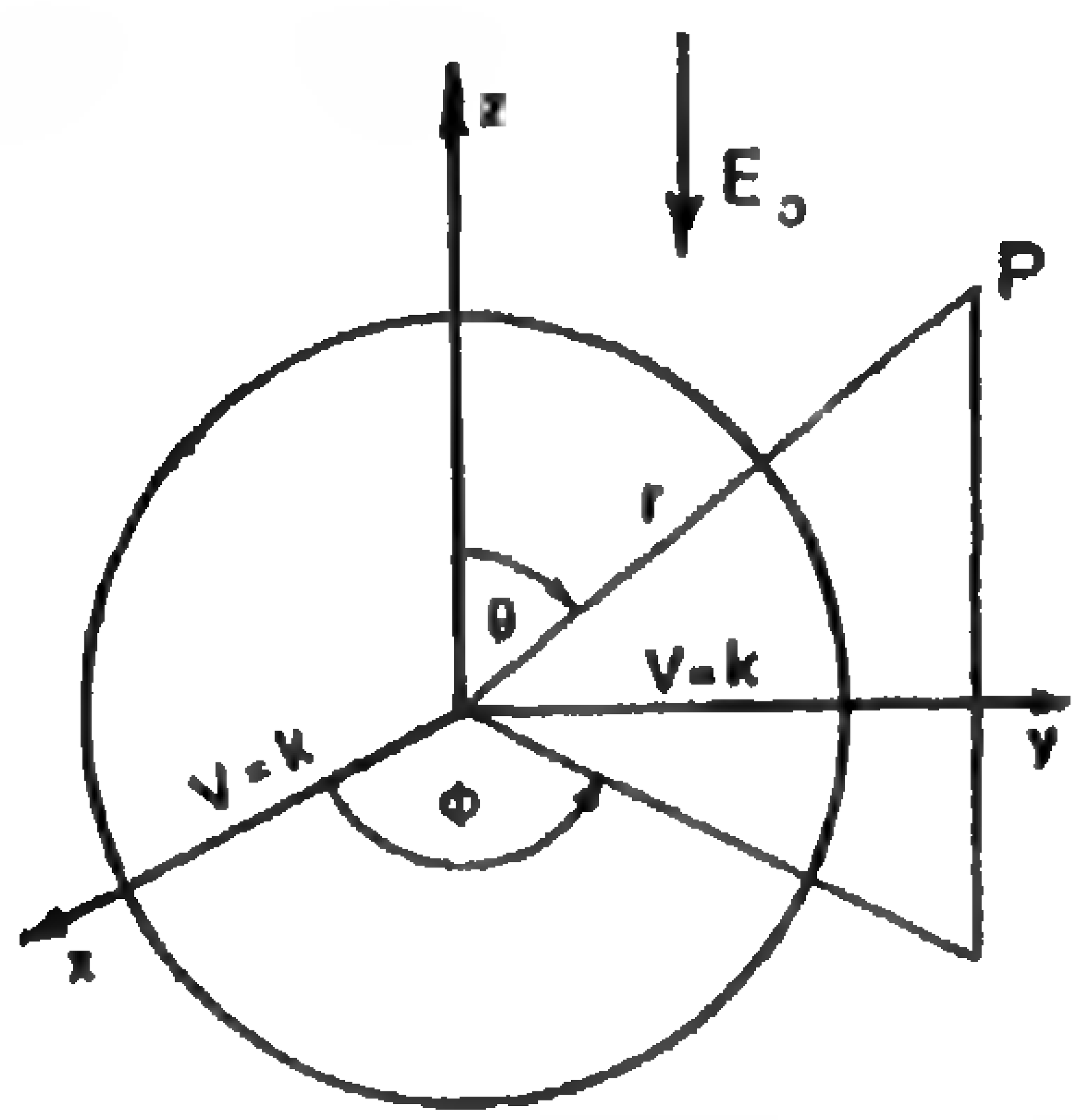


Fig. 10.16.

As shown in Fig. 10.16 the original field is taken in the negative z-direction. The potential of the sphere is ϕ_s. At large distances from the sphere the field is uniform and,

$$\mathbf{E} = \mathbf{E}_0 \tag{1}$$

$$V = E_0 z + K = E_0 r \cos\theta + K \tag{2}$$

where K is the value of the potential at $z = 0$ in the undistorted field. Since we have symmetry the solution of Laplace's equation at any

point outside the sphere is,

$$V = \Sigma \; (A_n r^n + B_n r^{-n-1}) \; P_n \; (\cos \theta) \tag{3}$$

At large values of r this must be equal to the potential of equation (2) so that,

$$E_o \; r \cos \theta + K = \Sigma A_n \; r^n \; P_n \; (\cos \theta)$$

which gives,

$$A_o = K, \; A_1 = E_o, \; A_n = 0, \; n \neq 0, 1$$

so that (3) becomes

$$V = K + E_o r \cos \theta + \sum_{n=1}^{\infty} B_n \; r^{-n-1} \; P_n \; (\cos \theta) \tag{4}$$

The constants B_n, $n=1, 2, 3, \ldots$ can be determined from the boundary condition at $r = a$, namely

$$V = \phi_s \text{ at } r = a$$

using (4)

$$\phi_s = K + E_o \; a \cos \theta + \sum_{n=1}^{\infty} B_n \; a^{-n-1} \; P_n \; (\cos \theta)$$

Then,

$$(K - \phi_s + B_o/a) + (E_o a + B_1/a^2) \cos \theta + (B_2/a^3) \; P_2 (\cos \theta) + \ldots = 0$$

Since this must hold for any θ, the sum can be zero in general only if each coefficient is zero. Therefore,

$$B_o = a \; (\phi_s - K), \quad B_1 = -E_o \, a^3, \quad B_n = 0, \; n > 1. \tag{5}$$

Thus the potential at points exterior to the sphere becomes,

$$V = \phi_s a/r + K(1 - a/r) + E_o \, r \; [1 - (a/r)^3] \cos \theta, \quad r \geq a \tag{6}$$

For the special case $K = \phi_s$, (6) reduces to,

$$V = \phi_s + E_o\, r \cos\theta\, [1-(a/r)^3] \tag{7}$$

This field distribution is plotted in Fig. 10.17. The electric field strength is given by

$$\mathbf{E} = -(\partial V/\partial r)\,\mathbf{a}_r - (\partial V/r\,\partial\theta)\,\mathbf{a}_\theta$$

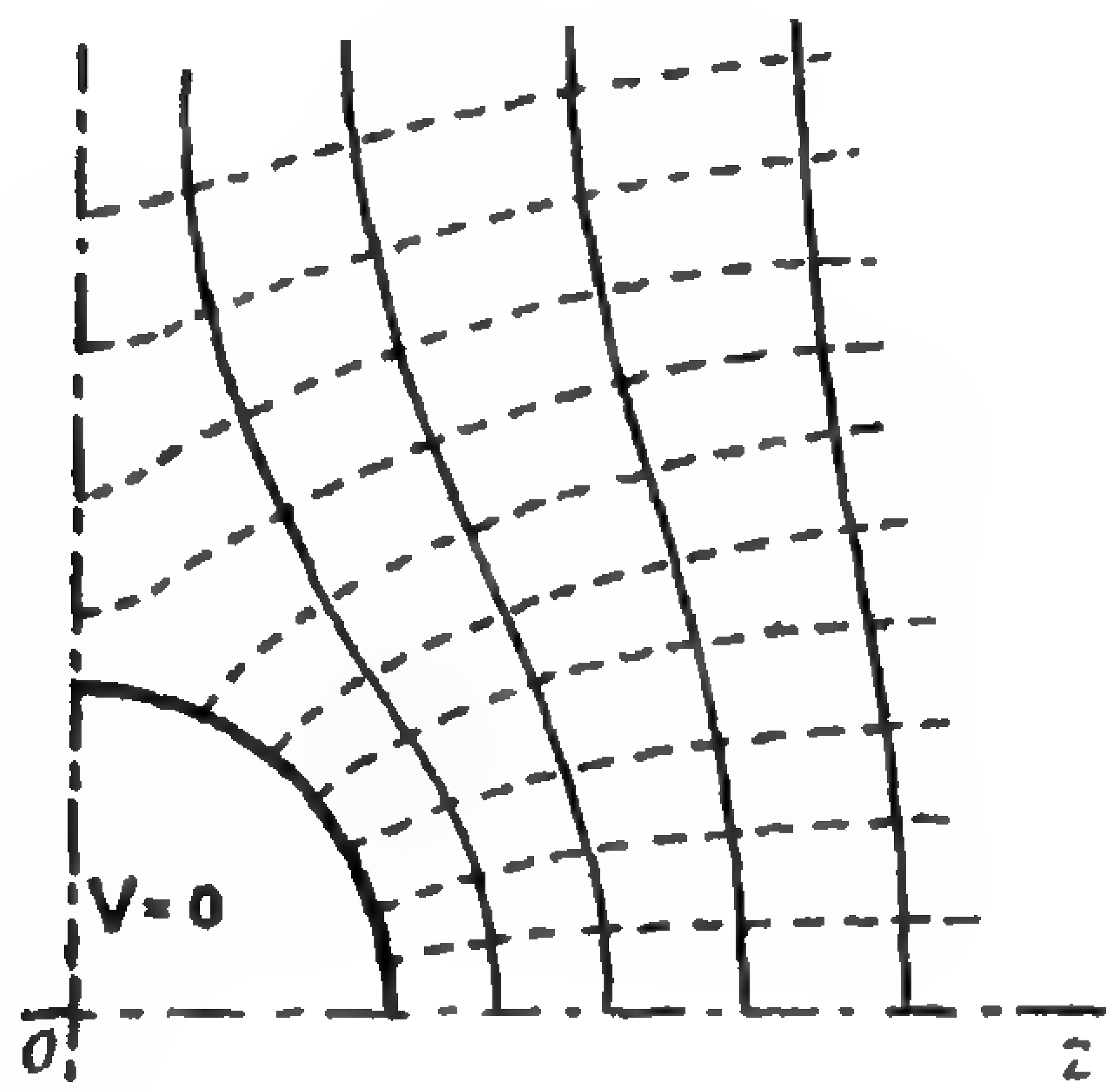


Fig. 10.17.

from (6),

$$\mathbf{E} = -\left\{(K-\phi_s)\,a/r^2 + E_o\,[1+2\,(a/r)^3]\cos\theta\right\}\mathbf{a}_r$$
$$+\;E_o\,[1-(a/r)^3]\sin\theta\;\mathbf{a}_\theta \tag{8}$$

while from (7),

$$\mathbf{E} = -E_o\,[1+2(a/r)^3]\cos\theta\,\mathbf{a}_r + E_o\,[1-(a/r)^3]\sin\theta\,\mathbf{a}_\theta \tag{9}$$

Maximum field strength occurs near the surface of the sphere. When $r = a$, only the radial field remains

$$E_r = (K-\phi_s)/a + 3\,E_o\cos\theta \tag{10}$$

If $K = \varphi_s$

$$E_r = 3\,E_o \cos\theta \tag{11}$$

This indicates that the maximum field strength occurs at the top and bottom of the sphere and the maximum value of E is exactly three times the undistorted field E_o, regardless of the size of the sphere,

$$E_{max} = 3\,E_o \tag{12}$$

20. *A block of cast brass carries a current in the negative z–direction. The presence of an air bubble in the molten metal left a spherical cavity of radius a in the casting. In what manner does this cavity distort the field.*

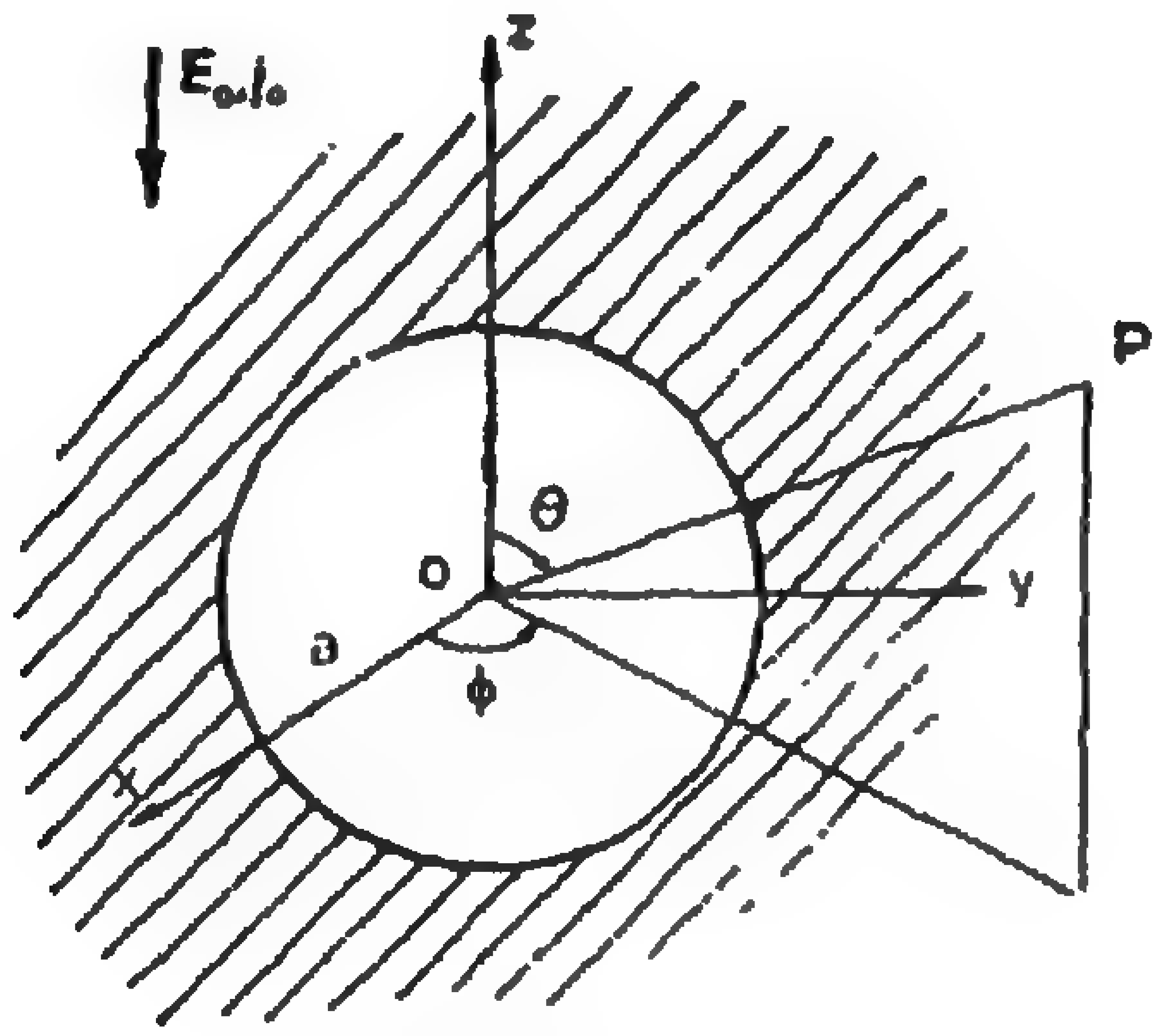


Fig. 10.18.

The cavity will be assumed small enough so that it causes no appreciable distortion of the uniform field near the outside of the block. Also without loss of generality the potential may be taken zero for $z = o$. The boundary conditions are thus, (see Fig. 10.18),

$$\partial\phi/\partial r = 0 \quad \text{at } r = a.$$

$$\phi = E_o\,z = E_o r \cos\theta \qquad \text{as } r \text{ tends to infinity}$$

A solution of Laplace's equation in the metal part is obtained as in the previous problem, equation (4),

$$\phi = A_o + A_1 \, r \cos \quad + \sum_{n=o}^{\infty} B_n \, r^{-n-1} \, P_n(\cos \quad)$$

The boundary condition when r is large gives,

$$A_o = 0, \qquad A_1 = E_o$$

Then

$$\partial\phi/\partial r = E_o \cos\theta + \sum_{n=o}^{\infty} -(n+1) \, B_n \, r^{-n-2} \, P_n(\cos\theta)$$

This is zero at $r = a$ so that,

$$-B_o/a^2 + (E_o - 2B_1/a^3) \cos\theta - (3B_2/a^4) \, P_2(\cos\theta) + \ldots = 0$$

since the Legendre functions are linearly independent we get,

$$B_o = 0, \;\; B_1 = E_o \, a^3/2 \;\; , \;\; B_n = 0, \;\; n>1.$$

Therefore the solution is

$$\phi = E_o \, r \, [1+½ \, (a/r)^3] \cos\theta$$

The equipotentials and flow lines are similar to those shown in Fig. 10.17. The electric field strength is,

$$\mathbf{E} = -\partial\phi/\partial r \; \mathbf{a}_r - \partial\phi/r\partial\theta \; \mathbf{a}_\theta$$

$$= -E_o \, [1-(a/r)^3] \cos\theta \, \mathbf{a}_r + E_o \, [1+½(a/r)^3 \sin\theta] \, \mathbf{a}_\theta$$

when r tends to a

$$\mathbf{E} = (3/2) \, E_o \sin\theta \;\; \mathbf{a}_\theta$$

and the maximum field strength is,

$$E_{max} = (3/2) \, E_o$$

Without the existence of the bubble the current density inside the cast is E_0/ρ while when the bubble exists the maximum current density is,

$$J_{max} = E_{max} / \rho = 3E_0/2\rho$$

which occurs when $\theta = \pi/2$, where ρ is the resistivity. This increase in the current density in the plane $\theta = 0$ at $r = a$ may cause heat concentration and consequently very high mechanical stresses inside the cast.

A comparison with the previous problem shows that in both cases the field is distorted and the maximum gradient *i.e.* the field, is raised because of the sphere. In the previous example, the maximum gradient was three times the undistorted gradient, while with the cavity the maximum is $3E_0/2$ independent of the size of the sphere.

21. *A thin spherical conducting shell of radius a is earthed and placed in a uniform electric field E_0 (Fig. 10.19). If the shell is cut into two halves by a plane parallel to E_0, show that the force acting on each half is $(9/8)\ \pi\ \epsilon_0\ a^2\ E_0^2$.*

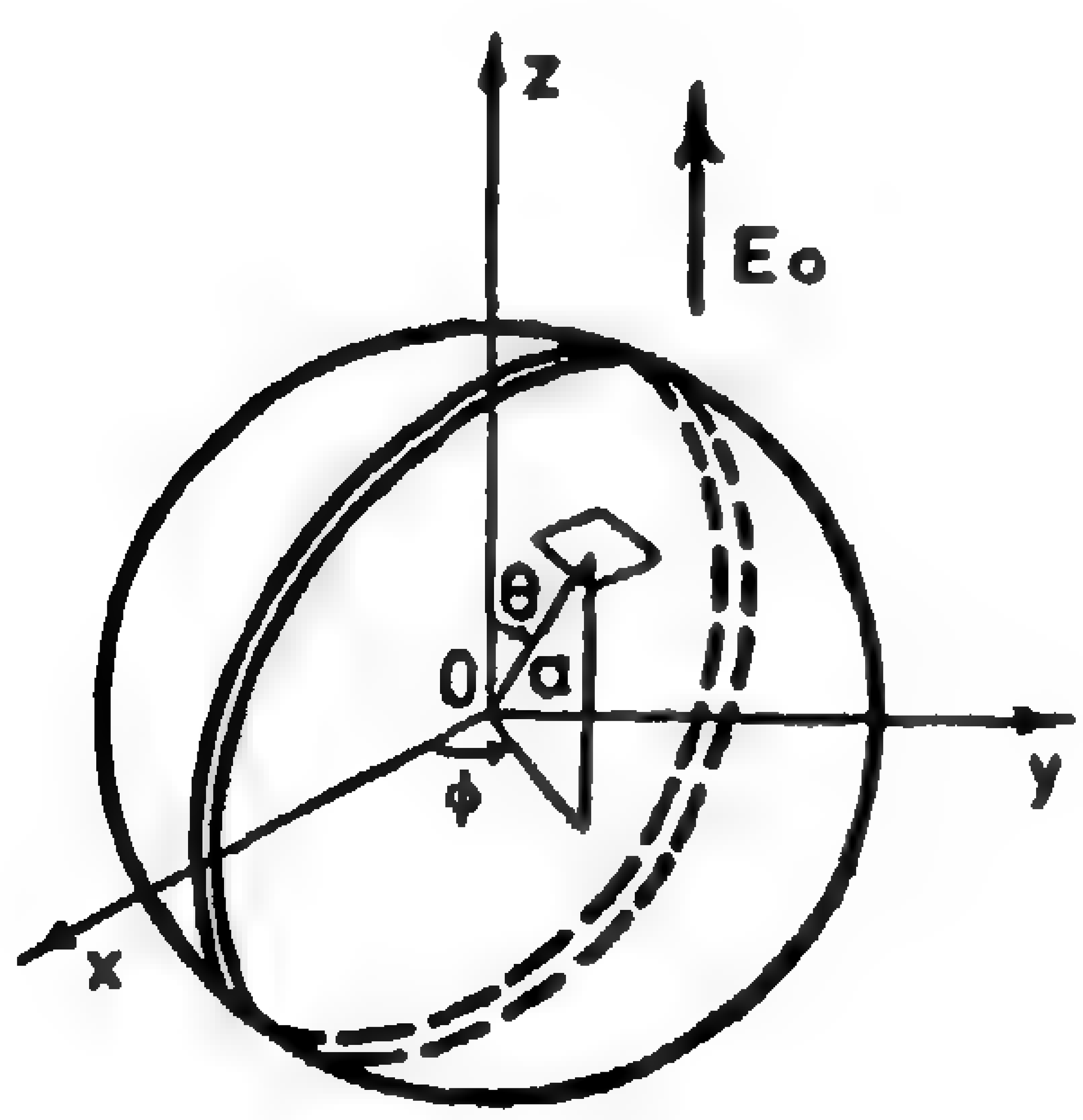


Fig. 10.19.

The same steps of problem 10,17 will be followed with $V_o = 0$.

$$\sigma = 3\, \epsilon_o E_o \cos\theta \qquad (1)$$

The force acting on one half of the shell is,

$$F = \int_0^{\pi} \int_0^{\pi} (\sigma^2/2\epsilon_o)\, a^2 \sin^2\theta \sin\phi \, d\theta \, d\phi$$

Substituting for σ and integrating we get,

$$F = (9\pi/8)\, \epsilon_o\, a^2\, E_o^2$$

which is the required result.

22. *A uniform dielectric sphere of permittivity ϵ and radius a carries a surface charge density $\sigma = \sigma_o + \sigma_1 \cos\theta$. Prove that the interior of the sphere contributes an amount to the electrostatic energy given by $2\pi a^3 \epsilon \sigma_1^2 / 3\, (2\epsilon_o + \epsilon)^2$.*

The problem has axial symmetry so that the $P_n(\cos\theta)$ functions only exist. Also the solution is finite at $r = o$ and at $r = \infty$. Thus suitable solutions to Laplace's equation inside and outside the sphere is,

$$V_1 = \Sigma\; A_n r^n P_n(\cos\theta) \qquad (1)$$

$$V_2 = \Sigma\; B_n r^{-n-1} P_n(\cos\theta) \qquad (2)$$

The coefficients A_n and B_n are determined from the boundary conditions at $r = a$, namley,

(i) $V_1 = V_2$ at $r = a$

(ii) $\epsilon\, \partial V_1/\partial r - \epsilon_o\, \partial V_2/\partial r = \sigma_o + \sigma_1 \cos\theta$ at $r = a$

The first condition gives,

$$A_n = a^{-2n-1}\; B_n \qquad (3)$$

Thus,

$$V_2 = \Sigma \; A_n \, a^{2n+1} \, r^{-n-1} \, P_n \,(\cos\theta) \tag{4}$$

Applying the second boundary condition to (2) and (4) we get that,

$$A_o = \sigma_o/a\epsilon_o \;,\; A_1 = \sigma_1/(\epsilon+2\epsilon_o) \;, \text{ and } A_n = 0, \; n>1$$

Thus we have,

$$V_1 = (\sigma_o/a\epsilon_o) + [\sigma_1/(\epsilon+2\epsilon_o)] \; r \cos\theta \tag{5}$$

$$V_2 = (\sigma_o/\epsilon_o r) + [\sigma_1/(\epsilon+2\epsilon_o)] \; (a^3/r^2) \cos\theta \tag{6}$$

The energy stored in the interior of the sphere is,

$$W = \int \tfrac{1}{2} \, \epsilon E^2 \, dv = \tfrac{1}{2} \, \epsilon \int \nabla V_1 \cdot \nabla V_1 \; dv$$

using the identity

$$\nabla \cdot (U \nabla V_1) = \nabla U \cdot \nabla V_1 + U \nabla^2 V_1$$

with $U = V_1$ and noting that $\nabla^2 V_1 = 0$ we get,

$$W = \tfrac{1}{2} \, \epsilon \int \nabla \cdot (V_1 \nabla V_1) \, dv$$

using Gauss's theorem

$$W = \tfrac{1}{2} \, \epsilon \int V_1 \nabla V_1 \cdot d\mathbf{S}$$

$$= \tfrac{1}{2} \, \epsilon \int V_1 (\partial V_1/\partial r) \, dS$$

Substituting for V_1 from (5) and $dS = 2\pi \, a^2 \sin\theta \, d\theta$ we get,

$$W = 2\pi a^3 \, \epsilon \sigma_1^2 / 3 \; (2\epsilon_o + \epsilon)^2$$

23. *A uniformly charged conducting spherical shell of radius a rotates about a diameter with a uniform angular velocity ω. Find an expression for the magnetic induction field* **B** *at any point.*

The rotating surface charge produces a surface current that will in turn produce a magnetic induction. As shown in Fig. 10.30 the surface current density at point A is,

$$\mathbf{K} = \sigma\omega \times \mathbf{a} = \sigma\omega a \sin\theta \, \mathbf{a}_\phi$$

The magnetic scalar potential V_m satisfies Laplace's equation,

$$\nabla^2 V_m = 0$$

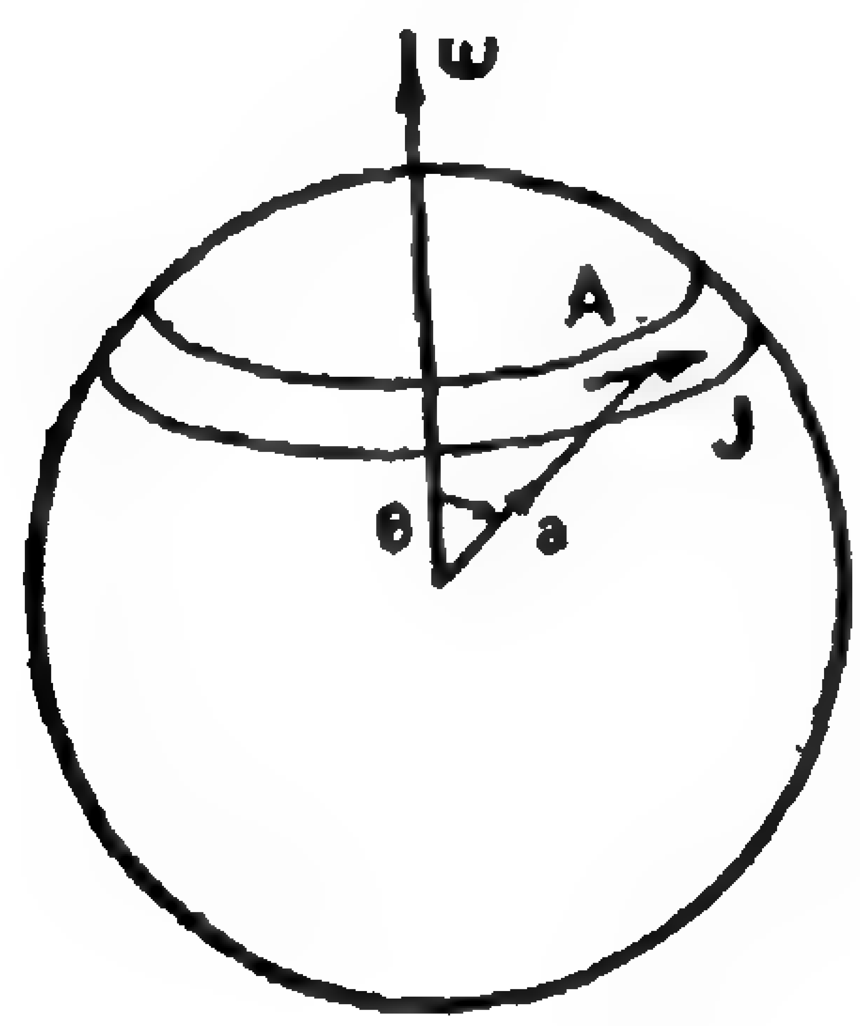


Fig. 10.20.

A suitable solution of this equation is,

$$V_1 = \sum_0^\infty A_n r^n P_n(\cos\theta) \qquad r \leq a \qquad (1)$$

$$V = \sum_0^\infty B_n r^{-n-1} P_n(\cos\theta) \qquad r \geq a \qquad (2)$$

The constants A_n and B_n can be obtained from the boundary conditions at the spherical surface.

(i) $\mathbf{B}.\mathbf{n} = -\mu_0 \partial V_m/\partial r$

is continuous at $r=a$. Using (1) and (2) we get,

$$B_o = 0, \; B_n = -[n/(n+1)] \, a^{2n+1} A_n, \; n>o \qquad (3)$$

(ii) $\mathbf{n} \times (\mathbf{H}_2 - \mathbf{H}_1) = \mathbf{J}$ at $r=a$

This gives

$$(1/a)\,(\partial V_1/\partial\theta - \partial V_2/\partial\theta) = \omega\,\sigma\,a\,\sin\theta$$

Substituting for V_1 and V_2 we get,

$$(1/a) \sum_{o}^{\infty} A_n \, a^n \, (d/d\theta) \, P_n(\cos\theta) \; -$$

$$(1/a) \sum_{1}^{\infty} B_n \, a^{-n-1} \, (d/d\theta) \, P_n(\cos\theta) = \omega\,\sigma\,a\,\sin\theta \qquad (4)$$

since,

$$(d/d\theta)\, P_n(\cos\theta) = P_n^1(\cos\theta)$$

where P_n^1 is the associated Legendre polynomial. Substituting for B_n from (3) and noting that $P_1^1 = -\sin\theta$ we get,

$$\sum_{0}^{\infty} A_n a^n \, P_n^1(\cos\theta) + \sum_{1}^{\infty} A_n a^n \, [n/(n+1)] \, P_n^1(\cos\theta)$$

$$= -\,\omega\,\sigma\,a^2\,P_1^1(\cos\theta)$$

Since the associated Legendre polynomials form a linearly independent set, we equate the coefficients of the corresponding polynomial,

$$A_o = 0, \; A_1 = -(2/3)\,a\,\omega\,\sigma\,, \; A_n = B_n = 0, \; n>1$$

Thus the magnetic potential becomes

$$V_1 = -(2/3)\,\omega\,\sigma\,a\,r\cos\theta \qquad r \leq a$$

$$V_2 = (\omega\,\sigma\,a^4/3\,r^2)\cos\theta \qquad r \geq a$$

The magnetic induction field follows from the relation $\mathbf{B} = -\mu_o \nabla V$,

$$\mathbf{B}_1 = (2/3)\, \mu_o\, \omega\, \sigma\, a\ \mathbf{a}_z \ , \qquad r \leq a$$

$$\mathbf{B}_2 = (2/3)\, \mu_o\, \omega\, \sigma\, a^4\, r^{-3} \cos\theta\ \mathbf{a}_r$$

$$+ (1/3)\, \mu_o\, \omega\, \sigma\, a^4 r^{-3} \sin\theta\ \mathbf{a}_\theta \ , \qquad r \geq a$$

This shows that the magnetic field inside the shell is uniform and in the same direction as vector ω.

24. *Solve the previous problem if the surface charge density at any point on the spherical shell is $\sigma = \Sigma A_n P_n(\cos\theta)$, where A_n are constants and θ is the angle with the diametrical axis of rotation.*

As shown in Fig. 10.20 the surface current at point A is,

$$K = \sigma\, \omega\, a \sin\theta \qquad \text{in the } \phi\text{—direction}$$

$$= \omega a\ \Sigma\ A_n \sin\theta\, P_n(\cos\theta)$$

By using the relation,

$$(2n+1)\, P_n(\alpha) = P'_{n+1}(\alpha) - P'_{n-1}(\alpha)$$

where the prime represents $d/d\alpha$,

$$(2n+1) \sin\theta\, P_n(\cos\theta) = \sin\theta\, P'_{n+1}(\cos\theta) - \sin\theta\, P'_{n-1}(\cos\theta)$$

$$= (d/d\theta)\ [P_{n-1}(\cos\theta) - P_{n+1}(\cos\theta)]$$

$$= P_{n-1}^1(\cos\theta) - P_{n+1}^1(\cos\theta)$$

So that we can write,

$$K = \omega a \sum_0^\infty [A_n/(2n+1)]\ (P_{n-1}^1 - P_{n+1}^1)$$

$$= \omega a \sum_0^\infty [A_{n+1}/(2n+3) - A_{n-1}/(2n-1)]\ P_n^1$$

$$= \omega a\ \Sigma\ B_n\, P_n^1$$

Suitable solutions for Laplace's equation inside and outside the spherical shell are,

$$V_1 = \Sigma\ D_n\ r^n\ P_n\ (\cos\ \theta) \qquad r \lesssim a$$

$$V_2 = \Sigma\ C_n\ r^{-n-1}\ P_n\ (\cos\ \theta) \qquad r \geq a$$

The boundary conditions (i) and (ii) of the previous problem give,

$$D_n = [(n+1)\ /\ (2n+1)]\ \omega\ a^{-n+2}\ B_n$$

$$C_n = -[n\ /\ (2n+1)]\ \omega\ a^{n+3}\ B_n$$

Thus we have,

$$V_1 = \omega a^2 \sum_0^{\infty} [(n+1)\ /\ (2n+1)]\ (r/a)^n\ B_n\ P_n\ (\cos\theta)$$

$$V_2 = -(\omega\ a^3/r) \sum_0^{\infty} [n/(2n+1)]\ (a/r)^n\ B_n\ P_n\ (\cos\ \theta)$$

The magnetic induction field can be obtained from the relation $\mathbf{B} = -\ \mu_0\ \nabla V$ as before.

25. *A point charge Q is located along the z–axis at z = a. Find an expression for the potential at a general point.*

We have symmetry about the z axis, see Fig. 10.21, so that the potential at $P\ (r,\ \theta,\ \phi)$ is,

$$V_p = Q/4\pi\ \epsilon_0\ R = (Q/4\pi\ \epsilon_0)\ (a^2+r^2-2\ ar\ \cos\ \theta)^{-1/2}$$

we shall consider two cases;

(i) $r \lesssim a$,

$$V_p = (Q/4\pi\ \epsilon_0\ a)\ [1-(2r/a)\ \cos\ \theta + (r/a)^2]^{-1/2}$$

$$= (Q/4\pi\ \epsilon_0\ a) \sum_0^{\infty} (r/a)^n\ P_n\ (\cos\ \theta) \qquad (1)$$

(ii) $r \geq a$

$$V_p = (Q/4\pi \epsilon_0 r) \, [1-(2a/r) \cos \theta + (a/r)^2]^{-1/2}$$

$$= (Q/4\pi \epsilon_0 r \sum_{n=0}^{\infty} (a/r)^n \, P_n (\cos \theta) \qquad (2)$$

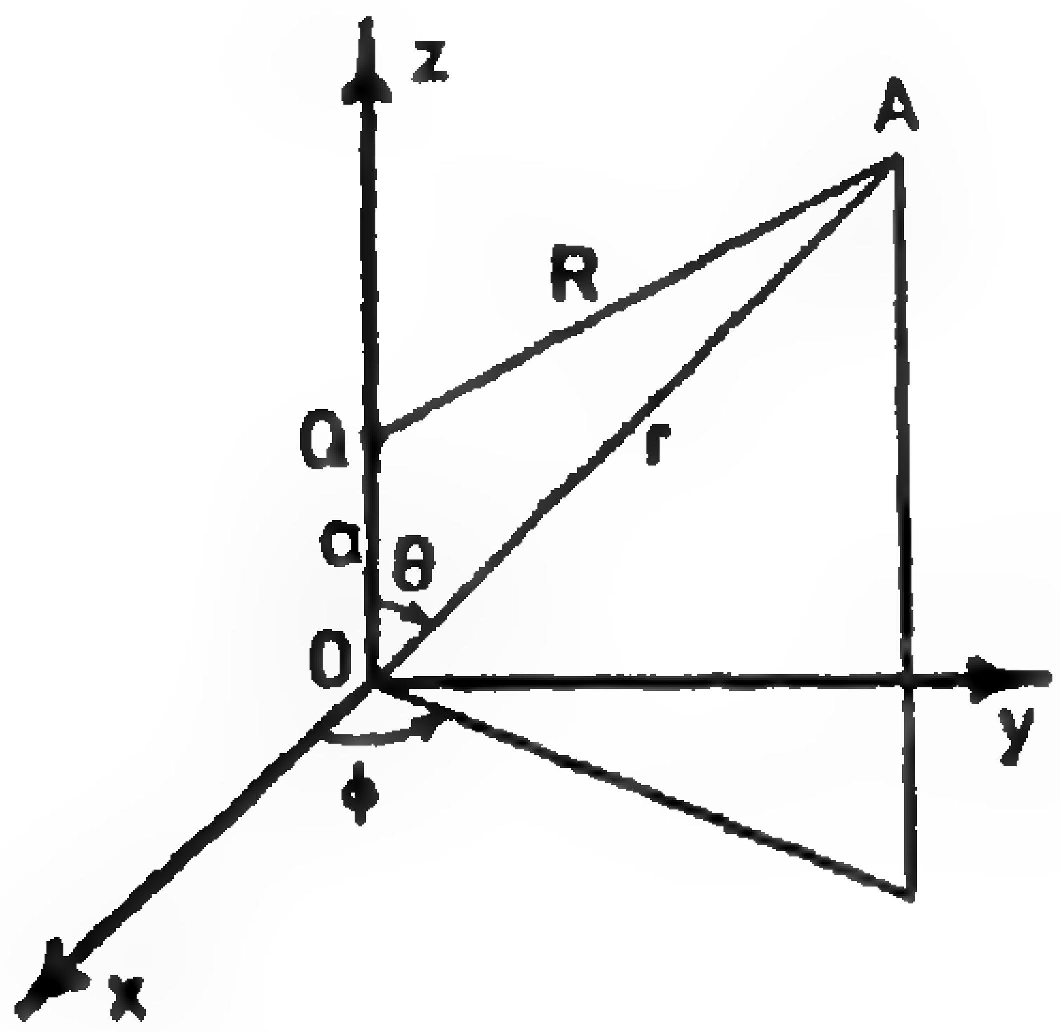


Fig. 10.21.

26. *Using the Legendre polynomials find; (i) the potential outside a uniformly charged sphere with density ρ_0, and (ii) the potential outside a uniformly charged spherical surface of density σ_0.*

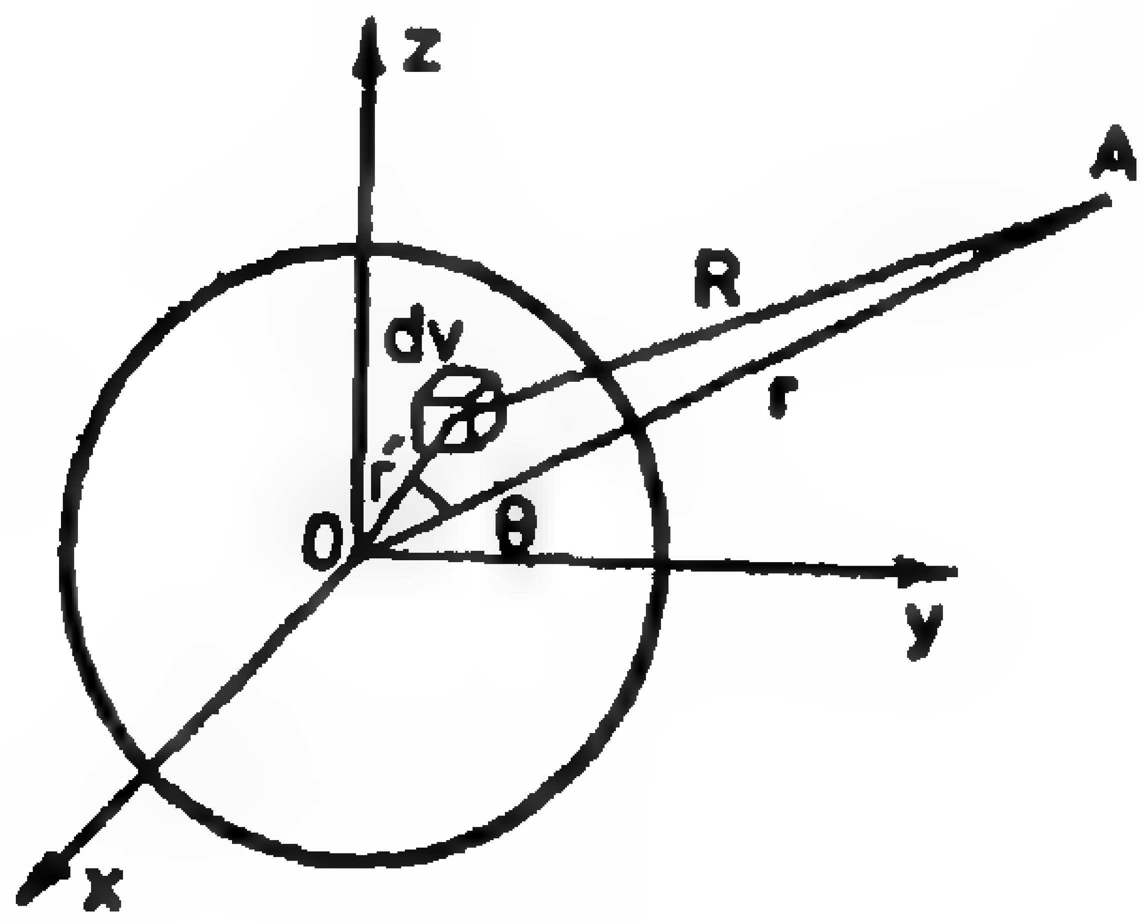


Fig. 10.22.

(i) Using the result (2) of the previous problem, the potential at a general point P due to an elementary volume charge $\rho_0\, dv$ is, (Fig. 10.22),

$$dV_p = (\rho_0\, dv/4\pi\, \epsilon_0\, r') \;\; \Sigma \;\; (r'/r)^{n+1}\; P_n\,(\cos\;\theta) \tag{1}$$

Since the problem has symmetry with respect to θ and ϕ the same result will be obtained if P is taken on the z–axis, (or if the z–axis is taken along OP),

$$V_p = (\rho_0/4\pi\,\epsilon_0)\;\Sigma\; r^{-n-1} \int_0^{\pi} \int_0^{a} 2\pi\, r'^{n+2}\, P_n(\cos\theta)\; \sin\theta\; d\,\theta\; dr'$$

$$= \;[\rho_0\; a^{n+3}\,/\,2\;\epsilon_0\,(n+3)\; r^{n+1}] \;\; \sum_0^{\infty} \int_{-1}^{1} P_n\,(\cos\;\theta)\; d\;(\cos\;\theta)$$

$$= \;\; a^3\;\rho_0\;/\;3\epsilon_0\, r \;=\; Q/4\pi\epsilon_0\; r$$

where Q is the total charge on the sphere.

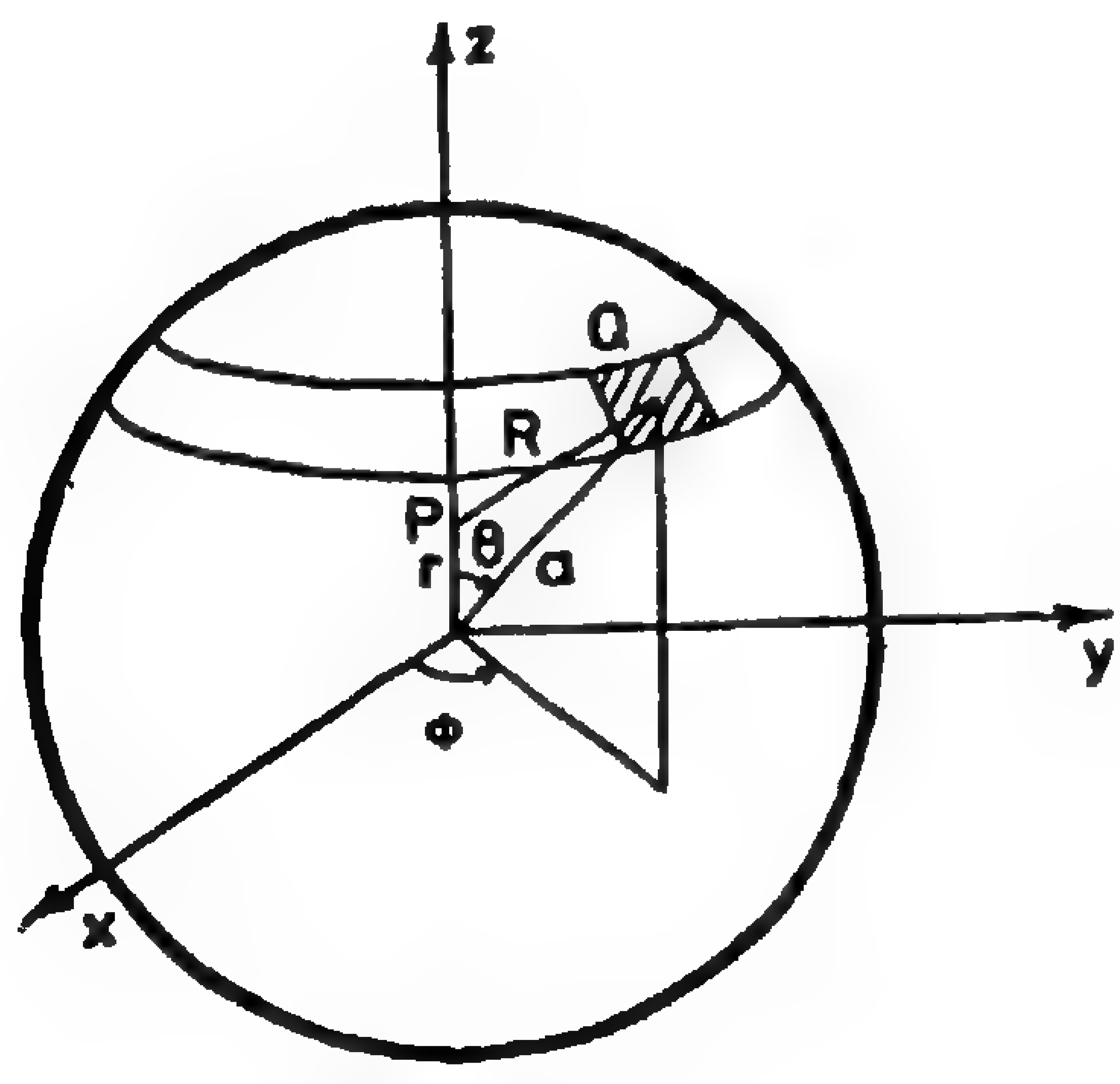


Fig. 10.23.

(ii) The point P may be taken on the z-axis without altering the results as in (i). Thus we have that (Fig. 10.23),

$$dV_p = (\sigma_o \, dS/4\pi\epsilon_o \, a) \sum_{0}^{\infty} (a/r)^{n+1} P_n (\cos \theta)$$

where, $dS = a^2 \sin \theta \, d\theta \, d\phi$. Integrating we get,

$$V_p = (\sigma_o/4\pi\epsilon_o \, a) \sum_{0}^{\infty} (a/r)^{n+1} \int_{0}^{\pi} \int_{0}^{2\pi} P_n (\cos \theta) \, a^2 \sin \theta \, d\theta \, d\phi$$

$$= (\sigma_o \, a/4\pi\epsilon_o) \sum_{0}^{\infty} (a/r)^{n+1} \int_{-1}^{1} P_n (\mu) \, d\mu$$

$$= \sigma_o \, a^2/\epsilon_o \, r = Q/4\pi\epsilon_o \, r$$

where Q is the total charge on the sphere.

27. *A point charge Q is placed at a distance b from the center of a conducting sphere of radius a. Determine the potential at any point given that the sphere is kept at potential V_o, the surface charge density at any point on the sphere, the total charge on the sphere, and the force acting on the charge Q.*

Consider the conducting sphere to be located with its center at origin and the charge Q at $z = b$ on the z-axis as shown in Fig. 10.24. It is clear that the potential at any point A (r, θ, ϕ) is the sum of two potentials. The first V_1 is due to Q alone while the second V_2 is due to the induced charges on the conducting sphere. Using the results of problem 10.25,

$$V_1 = (Q/4\pi\epsilon_0) \sum_{n=0}^{\infty} r^n b^{-n-1} P_n(\cos\theta) \qquad r \leq b \tag{1}$$

$$= (Q/4\pi\epsilon_0) \sum_{n=0}^{\infty} b_n r^{-n-1} P_n(\cos\theta) \qquad r \geq b \tag{2}$$

The potential V_2 satisfies Laplace's equation which in spherical coordinates, with symmetry about the z-axis, has solution,

$$V_2 = \Sigma \ (A_n r^n + B_n r^{-n-1}) P_n(\cos\theta) \tag{3}$$

Since V_2 must be finite at $r = \infty$, $A_n = 0$, so that,

$$V_2 = \sum_{n=0}^{\infty} B_n r^{-n-1} P_n(\cos\theta) \tag{4}$$

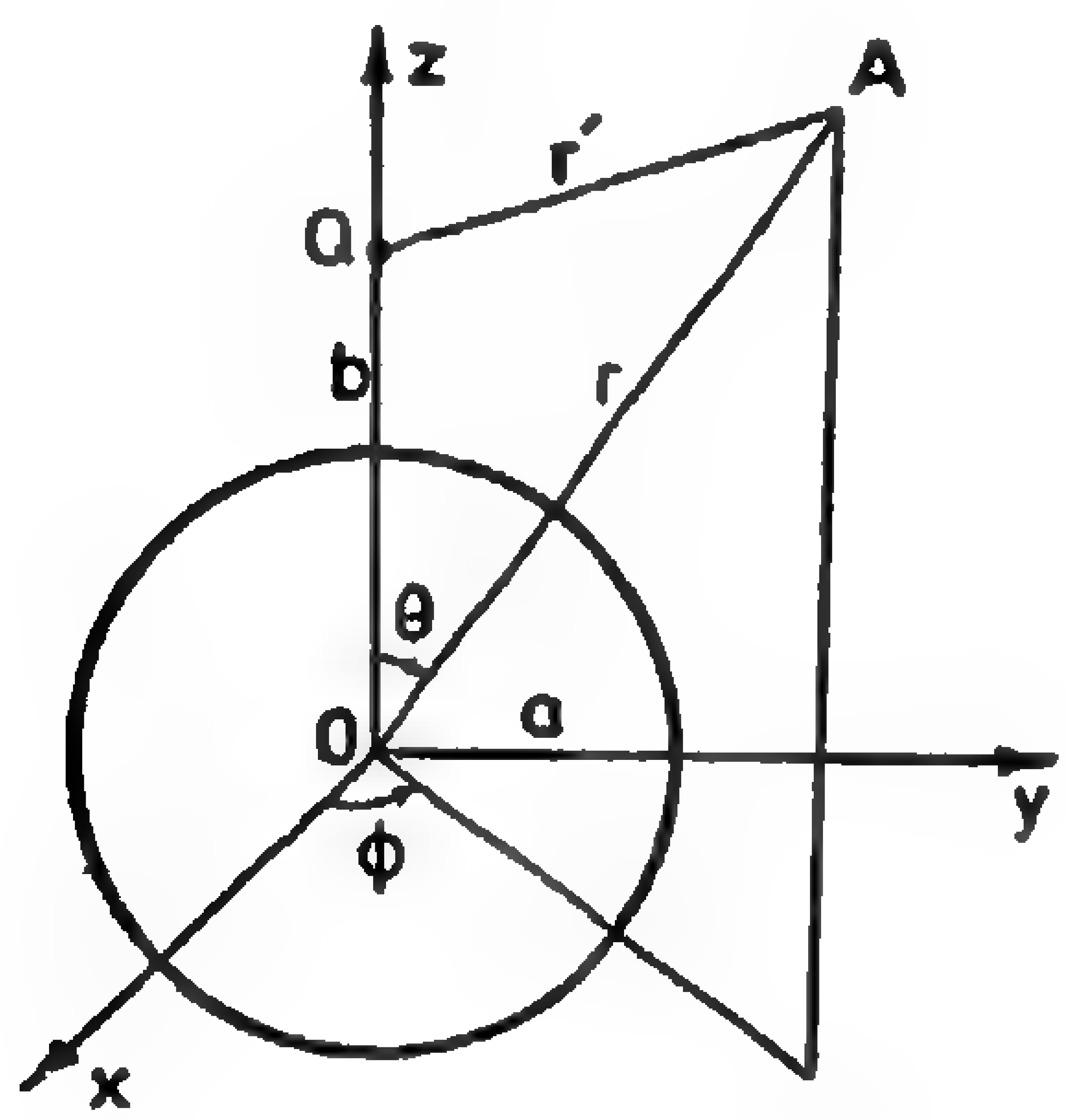


Fig. 10.24.

The constant B_n can be obtained from the boundary conditions at $r=a$, namely $V = V_0$ at $r = a$. For $r \leq b$ the total potential is,

$$V = V_1 + V_2$$

$$= (Q/4\pi\epsilon_0) \ \Sigma \ r^n b^{-n-1} P_n(\cos\theta) + \Sigma \ B_n r^{-n-1} P_n(\cos\theta) \tag{5}$$

Equating this with V_0 at $r = a$ we get,

$$B_0 = aV_0 - Qa/4\pi \epsilon_0 b$$

$$B_n = -Q a^{2n+1} / 4\pi\epsilon_0 b^{n+1}$$

Therefore the potential for $a<r \leq b$ becomes

$$V = (Q/4\pi \epsilon_0) \sum_{n=0}^{\infty} r^n b^{-n-1} P_n(\cos\theta) + aV_0/r - Qa/4\pi\epsilon_0 b r$$

$$-(Q/4\pi \epsilon_0) \sum_{n=1}^{\infty} a^{2n+1} b^{-n-1} r^{-n-1} P_n(\cos\theta) \qquad (6)$$

For $r \geq b$ only the first term on the right hand side of (6) is replaced by that given by (2). In order to determine the surface charge density on the sphere, we use the relation

$$\sigma = -\epsilon_0 \partial V/\partial r \qquad \text{at } r=a$$

Substituting from (6) we get,

$$\sigma = \epsilon_0 V_0/a - (Q/4\pi) \sum_{n=0}^{\infty} (2n+1) a^{n-1} b^{-n-1} P_n(\cos\theta) \quad (7)$$

The total charge on the sphere can be obtained by integrating σ over the surface of the sphere

$$q = \int_0^{\pi} \int_0^{2\pi} \sigma a^2 \sin\theta \, d\theta \, d\phi$$

$$= 2\pi a^2 \int_{-1}^{1} \sigma \, d\cos\theta$$

Substituting from (7) in (8) and using the relation

$$\int_{-1}^{1} P_n(\cos\theta)\, d\cos\theta = 0 \quad , \quad n > o$$

we get,

$$q = 4\pi a\, \epsilon_o\, V_o - Qa/b \tag{9}$$

If the sphere is earthed $q = -Qa/b$ which is the result obtained before in problem. 9.13.

The force F on the charge Q is that due to the field E_2 obtained from V_2 only,

$$F = QE_2 = -Q\, \nabla V_2 \qquad \text{at } r=b,\ \theta=0$$

It is directed along the z-axis

$$F = aV_oQ/b^2 - aQ^2/4\pi\,\epsilon_o\, b^3 - (aQ^2/4\pi\,\epsilon_o\, b^3) \sum_{n=1}^{\infty} (n+1)\, (a^2/b^2)^n \tag{10}$$

Consider the result

$$\sum_{n=1}^{\infty} (n+1)\, x^n = d/dx\, [\sum_{n=1}^{\infty} x^n] = d/dx\, [x^2/(1-x)], \qquad x < 1$$

Thus (10) can be written in the form

$$F = aV_o\, Q/b^2 - ab\, Q^2/4\pi\,\epsilon_o\, (b^2-a^2)^2 \tag{11}$$

which is the same result obtained before in problem 9.13.

28. *A point charge Q is placed at a distance b from the center of a grounded conducting sphere of radius a, ($b>a$) The region $a \leq r \leq c$ is occupied by a dielectric shell of permittivity ϵ. Find an expression for the potential in the dielectric shell.*

The potential due to the charge Q alone at a point $P(r, \theta)$ (Fig. 10.25), with $r \leq b$ is, (see problem 10.25).

$$V = Q/4\pi\epsilon_0 R = (Q/4\pi\epsilon_0 b) \; \Sigma \; (r/b)^n P_n (\cos\theta) \qquad (1)$$

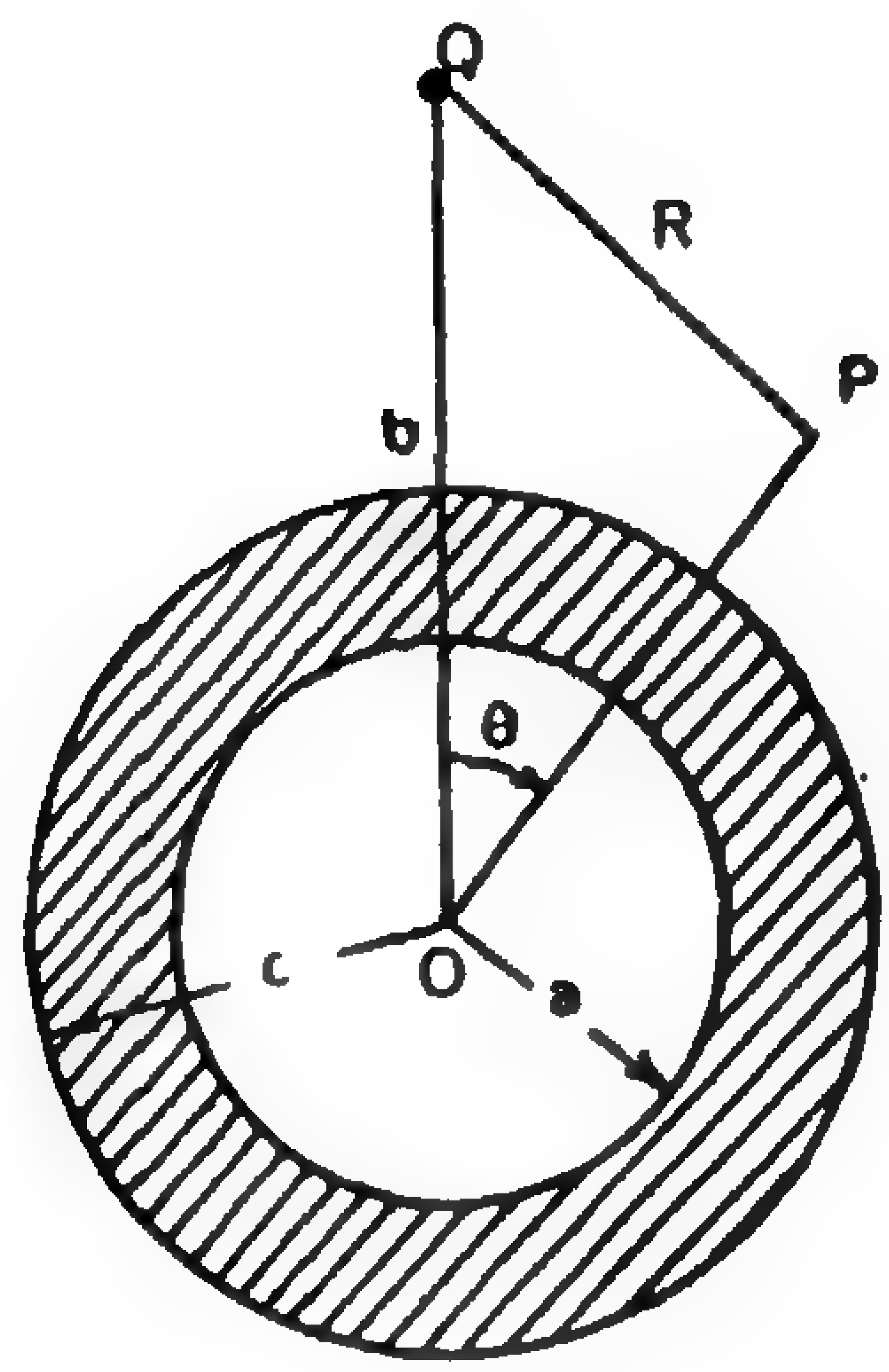


Fig. 10.25.

When the coated sphere is introduced in the field of the point charge, the potential in the region $a \leq r \leq c$ satisfies Laplace's equation. A suitable solution is,

$$V_1 = \sum_{0}^{\infty} (A_n r^n + B_n r^{-n-1}) P_n (\cos\theta) \qquad a \leq r \leq c \qquad (2)$$

Also the induced charges on the coated sphere produce a field whose potential is,

$$V'_2 = (Q/4\pi) \sum_{0}^{\infty} C_n r^{-n-1} P_n (\cos\theta) \qquad r \geq c \qquad (3)$$

Thus the total potential in the region $c \leq r \leq b$ is,

$$V_2 = V + V'_2 \tag{4}$$

The consts nts A_n, B_n, and C_n are obtained from the boundary conditions, namely,

(a) $V_1 = 0$ at $r=a$, (b) $V_1 = V_2$ at $r=c$

(c) $\epsilon\,(\partial V_1/\partial r) = \epsilon_0\,(\partial V_2/\partial r)$ at $r=c$

These give,

$$a^n A_n + a^{-n-1} B_n = 0 \tag{5}$$

$$c^n A_n + c^{-n-1} B_n = (Q/4\pi)\, c^{-n-1} C_n + (Q/4\pi\,\epsilon_0)\, c^n\, b^{-n-1} \tag{6}$$

$$n\epsilon A_n - (n+1)\ \epsilon c^{-2n-1} B_n = (nQ/4\pi)\, b^{-n-1} -$$

$$(Q\,\epsilon_0/4\pi)\,(n+1)\, c^{-2n-1} C_n \tag{7}$$

Eliminating C_n between (6) and (7) and then solving the resulting equation with (5) we get A_n and B_n.

$$A_n = [(2n+1)Q/4\pi\, b^{n+1}] \;/\; [\epsilon_0 + n\epsilon - (\epsilon_0 - \epsilon)(n+1)\,(a/c)^{2n+1}]$$

Substituting for A_n and $B_n = -a^{2n+1} A_n$ in (2) we get,

$$V_1 = (Q/4\pi) \sum_0^{\infty} \frac{(2n+1)Q[r^n - a^{2n+1}/r^{n+1}]}{b^{n+1}\,[\epsilon_0 + n\epsilon + (\epsilon - \epsilon_0)(n+1)(a/c)^{2n+1}]}$$

$$a \leq r \leq c \tag{8}$$

29. *A point charge* Q *is placed at a distance c from the common center of two earthed concentric spherical conductors of radii a and b.* $(a < c < b)$. *Find an expression for the potential between the two spheres.*

The potential at point $P\,(r, \theta)$, Fig. 10.26, consists of two parts. The first part is due to Q alone,

$$V_1 = Q/4\pi\epsilon r' = Q/4\pi\epsilon\,(c^2+r^2-2\,cr\cos\theta)^{1/2}$$

$$= (Q/4\pi\epsilon r)\sum_0^{\infty}(c/r)^n P_n(\cos\theta) \quad , \; r \geq c \qquad (1)$$

$$= (Q/4\pi\epsilon\, c)\;\Sigma\;(r/c)^n P_n(\cos\theta) \quad , \; r \leq c \qquad (2)$$

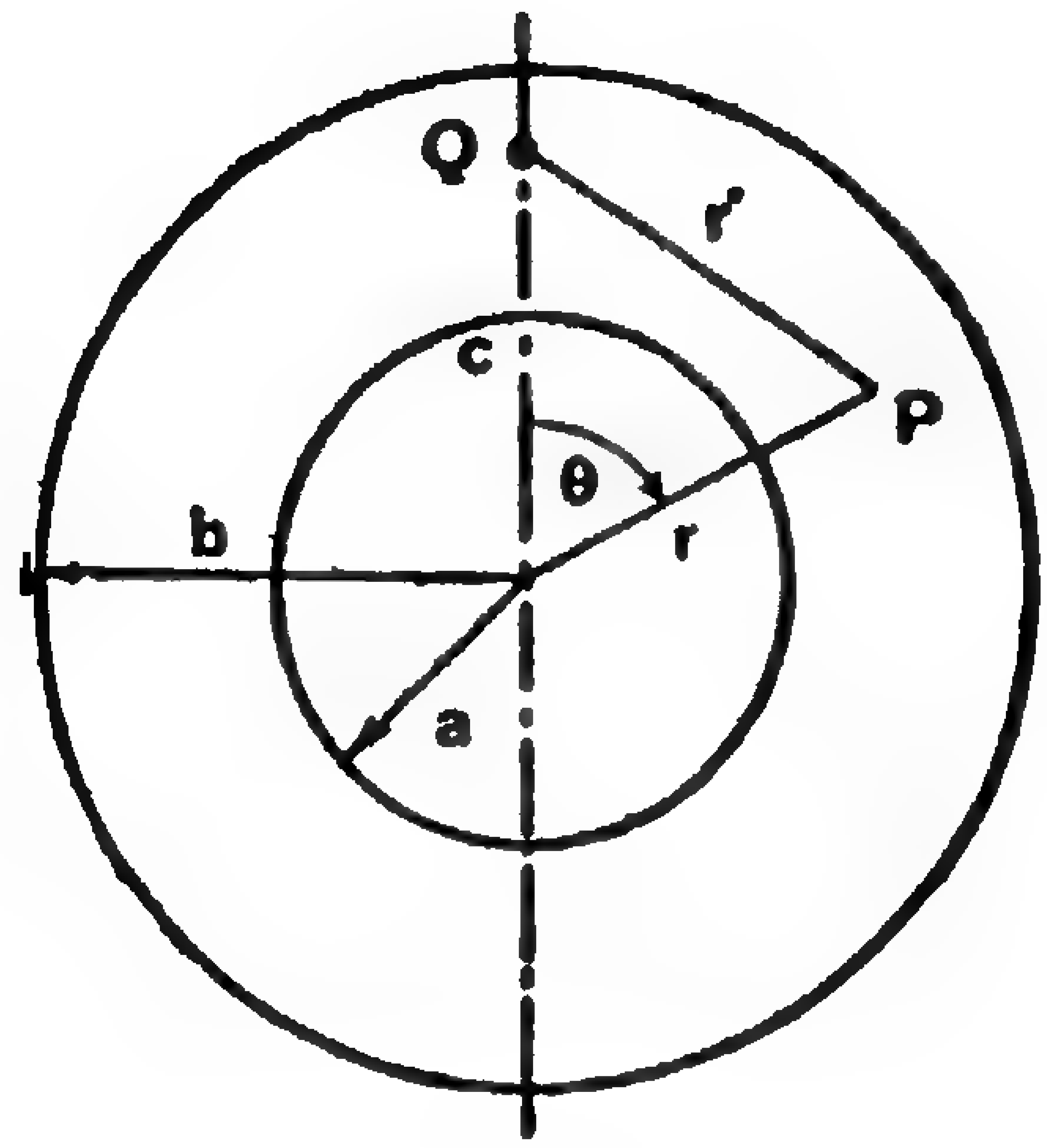


Fig. 10.26.

The second part is due to the induced charges on the two spheres. It satisfies Laplace's equation. A suitable solution is,

$$V_2 = \sum_0^{\infty}(A'_n r^n + B'_n r^{-n-1})\,P_n(\cos\theta) \qquad (3)$$

Thus the potential is

$$V' = (Q/4\pi\epsilon)\sum_0^{\infty}[A_n r^n + (B_n + c^n)\,r^{-n-1}]\;P_n(\cos\theta) \quad r \geq c \quad (4)$$

$$V'' = (Q/4\pi\,\epsilon)\sum_0^{\infty}[(A_n + c^{-n-1})r^n + B_n r^{-n-1}]\,P_n(\cos\theta) \quad r \leq c \; (5)$$

The two constants A_n and B_n are to be determined from the boundary conditions at $r=a$, b.

(a) $V' = 0$ at $r=b$, (b) $V''=0$ at $r=a$.

These give,

$$A_n\, b^n+(B^n+c^n)\, b^{-n-1} = 0$$

$$(A_n+c^{-n-1})\, a^n+B_n\, a^{-n-1} = 0$$

Solving for A_n and B_n

$$A_n = (c^{2n+1}-a^{2n+1}) \;/\; c^{n+1}\,(a^{2n+1}-b^{2n+1})$$

$$B_n = a^{2n+1}\,(b^{2n+1}-c^{2n+1}) \;/\; [c^{n+1}\,(a^{2n+1}-b^{2n+1})]$$

Thus (4) and (5) become,

$$V' = (Q/4\pi\epsilon) \sum_{0}^{\infty} (c^{2n+1}-a^{2n+1})\,(r^n-b^{2n+1}/r^{n+1})\,P_n(\cos\theta) \,/\, c^{n+1}\,(a^{2n+1}-b^{2n+1})$$

and

$$V'' = (Q/4\pi\epsilon) \sum_{0}^{\infty} (c^{2n+1}-b^{2n+1})\,(r^n-a^{2n+1}/r^{n+1})\,P_n(\cos\theta) \,/\, c^{n+1}\,(a^{2n+1}-b^{2n+1})$$

30. *A charge Q is located within a spherical conducting shell of radius b at a distance a, a<b, from center. Find the pot ntial at any point and the surface charge density on the inner and outer surfaces of the shell consider the cases (i) the spherical shell is earthed, and (ii) the spherical shell is insulated.*

This problem has been solved before in problem 9.11 by the method of images. In the present solution the spherical harmonics

will be used. As shown in Fig. 10.27 the potential at point P inside the shell is,

$$V \approx V_1 + V_o$$

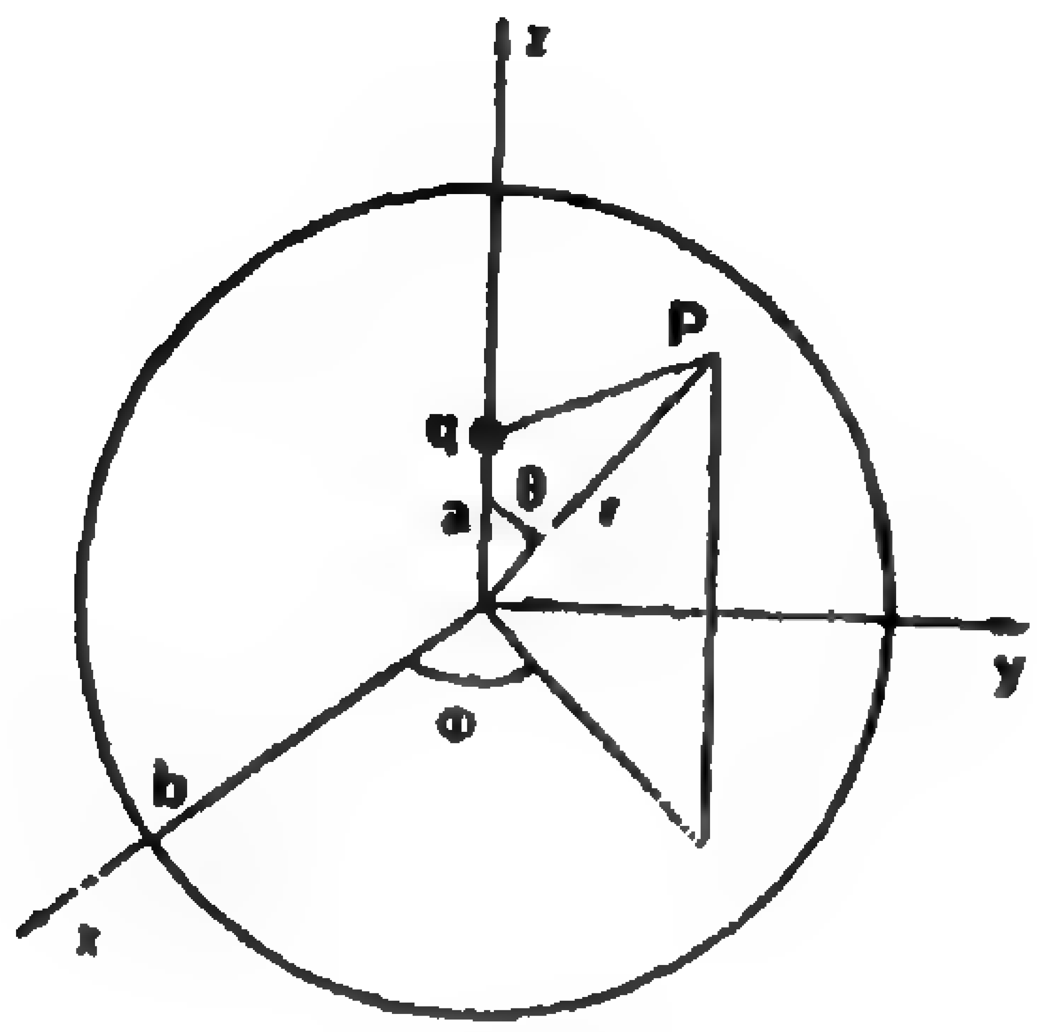


Fig. 10 27.

where V_1 is the potential at P due to the induced charges on the internal surface of the shell.

$$V_1 = \sum_{n=o}^{\infty} \sum_{m=o}^{n} [A_{nm} r^n + B_{nm} r^{-n-1}] P_n^m (\cos \theta) e^{jm\phi}$$

While V_o is the potential due to the charge Q alone. Using the results of problem 10.25 the symmetry around the z-axis, and the fact that the solution is to be finite at origin the potential at point P is,

$$V = \sum_{n=o}^{\infty} [(q/4\pi \epsilon_o) a^n r^{-n-1} + A_n r^n] P_n (\cos \theta) \quad a<r\leq b$$

$$= \sum_{n=o}^{\infty} [(q/4\pi \epsilon_o) r^n a^{-n-1} + A_n r^n] P_n (\cos \theta) \quad r<a$$

The constants A_n, $n=0, 1, 2 \ldots$ can be obtained from the boundary condition at $r=b$,

$$V(b, \theta) = V_o \quad o\leq\theta<\pi$$

This gives

$$(q/4\pi\epsilon_o)\, a^n\, b^{-n-1} + A_n\, b^n = 0, \qquad n>o$$

$$q/4\pi\epsilon_o\, b + A_o = V_b$$

These two equations can be solved to give

$$A_o = V_b - q/4\pi\epsilon_o\, b$$

$$A_n = -(q/4\pi\epsilon_o)\, a^n/b^{2n+1}$$

Substitution for A_n in the equation of the potential gives,

$$V = \sum_{n=1}^{\infty} (q/4\pi\epsilon_o)\, [r^{-n-1} - r^n/b^{2n+1}]\, a^n\, P_n(\cos\theta)$$
$$+ (q/4\pi\epsilon_o)\,(1/r - 1/b) + V_b \quad , \quad a<r\leq b$$

$$V = \sum_{n=1}^{\infty} (q/4\pi\epsilon_o)\,(a^{-n-1} - a^n/b^{2n+1})\, r^n\, P_n(\cos\theta)$$
$$+ (q/4\pi\epsilon_o)\,(1/a - 1/b) + V_b \quad , \quad r<a$$

For $a<r\leq b$ the electric field is,

$$E_{bi} = -(\partial V/\partial r)\Big|_{r=b} = \sum_{n=1}^{\infty} (q/4\pi\epsilon_o)(2n+1)(a^n/b^{n+2})\, P_n(\cos\theta)$$
$$+ q/4\pi\epsilon_o\, b^2$$

and the density of charge at the inner surface is

$$\sigma_{bi} = -\epsilon_o\, E_{bi}$$

$$\sigma_{bi} = -\sum_{n=1}^{\infty} (2n+1)\, a^n\, b^{-n-2}(q/4\pi)\, P_n(\cos\theta) - q/4\pi b^2$$

Thus the total charge on the internal surface of the sphere is,

$$Q_{bi} = \int_0^{\pi}\int_0^{2\pi} \sigma_{bi}(\theta)\, b^2 \sin\theta\, d\theta\, d\phi$$

$$= 2\pi b^2 \int_{-1}^{1} \sigma_i(\cos\theta)\, d(\cos\theta)$$

Substituting for σ_b $(\cos\theta)$ and using the orthogonality property of the Legendre polynomials we get,

$$Q_b = -q$$

If the sphere is uncharged and insulated we must have a zero total charge on the spherical shell. Thus we conclude that the charge induced on the outer surface of the shell is q. Therefore for points external to the spherical shell we substitute the spherical shell by a point charge q located at its center.

$$V(r) = q / 4\pi\epsilon_0 r \quad , \quad r \geq b$$

$$\mathbf{E}(r) = (q/4\pi\epsilon_0 r^2)\,\mathbf{a}_r \quad , \quad r \geq b$$

If the shell is at zero potential, $V_b = 0$ and no charge exists on the shell. For the insulated case the charge on the outer surface is distributed with uniform density

$$\sigma_b = q/4\pi b^2$$

31. *Find an expression for the potential inside and outside a dielectric sphere of radius a placed at a distance b (>a) in the field of a point charge* Q. *Also find the force attracting the charge* Q *towards the dielectric sphere.*

Referring to Fig. 10.28 let the sphere have a permittivity ϵ_1 and that of the surrounding medium be ϵ_2. The potential due to the charge Q is,

$$V_1 = Q/4\pi\epsilon_2 r' = Q/4\pi\epsilon_2 (b^2+r^2-2br\cos\theta)^{1/2} \quad .$$

$$= (Q/4\pi\epsilon_2 b) \sum_0^{\infty} (r/b)^n P_n(\cos\theta) \qquad \text{for } r \leq b \qquad (1)$$

$$= (Q/4\pi\epsilon_2 b) \sum_0^{\infty} (b/r)^{n+1} P_n(\cos\theta) \qquad \text{for } b \leq r \qquad (2)$$

The disturbance potential due to the presence of the sphere is,

$$V_2 = \sum_{0}^{\infty} A_n r^n P_n (\cos \theta) \qquad r \leq a \qquad (3)$$

$$= \sum_{n=0}^{\infty} B_n r^{-n-1} P_n (\cos \theta) \qquad r \geq a \qquad (4)$$

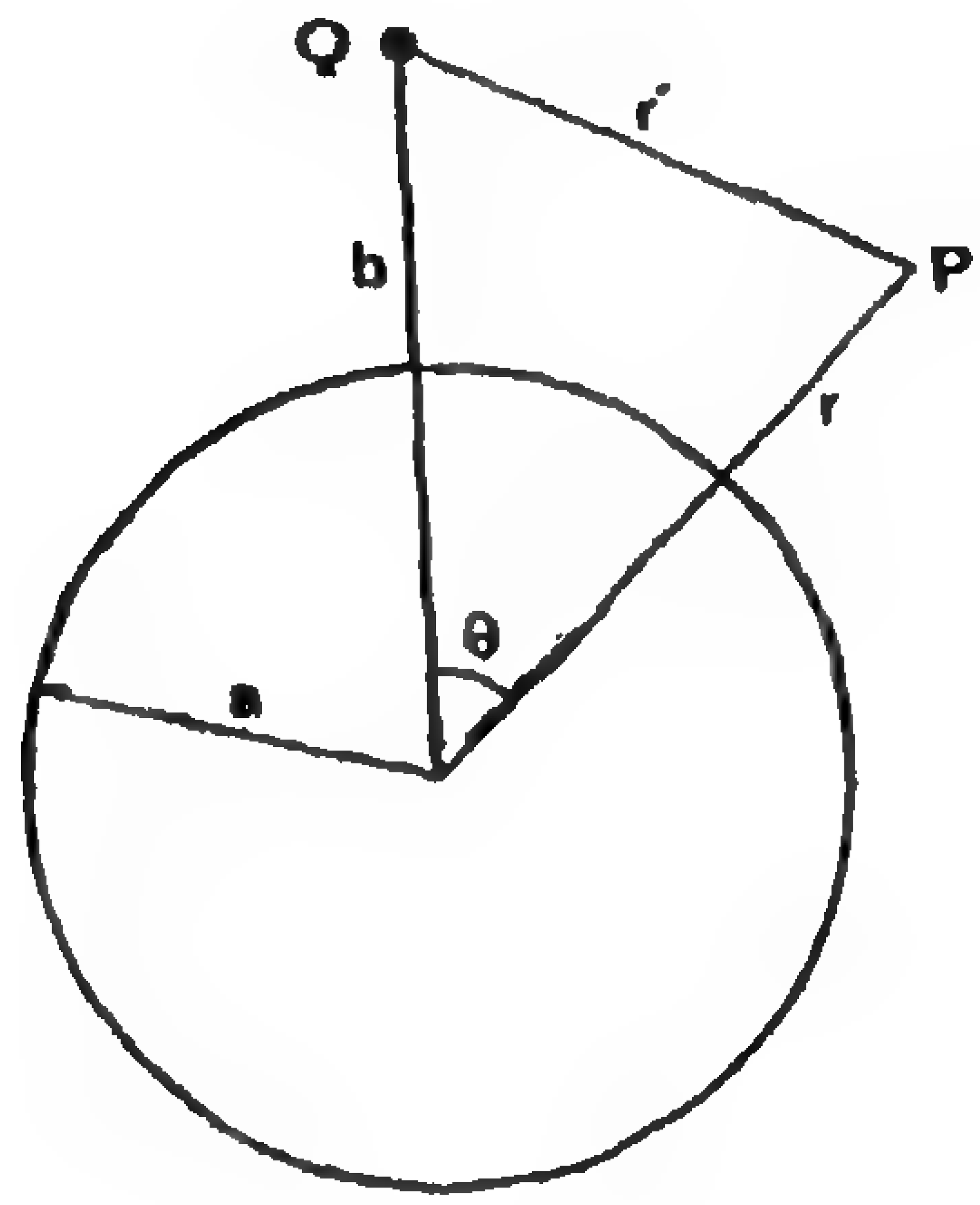


Fig. 10.28.

The coefficients A_n and B_n of (3) and (4) can be determined from the boundary conditions at the surface of the dielectric sphere, namely,

(i) Continuity of V_2 at $r = a$.

(ii) Continuity of the displacement density at $r=a$.

The first condition gives,

$$B_n = a^{2n+1} A_n \qquad (5)$$

while the second condition gives,

$$A_n = Qn (\epsilon_2 - \epsilon_1) \; / \; 4\pi \, b^{n+1} \epsilon_2 \, [\epsilon_1 n + (n+1) \epsilon_2] \qquad (6)$$

The potential in the region $r \leq a$ is that of (1) plus that of (3), the potential in the region $a < r \leq b$ is that given by (1) plus (4), and the potential in the region $r \geq b$ is given by (2) plus (4). The force on the charge Q is

$$F = -Q \nabla V_2 \qquad \text{at point } z = b$$

$$= -Q\, \partial V_2/\partial r \qquad \text{at } r = b,\ \theta = 0$$

$$= [(\epsilon_2 - \epsilon_1)\, Q^2/4\pi\, \epsilon_2\, b^2] \sum_{1}^{\infty} n(n+1)\, (a/b)^{2n+1} /$$

$$[n\epsilon_1 + (n+1)\, \epsilon_2]$$

32. *Use the result of the previous problem to obtain an expression for the electrostatic potential outside the dielectric sphere due to an electric dipole of moment p, distance b from the center, if the dipole axis is is in a radial direction. Find the force and torque on the dipole.*

The potential at any point can be obtained by considering the dipole to consist of two point charges $-Q$ at b and $+Q$ at $b+\delta b$ and then proceeding to the limit when δb tends to zero.

$$V' = \lim.\, [V(b+\delta b) - V(b)]$$

$$= (\partial V/\partial b)\, \delta b = (Q\, \delta b)\, \partial\, (V/Q) / \partial b$$

$$= p\, \partial\, (V/Q) / \partial b \tag{1}$$

The required values of potential follows from equations (1)—(4) of the previous problem. Now the forces on $-Q$, $+Q$ are radial so that there is no torque acting on the dipole. The force on the dipole is,

$$F = (\mathbf{p}\cdot\nabla)\, \mathbf{E}_2 \quad \text{in the radial direction}$$

$$= -p\, \partial^2 V_2' / \partial r^2 \qquad \text{at } r=b,\ \theta=0$$

$$= -p^2\, \partial^3 (V_2/Q) / \partial r^2\, \partial b \quad \text{at } r=b,\ \theta=0$$

Substituting for V_2 from equation (4) of the previous problem we get,

$$F = [p^2 (\epsilon_2-\epsilon_1) / 4\pi \epsilon_2] \sum_{1}^{\infty} n(n+1)^2 (n+2)\, a^{2n+1} / b^{2n+5} [\epsilon_1 n+ (n+1) \epsilon_2]$$

33. *A spherical air bubble of radius a is surrounded by a liquid of infinite extent and relative parmittivity ϵ_r. Show that if a point charge Q is placed at a distance b>a from the center of the bubble, the bubble will experience a force*

$$F = a^3 (\epsilon_r-1) Q^2/2\pi \epsilon_0 \epsilon_r (2 \epsilon_r + 1) b^5$$

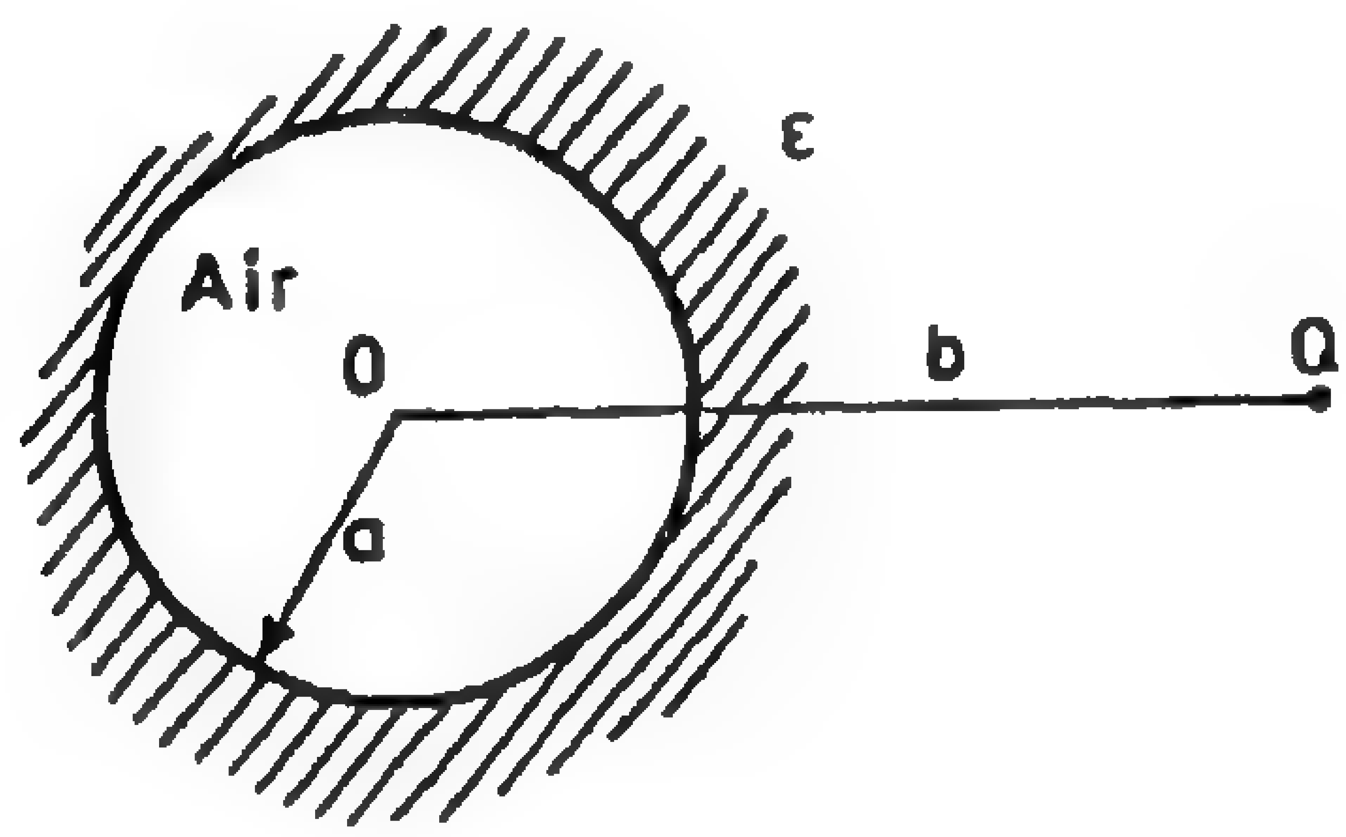


Fig. 10.29.

The potential outside the air bubble is same as that of equation (4) of problem 10.13. The second part of this equation gives the potential due to the induced charges on the bubble. With $\theta=0$, $r=b$, $\epsilon_1=\epsilon_0$, and $\epsilon_2 = \epsilon_r \epsilon_0$ we get, (Fig. 10.29),

$$V_A=[(1-\epsilon_r) Q/2\pi \epsilon_r \epsilon_0] \sum_{n=0}^{\infty} n\, a^{2n+1} / [n+(n+1)\epsilon_r]\, b^{2n+2} \qquad (1)$$

Since $b>>a$ we can neglect terms with $n\geq 2$ so that (1) gives,

$$V_A = a^3 (1-\epsilon_r) Q/2\pi \epsilon_0 \epsilon_r (1+2 \epsilon_r) b^4$$

The force acting on the charge Q is,

$$F = QE = -Q\,(\partial V_A/\partial b)$$

$$= a^3\,(\epsilon_r-1)\,Q^2/2\pi\epsilon_0\epsilon_r\,(2\,\epsilon_r+1)\,b^4$$

34. *A hollow dielectric sphere of radii a and b (b>a) and relative permittivity ϵ_r is placed in an originally uniform electric field. Show that inside the sphere the original field is reduced by the ratio $9\epsilon_r / [9\epsilon_r-2\,(\epsilon_r-1)^2\,(a^3/b^3-1)]$*

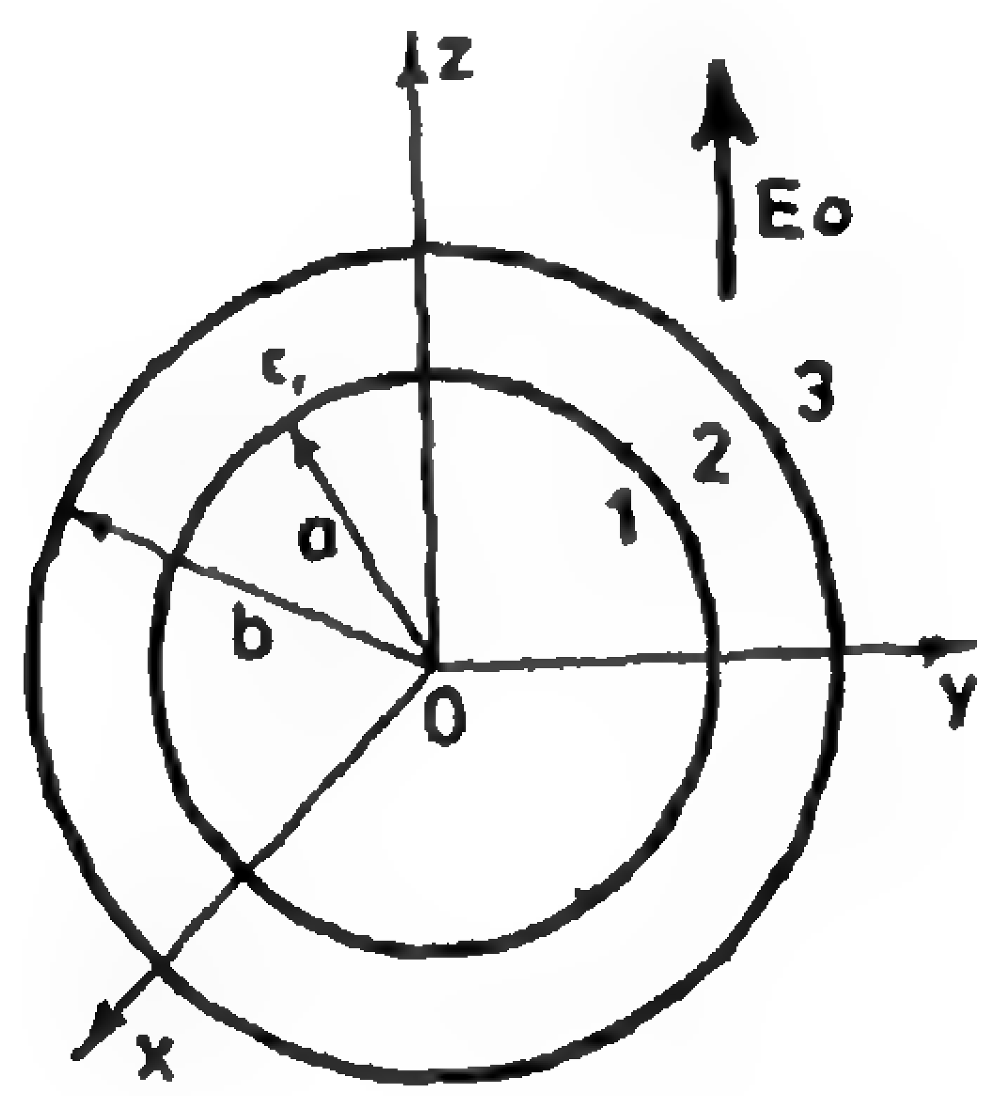


Fig. 10 30.

If the original field E_0 is taken along the z-axis, as shown in Fig. 10.30, a suitable solution to Laplace's equation in any of the regions 1, 2, and 3 is of the form (Appendix II),

$$V = [A\,r^n + B\,r^{-n-1}]\,P_n\,(\cos\,\theta) \tag{1}$$

where n is to be determined according to the boundary conditions,

(i) $V_3 \longrightarrow -E_0\,r\cos\theta$ as $r \longrightarrow \infty$, and V_1 finite as $r \longrightarrow 0$.

(ii) $V_2 = V_3$ at $r = b$, and $V_1 = V_2$ at $r = a$.

(iii) $\partial V_3/\partial r = \epsilon_r \, \partial V_2/\partial r$ at $r = b$.

(iv) $\epsilon_r \, \partial V_2/\partial r = \partial V_1/\partial r$ at $r = a$.

From (1) and (i) we conclude that,

$$V_3 = (Ar+B/r^2) \, P_1 (\cos \theta) = (Ar+B/r^2) \cos \theta$$

But as $r >>$, $Ar \cos \theta = -E_o r \cos \theta$ so that $A = - E_o$ and

$$V_3 = (-E_o r + B/r^2) \cos \theta \tag{2}$$

Also $V_2 = (Cr + D/r^2) \cos \theta$ (3)

and $V_1 = G r \cos \theta$ (4)

The constants B, C, D, and G can be determined from the boundary conditions (ii) to (iv), these give,

$$-E_o b + B/b^2 = Cb + D/b^2 \tag{5}$$

$$Ga = Ca + D/a^2 \tag{6}$$

$$-E_o - 2B/b^3 = \epsilon_r C - 2\epsilon_r D/b^3 \tag{7}$$

$$\epsilon_r C - 2\epsilon_r D/a^3 = G \tag{8}$$

Solving (5) to (8) we get,

$$G = -9\epsilon_r E_o \,/\, [9\epsilon_r - 2(\epsilon_r - 1)(a^3/b^3 - 1)]$$

The magnitude of the field inside the shell is,

$$E = |-\nabla V_1| = G \, [(\partial V_1/\partial r)^2 + (\partial V_1/r\partial\theta)^2]^{1/2}$$

$$= G$$

So that the field is reduced by the ratio G/E_o.

35. *A long hollow cylinder of relative permeability μ_r and radii a and b ($b>a$) is placed with its axis perpendicular to an initially uniform*

magnetic field (Fig. 10.31). Show that inside the cylinder the original field is reduced by the ratio $4\mu_r/[4\mu_r-(\mu_r-1)^2\,(a^2/b^2-1)]$.

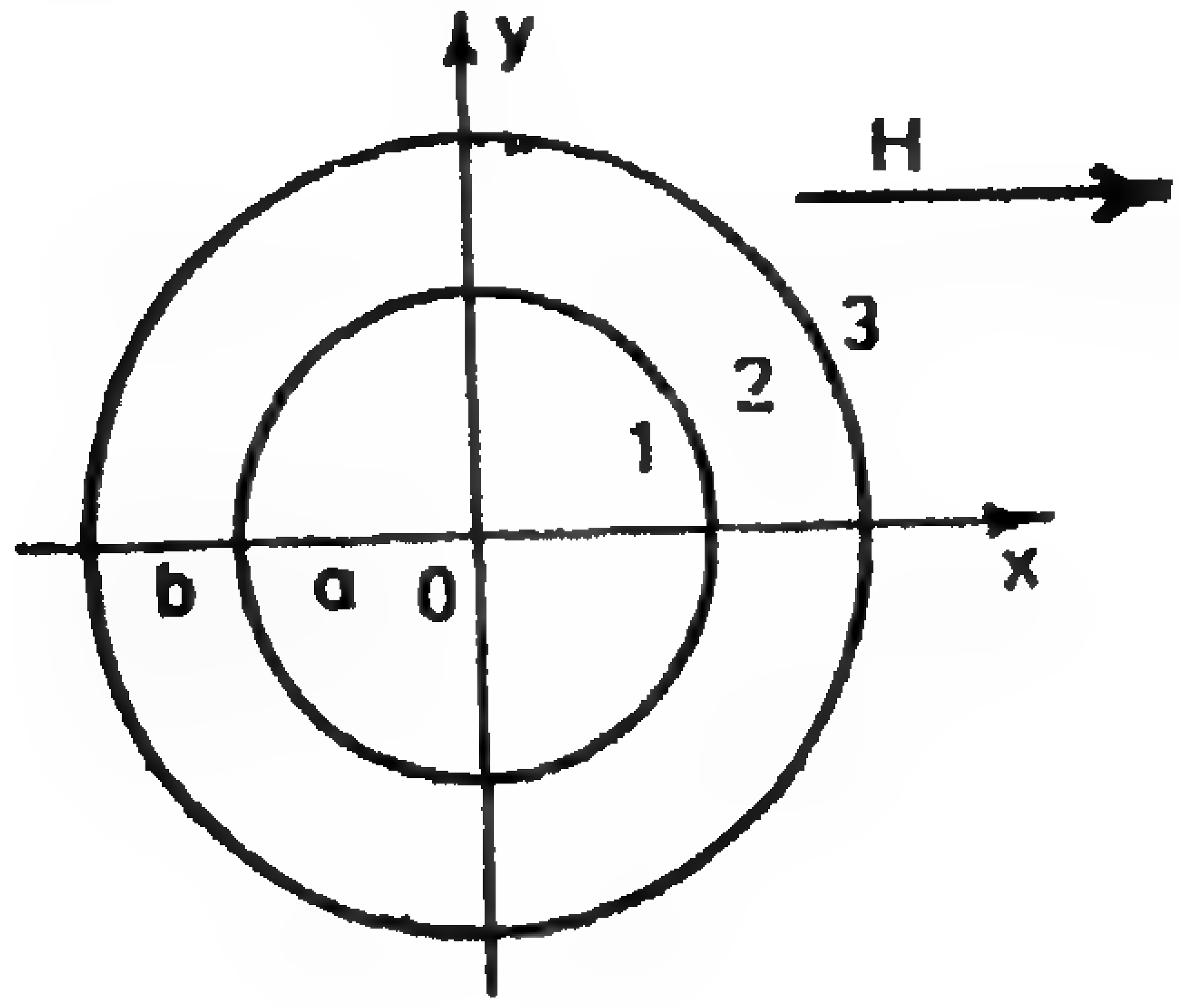


Fig. 10 31.

Since there is no variation along the axis of the cylinder we use the cylindrical coordinates in the plane polar form. The solution of Laplace's equation in any of the regions 1, 2, or 3 can be written in the form (see Appendix II),

$$\phi = (A\,r^n+B/r^n)\cos n\theta \tag{1}$$

where E is taken zero since we have symmetry about the $\theta=0$ axis, and D is absorbed in A and B. In this problem we have the boundary conditions on the magnetic potential function ϕ given by

(i) $\phi_3 \longrightarrow -H_0\, r\cos\theta$ as $r \longrightarrow \infty$ and ϕ_1 finite as $r \longrightarrow 0$

(ii) $\phi_2 = \phi_3$ at $r=b$, and $\phi_1 = \phi_2$ at $r=a$.

(iii) $\partial\phi_3/\partial r = \mu_r\,\partial\phi_2/\partial r$ at $r=b$

(iv) $\mu_r\,\partial\phi_2/\partial r = \partial\phi_1/\partial r$ at $r=a$

These boundary conditions give,

$$\phi_3 = (-E_0\,r+B/r)\cos\theta \tag{2}$$

$$\phi_2 = (Cr+D/r)\cos\theta \tag{3}$$

$$\phi_1 = Gr\cos\theta \tag{4}$$

Similar arguments as the previous problem give

$$G = 4\mu_r H_o \,/\, [4\mu_r - (\mu_r - 1)^2 \, (a^2/b^2 - 1)]$$

so that the original field H_o is reduced by the ratio G/H_o which again gives the required result.

36. *Prove that a distribution of surface density $\sigma = A\ P_n\,(\cos\theta)$ on a sphere gives rise to a potential which is at every point proportional to $A\ P_n\,(\cos\theta)$.*

Let the sphere be located with its center at origin, Fig. 10.23. First determine the potential at any point A on the z–axis.

$$V_A = \int (\sigma/4\pi\,\epsilon_o R)\; dS$$

$$= (1/4\pi\,\epsilon_o) \int_0^{\pi} [A\ P_n\,(\cos\ \theta)/R]\ 2\pi\ a^2\ \sin\theta\ d\,\theta$$

$$= (a^2A/2\epsilon_o) \int_{-1}^{1} [a^2+r^2-2ar\ \cos\theta]^{-1/2}\, P_n\,(\cos\theta)\, d\,(\cos\theta)$$

If point A is internal to the sphere, $r \lesssim a$.

$$V_A = (aA/2\epsilon_o) \int_{-1}^{1} P_n\,(\cos\theta)\ \ \Sigma\ \ (r/a)^m\ P_m\,(\cos\theta)\ d\,(\cos\theta)$$

$$= [aA/\epsilon_o\ (2n+1)]\ (r/a)^n \tag{1}$$

If the point A is external to the sphere,

$$V_A = (aA/2\epsilon_o) \int_{-1}^{1} P_n\,(\cos\theta)\ \sum_0^{\infty}\ (a/r)^{m+1}\, P_m(\cos\theta)\ d\,(\cos\theta)$$

$$= [Aa/\epsilon_o\ (2n+1)]\ (a/r)^{n+1} \tag{2}$$

When A is a general point the potentials are same as (1) and (2) after multiplying by $P_n(\cos\theta)$,

$$V_{A_1} = [Aa/\epsilon_0(2n+1)]\,(r/a)^n\,P_n(\cos\theta)\,, \qquad r \lesssim a \qquad (3)$$

$$V_{A_2} = [Aa/\epsilon_0(2n+1)]\,(a/r)^{n+1}\,P_n(\cos\theta)\,, \qquad r \gtrsim a \qquad (4)$$

This is true since both (3) and (4) satisfy Laplace's equation and at the same time satisfy the boundary condition at $r=a$,

$$\epsilon_0\,\partial V_{A_2}/\partial r - \epsilon_0\,\partial V_{A_1}/\partial r = \sigma = A\,P_n(\cos\theta)$$

at $r = a$.

37. *A sphere of radius a is charged with surface density* $\sigma = \sum_0^{\infty} A_n P_n(\cos\theta)$. *Find an expression for the potential at any point.*

Using the results (3) and (4) of the previous problem the potential at any general point is,

$$V = (a/\epsilon_0)\sum_0^{\infty}(2n+1)^{-1}(r/a)^n A_n P_n(\cos\theta) \qquad r \lesssim a \qquad (1)$$

and

$$V = (a/\epsilon_0)\sum_0^{\infty}(2n+1)^{-1}(a/r)^{n+1} A_n P_n(\cos\theta) \qquad r \gtrsim a \qquad (2)$$

38. *Charge is distributed over the surface of a sphere of radius a, such that, (Fig. 10.32).*

$$\sigma = \sigma_0 \qquad 0 \le \theta \le \pi/3$$

$$\sigma = 0 \qquad \pi/3 < \theta \le \pi$$

The given charge distribution can be expanded in a series of Legendre polynomials in the form,

$$\sigma = \sum_{m=0}^{\infty} A_m P_m (\cos \theta) \tag{1}$$

The coefficients A_n can be obtained by multiplying both sides of (1) by $P_n (\cos \theta)$ and integrating with respect to $\cos \theta$ from $\theta = 0$ to π. This gives,

$$\int_{-1}^{1} \sigma P_n(\cos \theta)\, d(\cos \theta) = \sum_{m=0}^{\infty} A_m \int_{-1}^{1} P_n(\cos \theta)\, P_m(\cos \theta)\, d \cos \theta$$

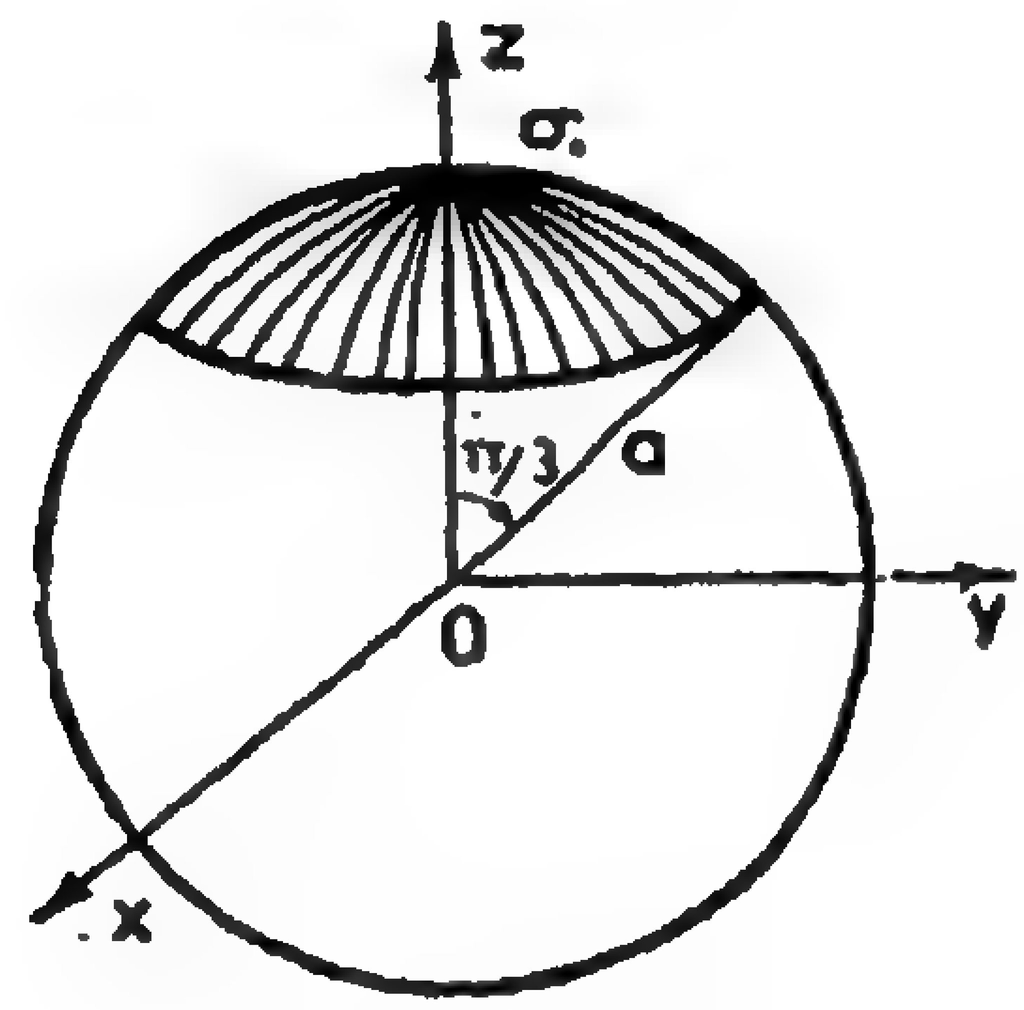


Fig. 10.32.

Due to the orthogonality of the Legendre polymonials the right hand side is $2A_n / (2n+1)$ so that,

$$A_n = \tfrac{1}{2} (2n+1) \int_{-1}^{1} \sigma P_n (\cos \theta)\, d \cos \theta$$

$$= \tfrac{1}{2} (2n+1) \int_{\cos(\pi/3)}^{1} \sigma_o P_n (u)\, d u \tag{2}$$

where, $u = \cos \theta$. This integral can be evaluated using the relation,

$$(d/du) \left[P_{n+1} (u) - P_{n-1} (u)\right] = (2n+1) P_n (u) \tag{3}$$

so that we get,

$$A_n = \tfrac{1}{2}\,\sigma_o\,[P_{n-1}(\tfrac{1}{2}) - P_{n+1}(\tfrac{1}{2})]\ , \qquad n>o \tag{4}$$

when $n = o$ we have [using (2)],

$$A_o = \tfrac{1}{2}\,\sigma_o \int_{1/2}^{1} du = \sigma_o/4 \tag{5}$$

Therefore the expansion of σ in terms of the Legendre polynomials gives,

$$\sigma = \sigma_o/4 + \tfrac{1}{2}\,\sigma_o \sum_{1}^{\infty} [P_{n-1}(\tfrac{1}{2}) - P_{n+1}(\tfrac{1}{2})]\; P_n(\cos\theta)$$

Using the result of the previous problem, the potential at any point is,

$$V = (\sigma_o a/2\epsilon_o)\left\{ \tfrac{1}{2} + \sum_{1}^{\infty} (2n+1)^{-1} [P_{n-1}(\tfrac{1}{2}) - P_{n+1}(\tfrac{1}{2})]\,(r/a)^n\, P_n(\cos\theta)\right\} \quad r \leq a$$

or

$$V = (\sigma_o a/2\epsilon_o)\left\{ a/2r + \sum_{1}^{\infty} (2n+1)^{-1} [P_{n-1}(\tfrac{1}{2}) - P_{n+1}(\tfrac{1}{2})]\,(a/r)^{n+1}\, P_n(\cos\theta)\right\} \quad r \geq a$$

39. *Determine the potential due to a uniformly charged circular wire of radius a.*

The potential at any point P, Fig. 10.33, must satisfy Laplace's equation and must be finite at origin and tend to zero as r tends to ∞. The two conditions at origin and at infinity cannot be satisfied simultaneously by a single potential function. Therefore the solution will be obtained in two separate regions $r \geq a$, and $r \leq a$. Suitable solutions of Laplace's equation in these regions are,

$$V_1 = \sum_{0}^{\infty} B_n\, r^{-n-1}\, P_n(\cos\theta) \qquad r \geq a \tag{1}$$

$$V_2 = \sum A_n\, r^n\, P_n(\cos\theta) \qquad r \leq a \tag{2}$$

The coefficients A_n and B_n are obtained from the boundary conditions; (i) $V_1 = V_2$ at $r = a$, $\theta \neq \pi/2$, and (ii) $\partial V_2/\partial r - \partial V_1/\partial r = \sigma/\epsilon_0$ at $r = a$. Where σ is the surface charge on the sphere of radius a when $\theta = \pi/2$. These boundary conditions give,

$$B_m = a^{2m+1} A_m \tag{3}$$

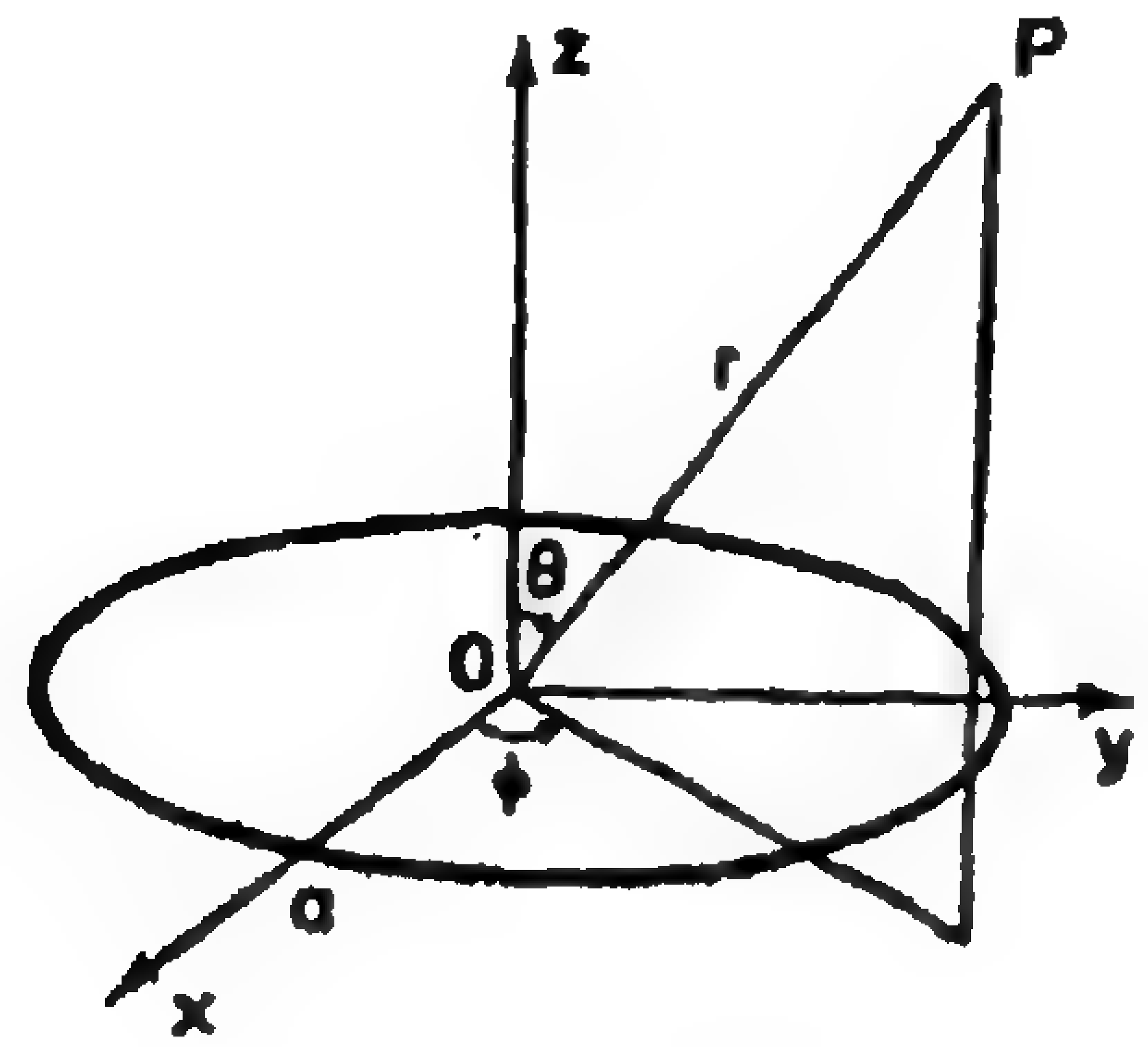


Fig. 10.33.

$$\Sigma (n+1) a^{-n-2} B_n P_n (\cos \theta) + \Sigma n a^{n-1} A_n P_n (\cos \theta)$$
$$= \sigma \delta (\theta - \pi/2) \tag{4}$$

where $\delta (\theta - \pi/2)$ is the Dirac delta function which is defined by,

$$\delta (\theta - \pi/2) = \begin{cases} 0 & \theta \neq \pi/2 \\ 1 & \theta = \pi/2 \end{cases}$$

The value of σ in the right hand side of (4) can be obtained from the total charge on the ring.

$$2\pi a\lambda = \int_0^\pi \sigma \delta (\theta - \pi/2) 2\pi a^2 \sin \theta \, d\theta$$

$$= 2\pi a^2 \sigma \quad \text{that is,} \quad \sigma = \lambda/a,$$

Another relation between A_m and B_m can be obtained by multiplying both sides of (4) by $P_m(\cos\theta)$ and integrating with respect to $\cos\theta$ from -1 to 1. Hence we get,

$$(m+1)\, a^{-m-2} B_m + m\, a^{m-1} A_m = [(2m+1)\lambda/2\epsilon_o a]\, P_m(0) \qquad (5)$$

Solving (3) and (5) for A_m and B_m we get,

$$A_m = \lambda\, P_m(o) / 2\, \epsilon_o\, a^m$$

$$B_m = \lambda\, P_m(o)\, a^{m+1}/2\, \epsilon_o$$

From the properties of the Legendre polynomials we have that,

$$P_{2m+1}(o) = 0\ , \text{ and } P_{2m}(o) = (-1)^m\, (\tfrac{1}{2})_m\ /\ m\,!.$$

where, $(\frac{1}{2})_i = \frac{1}{2}\,(\frac{1}{2}+1)\,(\frac{1}{2}+2)\ \ldots\ (\frac{1}{2}+i-1)$.

Thus the potential at any point is,

$$V_1 = (\lambda/2\epsilon_o) \sum_{o}^{\infty} X_m\, (a/r)^{2m+1}\, P_{2m}(\cos\theta)\ , \qquad r \geq a \qquad (6)$$

$$V_2 = (\lambda/2\epsilon_o) \sum_{o}^{\infty} X_m\, (r/a)^{2m}\, P_{2m}(\cos\theta)\ , \qquad r \leq a \qquad (7)$$

where,

$$X_m = (-1)^m\, (\tfrac{1}{2})_m\ /\ m\,!$$

40. *Use the results of the previous problem to deduce the potential if P lies on the z-axis. Compare the results when the potential is directly obtained.*

On the *z-axis*, $\theta=0$, and $r=z$ so that (12) and (13) of the previous problem becomes,

$$V_1 = (\lambda/2\epsilon_o)\ \Sigma\ X_m\, (a/z)^{2m+1} \qquad z \geq a \qquad (1)$$

$$V_2 = (\lambda/2\epsilon_o)\ \Sigma\ X_m\, (z/a)^{2m} \qquad z \leq a \qquad (2)$$

where, $X_m = (-1)^m\, (\frac{1}{2})_m\ /\ m\,!$

For direct determination of the potential consider a point on the z—axis,

$$V = (2\pi\, a\lambda/4\pi\,\epsilon_0)\,(a^2+z^2)^{-1/2}$$

For $z \geq a$

$$V = (\lambda/2\epsilon_0)\,(a/z)\,[1+(a^2/z^2)]^{-1/2} \quad (3)$$

while for $z \leq a$

$$V = (\lambda/2\,\epsilon_0)\,[1 + (z^2/a^2)]^{-1/2} \quad (4)$$

By using the binomial expansion we obtain V_1 and V_2 respectively.

41. *Find an expression for the potential on the axis of a uniformly charged disc of radius a and total charge Q. Deduce an expression for the potential at a general point.*

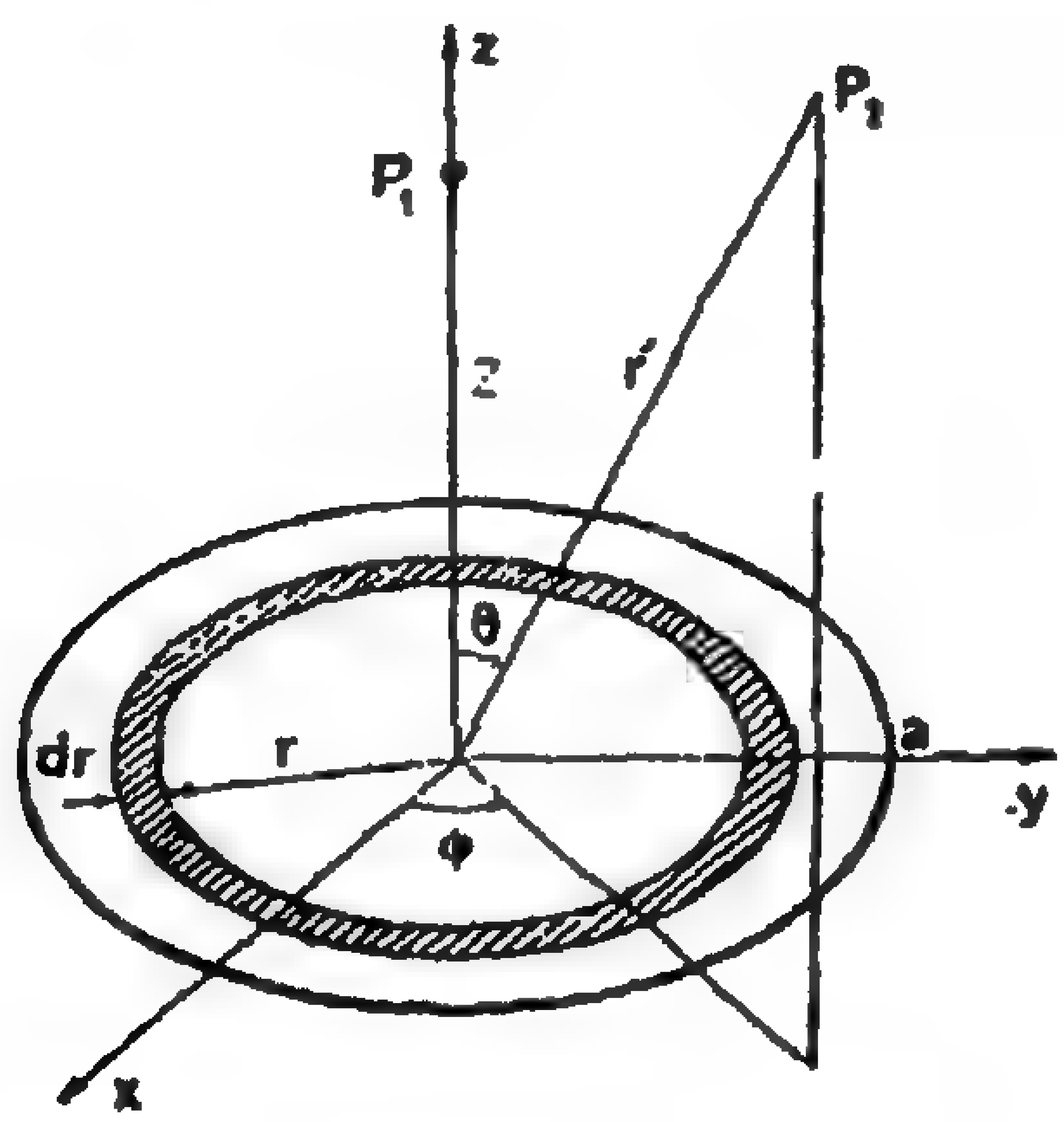


Fig. 10.34.

The potential at a point P_1 distance z from the disc due to an elementary ring of radius r and thickness dr is, (Fig. 10.34),

$$dV = \sigma 2\pi\, r dr/4\pi\,\epsilon_0\,(z^2+r^2)^{1/2}$$

Integrating we get the potential at point A (z, o)

$$V = (\sigma/2\epsilon_0)\,[(z^2+a^2)^{1/2}-z]$$

$$= (Q/2\pi\,\epsilon_0\,a^2)\,[(z^2+a^2)^{1/2}-z] \qquad (1)$$

for $z \leq a$,

$$V = (Q/2\pi\,\epsilon_0\,a)\,[(1+z^2/a^2)^{1/2}-z/a]$$

$$= (Q/2\pi\,\epsilon_0\,a)\,[1-z/a+\tfrac{1}{2}(z^2/a^2)-1/2.1/4.(z^2/a^2)^2+$$

$$1/2.1/4.3/6\,(z^2/a^2)^3+\ldots\ldots(-1)^{n-1}\,1/2.1/4.3/6.5/8.$$

$$(2n-3)/2n.(z^2/a^2)^n+\ldots] \qquad (2)$$

while for $z \geq a$, a similar procedure gives,

$$V = (Qz/2\pi\,\epsilon_0\,a^2)\,[(1+a^2/z^2)^{1/2}-1]$$

$$= (Qz/2\,\pi\,\epsilon_0\,a^2)\,[\tfrac{1}{2}(a^2/z^2)-1/2.1/4.(a^2/z^2)^2+$$

$$1/2.1/4.3/6.(a^2/z^2)^3+\ldots+(-1)^{n-1}\,1/2.1/4.3/6.5/8.$$

$$(2n-3)/(2n).(a^2/z^2)^n+\ldots] \qquad (3)$$

Since the problem has symmetry about the z-axis, the electrostatic potential at a general point P_2 (r, θ) is,

$$V_1 = \sum_{0}^{\infty} A_n\,r^n\,P_n(\cos\theta) \quad , \qquad r \leq a \qquad (4)$$

$$V_2 = \sum_{0}^{\infty} B_n\,r^{-n-1}\,P_n(\cos\theta) \quad , \qquad r \geq a \qquad (5)$$

The solutions (4) and (5) are identical to (2) and (3) at $\theta=0$; we have that,

$$A_0 = Q/2\pi\,\epsilon_0\,a,\ A_1 = -Q/2\pi\epsilon_0\,a^2,\ A_{2n+1} = 0,$$

$$A_{2n} = (-1)^{n-1}\,1.3.5.7.\ \ldots\ (2n-3)\,Q/2^{n+1}\,n\,!\,\pi\,\epsilon_0\,a^{2n}$$

$$B_0 = Q/4\pi\,\epsilon_0\ ,\ B_{2n+1} = 0,\ \text{and}$$

$$B_{2n} = (-1)^n\,1.3.5.\ (2n-1)\,a^{2n-1}\,Q/2^{n+2}\,(n+1)\,!\,\pi\,\epsilon_0$$

42. *Express the potential due to a uniformly charged ring in terms of elliptic integrals.*

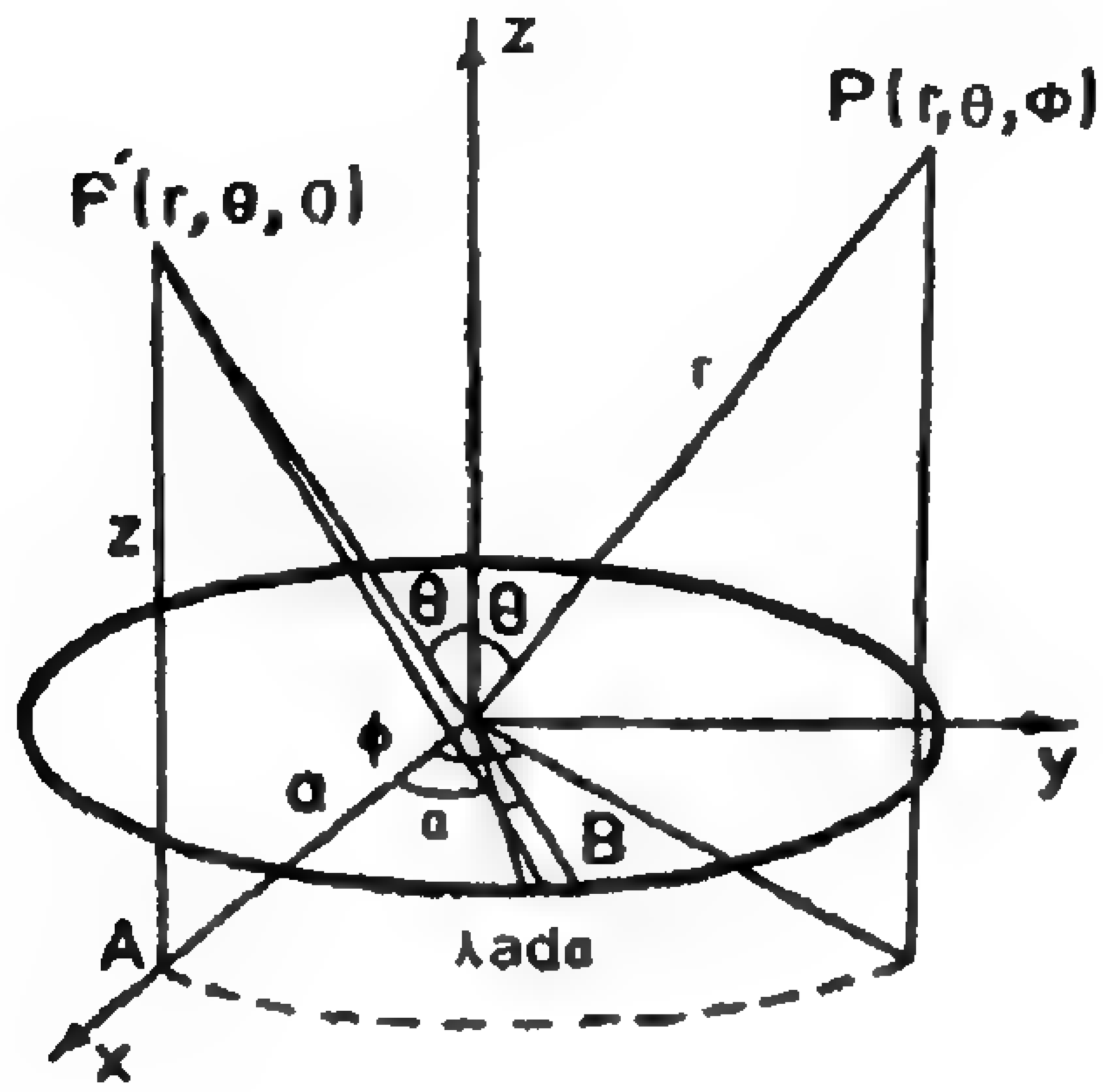


Fig. 10 35.

Let the ring be located in the xy–plane as shown in Fig. 10.35. Due to symmetry of the problem about the z–axis, the potential at any point P (r, θ, ϕ) is the same as that at point P' $(r, \theta, 0)$. Using the cylinrical coordinates consider an elementary charge $\lambda\ a\ d\ \alpha$. The potential at P' due to this element is

$$dV_p' = \lambda\ a\ d\ \alpha\ /\ 4\pi\ \epsilon_0\ (P'B)$$

$$= \lambda\ a\ d\ \alpha\ /\ 4\pi\ \epsilon_0\ [(P'A)^2+(BA)^2]^{1/2}$$

$$= \lambda\ a\ d.\ \alpha\ /\ 4\pi\ \epsilon_0\ [z^2+(a+r)^2-4ar\ cos^2\ \tfrac{1}{2}\ q]^{1/2}$$

Integrating we get, with $k = 2\ (ar)^{1/2}\ /\ [(a+r)^2+z^2]^{1/2}$,

$$V_p' = \{\ \lambda\ a/4\pi\epsilon_0\ [z^2+(a+r)^2]^{1/2}\ \} \int_0^{2\pi} d\alpha\ /\ (1-k^2cos^2\ \tfrac{1}{2}\alpha)^{1/2} \quad (1)$$

Let $\delta = \frac{1}{2}(\pi - \alpha)$, $d\delta = -\frac{1}{2} d\alpha$, and (1) reduces to,

$$V_P = V_P' = (\lambda k/2\pi\epsilon_0)(a/r)^{1/2} \int_0^{\pi/2} d\delta \,/\, (1-k^2 \sin^2\delta)^{1/2} \quad (2)$$

Thus the potential at P and P' has been obtained in terms of elliptic integral. The main objective of representation of the potential in this form is that the elliptic integral has been approximated recently by J. Hasting by a rational fraction

$$\int_0^{\pi/2} d\delta \,/\, (1-k^2 \sin^2\delta)^{1/2} = (1.38629 + 0.11197\eta + 0.07253\eta^2)$$

$$+ (0.5 + 0.12135\eta + 0.02887\eta^2) \log (1/\eta)$$

with $\eta = 1 - k^2$ and $0 \leq k < 1$.

Thus when it is required to determine the potential due to the circular ring, we do not need to evaluate an infinite series of Legendre polynomials but we use this approximation which is accurate to 10^{-4}.

43. *Find the energy of interaction and the force between two parallel coaxial rings of radii a and b, charged uniformly with densities λ_1 and λ_2 C/m. The distance between ring-planes is d.*

Let the two rings be located as shown in Fig. 10.36. The potential due to the ring in the xy-plane at the second ring is constant and has the value

$$V = (\lambda_1 k/2\pi\epsilon_0)(a/b)^{1/2} \int_0^{\pi/2} d\delta \,/\, (1-k^2 \sin^2\delta)^{1/2}$$

where $\quad k = 2\,[ab]^{1/2} \,/\, [(a+b)^2 + d^2]^{1/2}$

The potential energy due to an elementary charge $\lambda_2\, b\, d\,\phi$ is

$$dU = \tfrac{1}{2}\,(\lambda_2\, b\, d\phi)\, V$$

Integrating we get,

$$U = [\lambda_1\lambda_2\, k\, (ab)^{1/2} / 2\,\epsilon_o] \int_0^{\pi/2} d\,\delta \,/\, (1-k^2 \sin^2 \delta)^{1/2}$$

$$= [\lambda_1\lambda_2\, k\, (ab)^{1/2} / 2\,\epsilon_o]\; K\,(k)$$

where

$$K(k) = \int_0^{\pi/2} d\,\delta \,/\, (1-k^2 \sin^2 \delta)^{1/2}$$

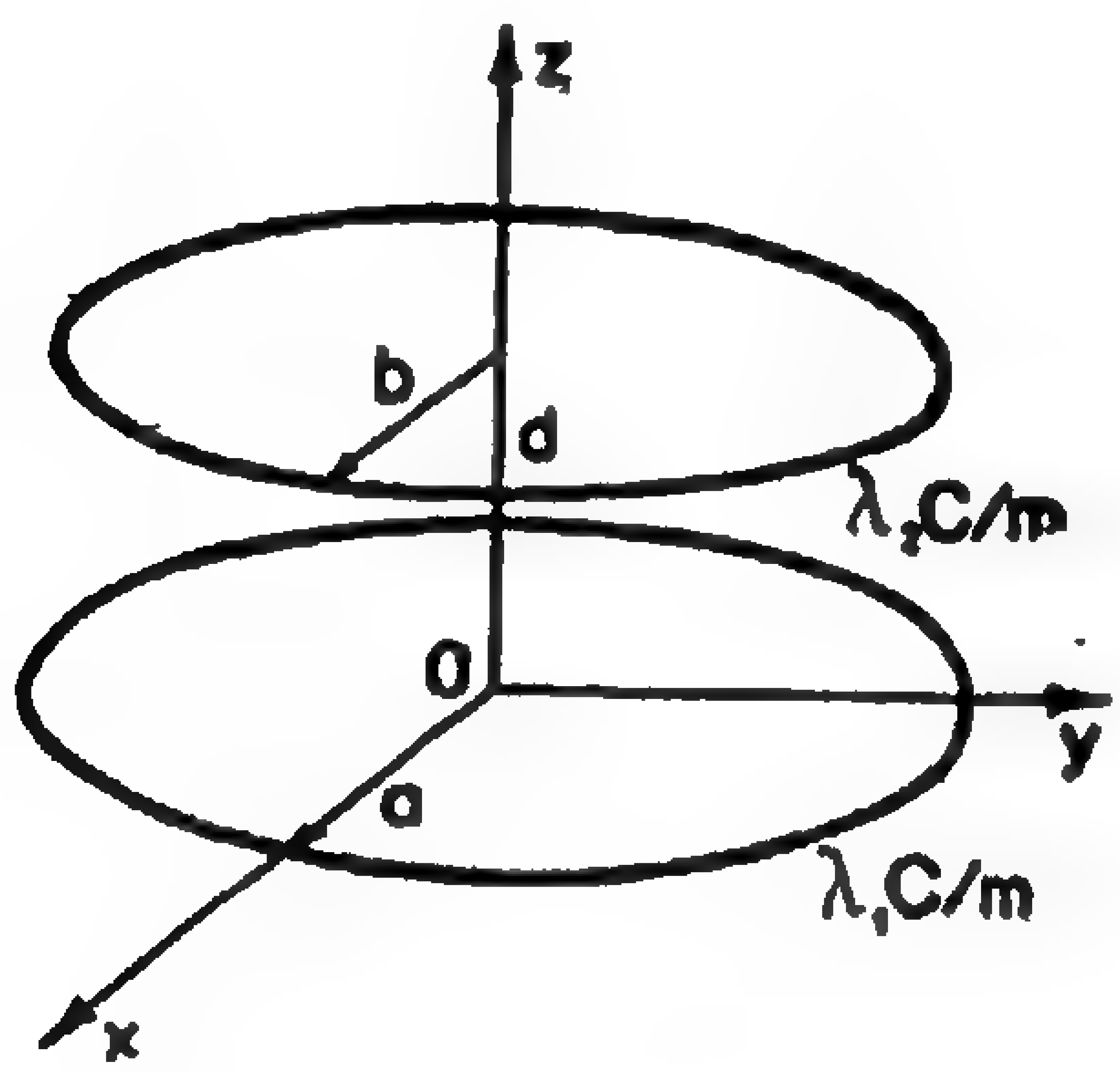


Fig. 10.36.

The force between the two rings is,

$$F = -\,\partial U/\partial d = -(\partial U/\partial k)\,(\partial k/\partial d)$$

$$= -\,(\partial k/\partial d)\,[\lambda_1\lambda_2\,(ab)^{1/2}/2\epsilon_o]\;[K(k)+k\,(\partial K/\partial k)]$$

$$= (\lambda_1\lambda_2\, d\, k^3/4\,\epsilon_o)\;[K\,(k) + k\;\partial K/\partial k]$$

From the properties of elliptic integrals,

$$K(k) + k\,(\partial K/\partial k) = K(k) + 2\,k^2\,(\partial K/\partial k^2)$$

$$= [1\,/\,(1-k^2)]\;G(k)$$

where

$$G(k) = \int_0^{\pi/2} (1-k^2 \sin^2 \delta)^{1/2}\, d\delta$$

Substituting in the force expression we get,

$$F = (\lambda_1 \lambda_2\, d/4\epsilon_0)\;[k^3/(1-k^2)]\;G(k)$$

in the z-direction.

As in the previous example the force can be numerically evaluated since $G(k)$ can be obtained numerically in the form,

$$G(k) \simeq (1.0 + 0.46301\,\eta + 0.10778\,\eta^2)$$

$$+ \;(0.24527\,\eta + 0.0412496\,\eta^2)\;\log\,(1/\eta)$$

with $o \leq k < 1$ and $\eta = 1-k^2$. When evaluated the error is less than 10^{-4}.

44. *A current I flows in a circular wire of radius a. A second circular wire of radius b ($b << a$) is concentric with and its plane perpendicular to that of the first wire. If the second wire carries a current I show that it will experience a couple of magnitude* $(\pi \mu_0\, I^2 b^2/2a)\;[1 + O\,(b^2/a^2)]$.

As shown in Fig. 10.37 the magnetic potential at point P is,

$$V_m = -\,(I/4\pi)\;\Omega = -\tfrac{1}{2}\,I\,(1-\cos\psi)$$

$$= -\tfrac{1}{2}\,I\;[1 - z\,/\,(z^2+a^2)^{1/2}]$$

when $z<a$

$$V_m = -\tfrac{1}{2} I \left[1-(z/a)\left(1+z^2/a^2\right)^{-1/2}\right]$$

$$= -\tfrac{1}{2} I \left[1-z/a+\tfrac{1}{2}(z/a)^3-(1.3/2.4)(z/a)^5+\ldots\right]$$

Since we have symmetry about the z–axis, the potential at a general point is,

$$V_m = -\tfrac{1}{2} I \left[1-(r/a)\, P_1(\cos\theta)+\tfrac{1}{2}(r/a)^3\, P_3(\cos\theta)-\ldots\right]$$

To find the force on the smaller loop we must determine the radial field component B_r in the plane of this loop at $r=b$.

$$B_r = -\mu_0\, \partial V_m/\partial r \qquad \text{at } r=b$$

$$= (\mu_0\, I/2a)\left[P_1(\cos\theta)-(3/2)(b/a)^2\, P_3(\cos\theta)+\ldots\right]$$

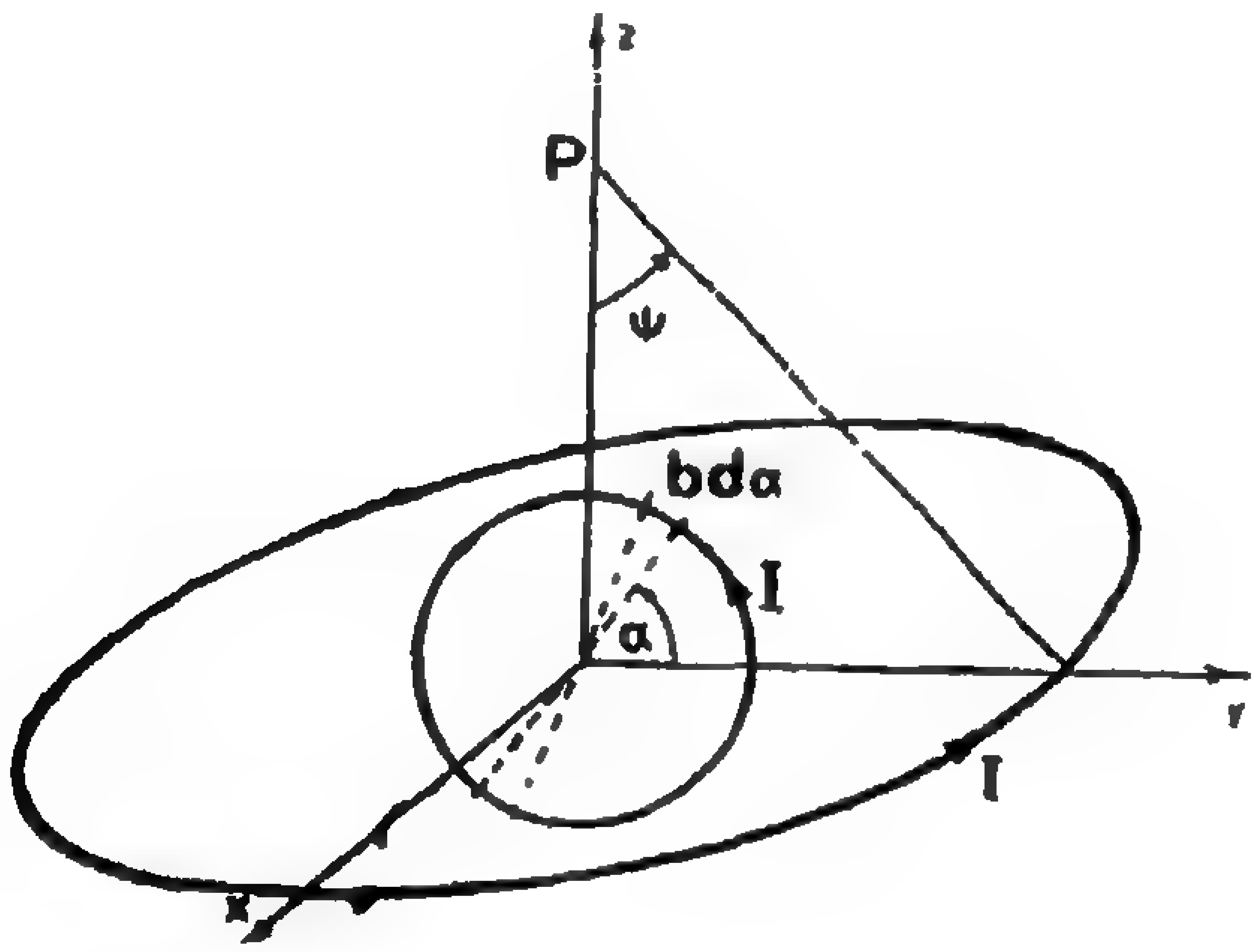


Fig.10.37.

The force on an elementary length $b\, d\, \alpha$ is

$$dF = I B_r\, b\, d\alpha$$

perpendicular to the loop plane. An equal but opposite force acts on the length $b\, d\, \alpha$ at angle $\alpha+\pi$. The two elementary forces have a

moment about a line perpendicular to the large loop.

$$dT = (2b \cos a)\, dF$$

Substituting for B, and integrating we get,

$$T = (\mu_o b^2 I^2/a) \int_0^{\pi} [P_1 - 1.5\,(b/a)^2 P_3 + \ldots] \cos\theta\, d\theta$$

$$= (\mu_o b^2 I^2/a) \left[\int_0^{\pi} \cos^2\theta\, d\theta + O\,(b^2/a^2) \right]$$

$$= (\pi \mu_o I^2 b^2/2a)\,[1 + O\,(b^2/a^2)]$$

15. *A uniform circular wire of radius a, charged with line density λ, surrounds a grounded concentric spherical conductor of radius b, Fig. 10.38. Determine the force acting on the spherical conductor.*

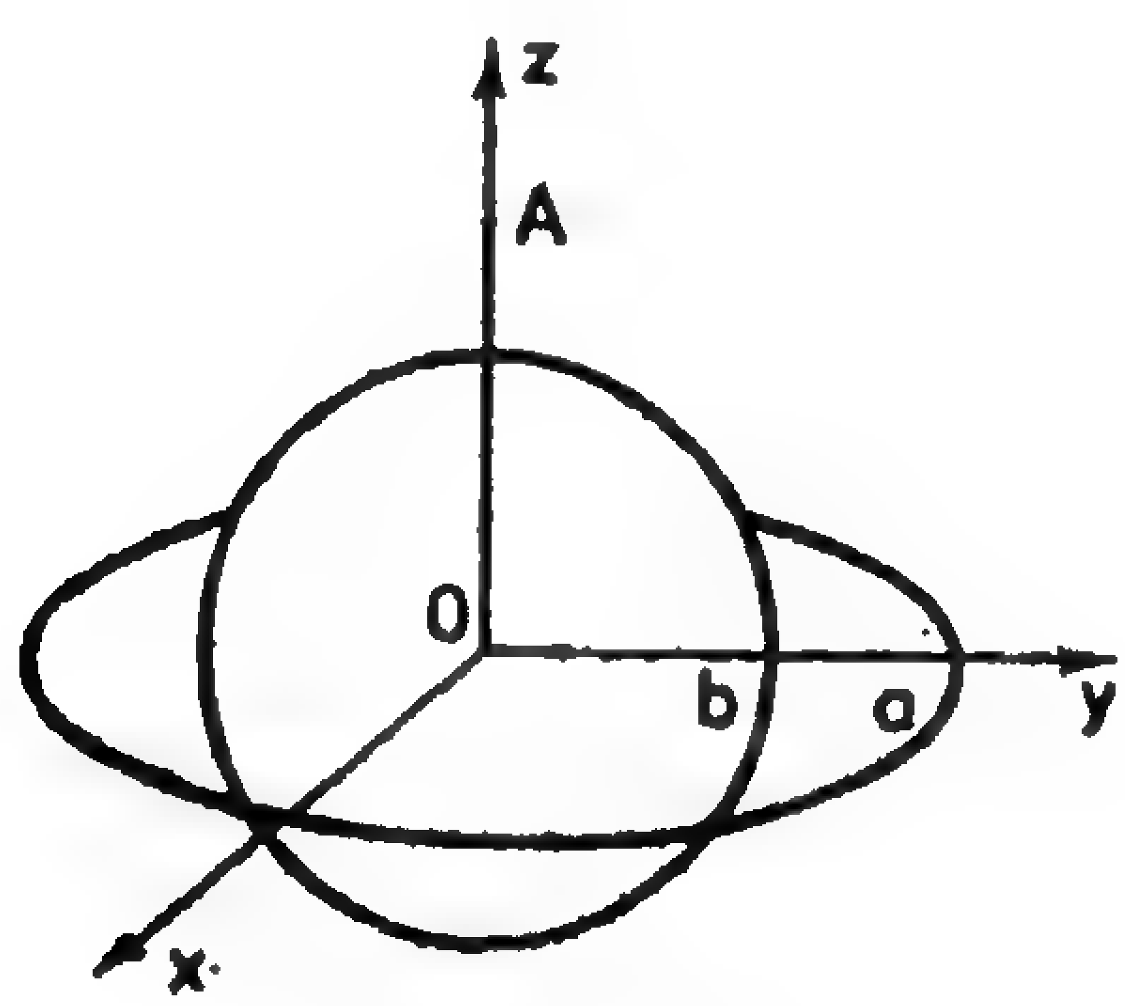


Fig. 10.38.

We have two potential functions. V_1 is valid in the region $b \leq r \leq a$, and V_2 is valid for $r > a$. From the results of problem 10.39 equations (12), (13), the solution contains only even harmonics, so that we can write,

$$V_1 = (\lambda/2\epsilon_0) \quad \Sigma \quad [X_n(r/a)^{2n} + A_n(r/a)^{2n} + B_n(b/r)^{2n+1}] \; P_{2n}(\cos\theta) \quad (1)$$

$$V_2 = (\lambda/2\epsilon_0) \quad \Sigma \quad [X_n(a/r)^{2n+1} + C_n \; (a/r)^{2n+1}] \; P_{2n} \; (\cos \; \theta) \qquad (2)$$

where,

$$X_n = (-1)^n \; (1/2)_n \; / \; n \; !$$

The coeffieicents A_n, B_n, and C_n are determined from the boundary conditions;

$$V_1 = 0 \text{ at } r = b \; , \; V_1 = V_2 \text{ and } \partial V_1/\partial r = \partial V_2/\partial r \text{ at } r = a.$$

These conditions give,

$$X_n (b/a)^{2n} + A_n (b/a)^{2n} + B_n = 0 \qquad (3)$$

$$A_n + B_n \; (b/a)^{2n+1} - C_n \quad = 0 \qquad (4)$$

$$X_n (4n+1) + 2nA_n - (2n+1) \; (b/a)^{2n+1} \; B_n = -(2n+1)C_n \quad (5)$$

Eliminating C_n between (4) and (5) we get that,

$$A_n = -X_n = -C_n \text{ , and } B_n = 0$$

Thus,

$$V_1 = (\lambda/2\epsilon_0) \; \sum_0^{\infty} \; X_n \quad (r/a)^{2n} - (b/r)^{2n+1} (b/a)^{2n}] \; P_{2n}(\cos \; \theta) \qquad (6)$$

The surface charge density on the sphere is,

$$\sigma = -\epsilon_0 \; \partial V_1/\partial r \qquad \text{at } r = b$$

$$\sigma = -(\lambda/2b) \quad \Sigma \quad X_n \; (4n+1) \; (b/a)^{2n} P_{2n} \; (\cos \; \theta) \qquad (7)$$

The force acting on the sphere is thus,

$$F = \int_0^{\pi} (\sigma^2/2\epsilon_0) \; 2\pi \; b^2 \sin \; \theta \cos \theta \; d\, \theta$$

in the z-direction. Substituting for σ we get,

$$F = (\pi\lambda^2/2\epsilon_0) \Sigma \Sigma X_n X_m (b^2/a^2)^{m+n} \int_{-1}^{1} P_{2n}(\mu) P_{2m}(\mu) P_1(\mu)\, d\mu$$

using the relation,

$$(2i+1)\ P_i\ P_1 = (i+1)\ P_{i+1} + i\ P_{i-1}$$

we get that,

$$\int_{-1}^{1} P_{2n}(\mu)\ P_{2m}(\mu)\ P_1(\mu)\ d\mu$$

$$= (2m+1) \int_{-1}^{1} P_{2m+1}(\mu)\, P_{2n}(\mu)\, d\mu + 2m \int_{-1}^{1} P_{2n}(\mu)\, P_{2m-1}(\mu)\, d\mu$$

$$= 0$$

Thus, the force acting on the sphere is zero.

46. *A uniformly charged circular wire of radius a and total charge* Q *surrounds a dielectric sphere of permittivity ϵ and radius b. If the plane of the wire passes by the center of the sphere find an expression for the potential at any point inside the dielectric sphere.*

The potential at point A, Fig. 10.38, due to the charged wire only is,

$$V_w = Q/4\pi\epsilon_0\, (a^2+z^2)^{1/2}$$

$$= (Q/4\pi\epsilon_0 b) \sum_{0}^{\infty} Y_{2n}\, (z/a)^{2n} \qquad z<a \tag{1}$$

where,

$$Y_{2n} = (-1)^n\, (2n)\,!\ /\ (2^{2n}\ n\,!\ n\,!) \tag{2}$$

Due to the axial symmetry of the problem, the potential at a general point $P\ (r,\ \theta)$ is;

$$V_w = (Q/4\pi\,\epsilon_o\,b)\ \sum_0^{\infty}\ Y_{2n}\ (r/a)^{2n}\ P_{2n}\ ,\quad r \leq a \qquad (3)$$

The induced and bound charges on the dielectric sphere give rise to potentials which satisfy Laplace's equation. A suitable solution to this equation is,

$$V_1 = \sum_0^{\infty} A_n\ r^{-n-1}\ P_n \qquad r \geq b \qquad (4)$$

$$V_2 = \sum_0^{\infty} B_n\ r^n\ P_n \qquad r \leq b \qquad (5)$$

Thus the potential due to the wire and the sphere is,

$$V_{in} = V_w + V_1 \qquad a \geq r \geq b \qquad (6)$$

$$V_{out} = V_w + V_2 \qquad r \leq b \qquad (7)$$

The constants A_n and B_n are obtained from the boundary conditions at $r = b$, namely,

(i) $V_{in} = V_{out}$, (ii) $\epsilon\,\partial V_{in}/\partial r = \epsilon_o\,\partial V_{out}/\partial r$

Since the potential vanishes at infinity, and since the potential of the charged wire alone contains only even polynomials, we conclude that $B_{2n+1} = A_{2n+1} = 0$. Equation (i) gives,

$$A_{2n} = a^{4n+1}\,B_n \qquad (8)$$

while (ii) gives,

$$A_{2n} = (Q/4\pi\epsilon_o)\ Y_{2n}\ b^{4n+1}(\epsilon_o - \epsilon)\ /\ [2n(\epsilon_o+\epsilon)+\epsilon_o]\ a^{2n+1}$$

Thus the potential inside the dielectric sphere is,

$$V_{in} = (Q/4\pi \epsilon_o a) \sum_0^{\infty} Y_{2n} (r/a)^{2n} P_{2n}$$

$$+ (Q/4\pi \epsilon_o a) \sum_0^{\infty} \frac{(\epsilon_o - \epsilon)\, Y_{2n}\, 2n\, (r/a)^{2n} P_{2n}}{[2n(\epsilon + \epsilon_o) + \epsilon_o]}$$

$$= (Q/4\pi \epsilon_o a) \sum_0^{\infty} Y_{2n} \{ (4n+1)\, \epsilon_o \,/\, [2n\, (\epsilon + \epsilon_o) + \epsilon_o] \} (r/a)^{2n} P_{2n}$$

47. *A dielectric of permittivity ϵ has a spherical cavity of radius a. A circular wire, whose equation relative to the center of cavity is $r = b$, $\theta = \alpha$, $(b < a)$, is charged with density λ. Find an expression for the force acting on the circular wire.*

The potential at any point inside the cavity is the sum of two parts. The first part is due to the charged wire only while the second part is due to the bound charges on the dielectric boundary. The potential at a point P, Fig. 10.39, on the z-axis is,

$$V_1 = q/4\pi \epsilon_o d$$

$$= (q/4\pi \epsilon_o) (b^2 + z^2 - 2bz \cos \alpha)^{-1/2}, \tag{1}$$

where $q = 2\pi (b \sin \alpha) \lambda$. For points with $z \geq b$ (1) can be written,

$$V_1 = (q/4\pi \epsilon_o z) \sum (b/z)^n P_n (\cos \alpha) \tag{2}$$

Since we have axial symmetry, the potential at a general point is,

$$V_1 = (q/4\pi \epsilon_o r) \sum (b/r)^n P_n (\cos \alpha) P_n (\cos \theta) \tag{3}$$

The potential due to the bound charges satisfies Laplace's equation. A suitable solution is,

$$V_2' = \sum_0^{\infty} A_n r^n P_n (\cos \theta) \qquad r \leq a \tag{4}$$

$$V_2'' = \sum_{0}^{\infty} B_n r^{-n-1} P_n (\cos\theta) \qquad r \geq a \qquad (5)$$

Thus we have that,

$$V_{in} = V_1 + V_2' \qquad b \leq r \leq a \qquad (6)$$

$$V_{out} = V_1 + V_2'' \qquad r \geq a \qquad (7)$$

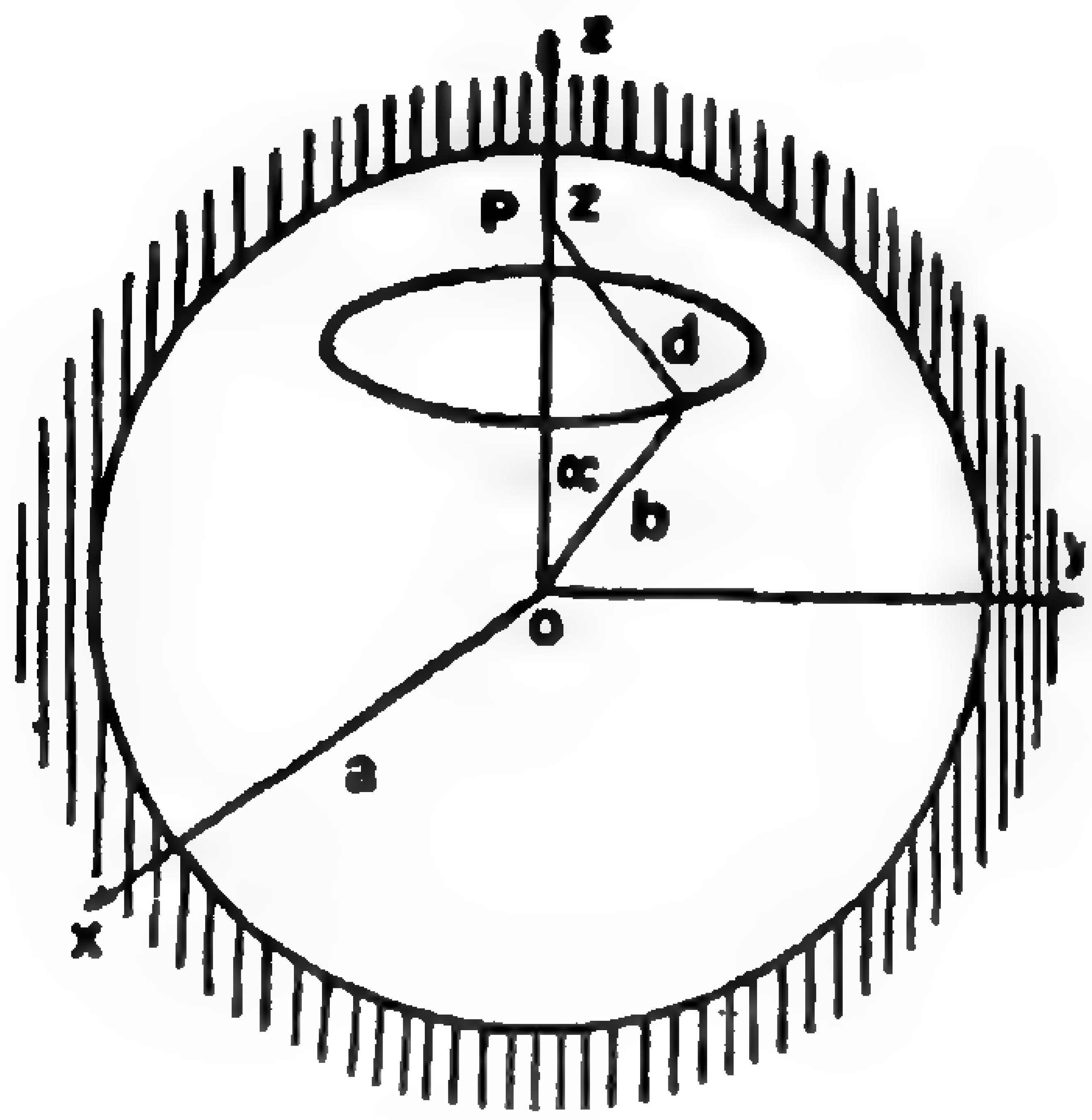


Fig. 10. 39.

The coefficients A_n, B_n are determined from the boundary conditions at $r = a$, namely, $V_{in} = V_{out}$, and $\epsilon\, \partial V_{out}/\partial r = \epsilon_0\, \partial V_{in}/\partial r$. Thus,

$$B_n = a^{2n+1} A_n = q\,(1-\epsilon/\epsilon_0)\,(n+1)\, b^n/4\pi\, [\epsilon+n(\epsilon+\epsilon_0)]$$

The force acting on the charged wire can be obtained from the potential energy of the wire,

$$\mathbf{F} = -q\, \nabla\, (V_{in}-V_1), \text{ at } r = b,\ \theta = \alpha.$$

$$= -q\, [(\partial V/\partial r) \cos\alpha - (\partial V/r\partial\theta) \sin\alpha]\ \mathbf{a}_z$$

Substituting for $\partial V/\partial r$, and $\partial V/r\partial\theta$ at $r = b$, $\theta = \alpha$, we get,

$$\mathbf{F} = -[q^2(\epsilon_0-\epsilon)/4\pi\,\epsilon_0\, a^2]\ \mathbf{a}_z\ \Sigma\ C_n\ (\cos\ \alpha\ P_n\ (\cos\ \alpha) -$$

$$(1-\cos^2\ \alpha)\ P_n'\ (\cos\ \alpha)/n]$$

where,

$$C_n = n\,(n+1)\ (b/a)^{2n-1}\ P_n\ (\cos\ \alpha)\ /\ [\epsilon+n\ (\epsilon+\epsilon_0)]$$

Using the relation,

$$(1-\mu^2)\ d\ P_n/d\mu = n\ P_{n-1} - \mu n\ P_n$$

we get,

$$\mathbf{F} \quad [(\epsilon-\epsilon_0)\ q^2/4\pi\ \epsilon_0\ a^2]\ \sum_{0}^{\infty}\ \{n(n+1)/\ [\epsilon+n(\epsilon+\epsilon_0)]\ \}$$

$$(b/a)^{2n-1}\ P_{n-1}\ (\cos\ \alpha)\ P_n\ (\cos\ \alpha)\ \mathbf{a}_z.$$

48. *A metal sphere of radius a is placed inside a hollow metal sphere of radius b. The distance c between the centers of the two spheres is so small that powers of c/a above the first may be neglected. The inner conductor is maintained at a potential V_0 and the outer is earthed. Find an expression for the potential in the space between the two conductors, the surface charge density on the inner conductor, the mechanical force on the inner conductor, and the correction ΔC to the capacitance due to the departure from exact concentricity between the two spheres.*

Since we have symmetry about the axis joining the centers of the spheres, the problem can be solved using spherical harmonics. As shown in Fig. 10.40 the equation of the surface S_1 is $r = a$. For S_2 we note that for a general point (r, θ) on this surface,

$$r = [b^2 + c^2 - 2bc\ \cos\ \theta]^{1/2}$$

Expanding we get,

$$r \;=\; b \Big/ \sum_{0}^{\infty} (c/b)^n \, P_n \,(\cos\theta) \qquad (1)$$

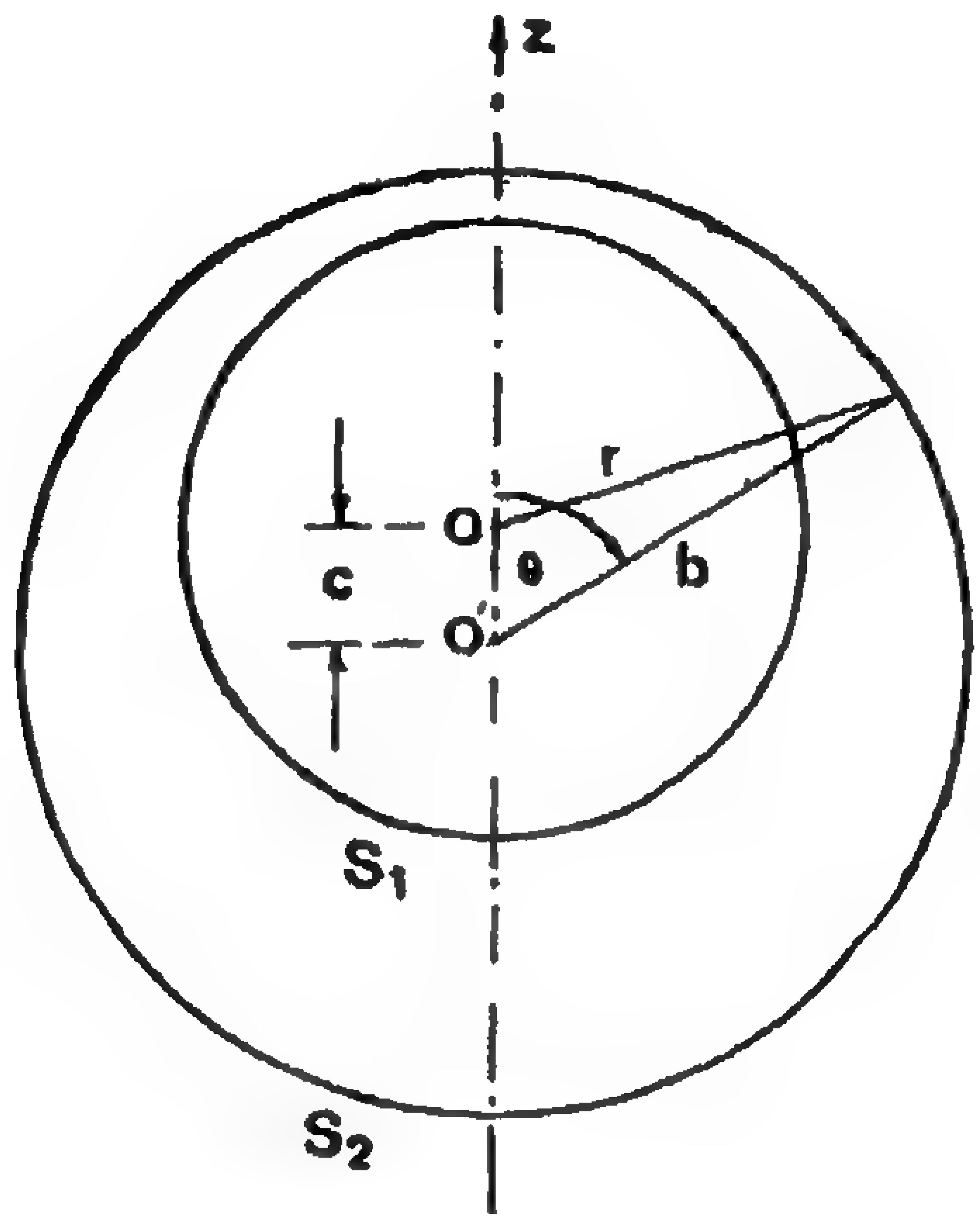


Fig. 10.40.

This represents the equation of the outer surface S_2 referred to the center of S_1. If neglecting higher order terms the equation of S_2 becomes

$$r \;=\; b - c \cdot P_1\,(\cos\theta) = b - c\,\cos\theta \qquad (2)$$

The form of this outer surface suggests that we seek a potential in the form of spherical harmonics and to retain only the first two terms,

$$V(r,\theta) = (A_1 + B_1/r) + (A_2 r + B_2/r^2)\,\cos\theta \qquad (3)$$

where, A_1, B_1, A_2, and B_2 are to be determined from the boundary conditions

$$V = V_0 \quad \text{at } r = a \;, \qquad V = 0 \quad \text{on } r = b - c\,\cos\theta \quad (4)$$

These give,

$$A_1 + B_1/a + (A_2 a + B_2/a^2) \cos\theta = V_o \qquad (5)$$

and

$$A_1 + B_1/b\,[1-(c/b)\cos\theta] + (A_2 b + B_2/b^2)\cos\theta = 0 \qquad (6)$$

neglecting the second order terms. Equations (5) and (6) give the values of the constants A_1, B_1, A_2, and B_2. Thus (3) becomes

$$V(r,\theta) = [V_o ab/(b-a)]\,[1/r-1/b-c\,(r-a^3/r^2)\cos\theta/(b^3-a^3)]$$

The surface charge density on the inner sphere is

$$\sigma = -\epsilon_o\,\partial V/\partial r \qquad \text{at } r = a$$
$$= [\epsilon_o\, b\, V_o/a\,(b-a)]\,[1+3a^2c\cos\theta/(b^3-a^3)]$$

The normal force per unit area on the inner sphere is $\sigma^2/2\epsilon_o$. Due to the symmetry about the z-axis the resultant force is along the z-axis in the positive direction,

$$F = \iint (\sigma^2/2\epsilon_o)\cos\theta\, dA$$
$$= 4\pi\, a^2\, b^2 c\, \epsilon_o\, V_o^2/(b-a)^2\,(b^3-a^3)$$

This force tends to decrease the distance between the two spheres which indicates that the equilibrium state ($c=o$) is unstable.

The capacity between the two spheres can be obtained by considering a virtual displacement δc of the inner sphere. The electrostatic energy before the displacement is $U = \frac{1}{2}\, Q^2/C$; after the displacement it increases to $U + \delta U$ with,

$$U-(U+\delta U) = F(c)\,\delta c$$

so that

$$(\tfrac{1}{2}\, Q^2/C^2)\,\delta C = F(c)\,\delta c$$
$$= 8\pi\, a^2\, b^2 c\, \epsilon_o\, V_o^2/[(b-a)^2\,(b^3-a^3)]\,\delta c$$

Integrating we get

$$Q^2/C - Q^2/C_o = -4\pi a^2 b^2 c \epsilon_o V_o^2 / [(b-a)^2 (b^3-a^3)]$$

where $C_o = 4\pi \epsilon_o ab / (b-a)$ is the capacity at $c=0$.

Thus

$$C = [4\pi \epsilon_o ab / (b-a)] [1 + a b c^2 / (b-a) (b^3-a^3)]$$

so that the increase ΔC in capacity is,

$$\Delta C = C - C_o = 4\pi \epsilon_o a^2 b^2 c^2 / [(b-a)^2 (b^3-a^3)]$$

49. *The equation of a nearly spherical conducting surface is $r = r_o [1+F(\theta)]$ where $F(\theta)$ is a function whose absolute value is less than a small constant k. If the surface is earthed and a point charge Q is placed at a point outside the conductor, find an expression for the total induced charge on the surface. First order terms in k are only considered.*

The problem can be solved using Green'sr eciprocation theorem. Here we consider two cases :

(i) The conductor is earthed and carries a charge Q′ and Q exists.

(ii) Q is absent and the conductor is at unit potential.
Green's reciprocation theorem gives,

$$1\,Q' + VQ = 0 \quad \text{Thus } Q' = -QV \tag{1}$$

where V is the potential at the location of the charge Q in (ii).

The potential outside the conductor satisfies Laplace's equation. A suitable solution is, Fig. 10.41,

$$V = \sum_{0}^{\infty} A_n r^{-n-1} P_n (\cos \theta) \tag{2}$$

The constant A_n can be determined from the boundary condition at the conductor surface $r = r_o [1+F(\theta)]$.

$$V = 1 \quad \text{at} \quad r = r_o [1+F(\theta)]$$

$$1 = \sum_{0}^{\infty} A_n r_o^{-n-1} [1+F(\theta)]^{-n-1} P_n(\cos\theta)$$

If first order terms are only considered,

$$1 = A_o r_o^{-1} [1-F(\theta)] + \sum_{1}^{\infty} A_n r_o^{-n-1} P_n(\cos\theta) \qquad (3)$$

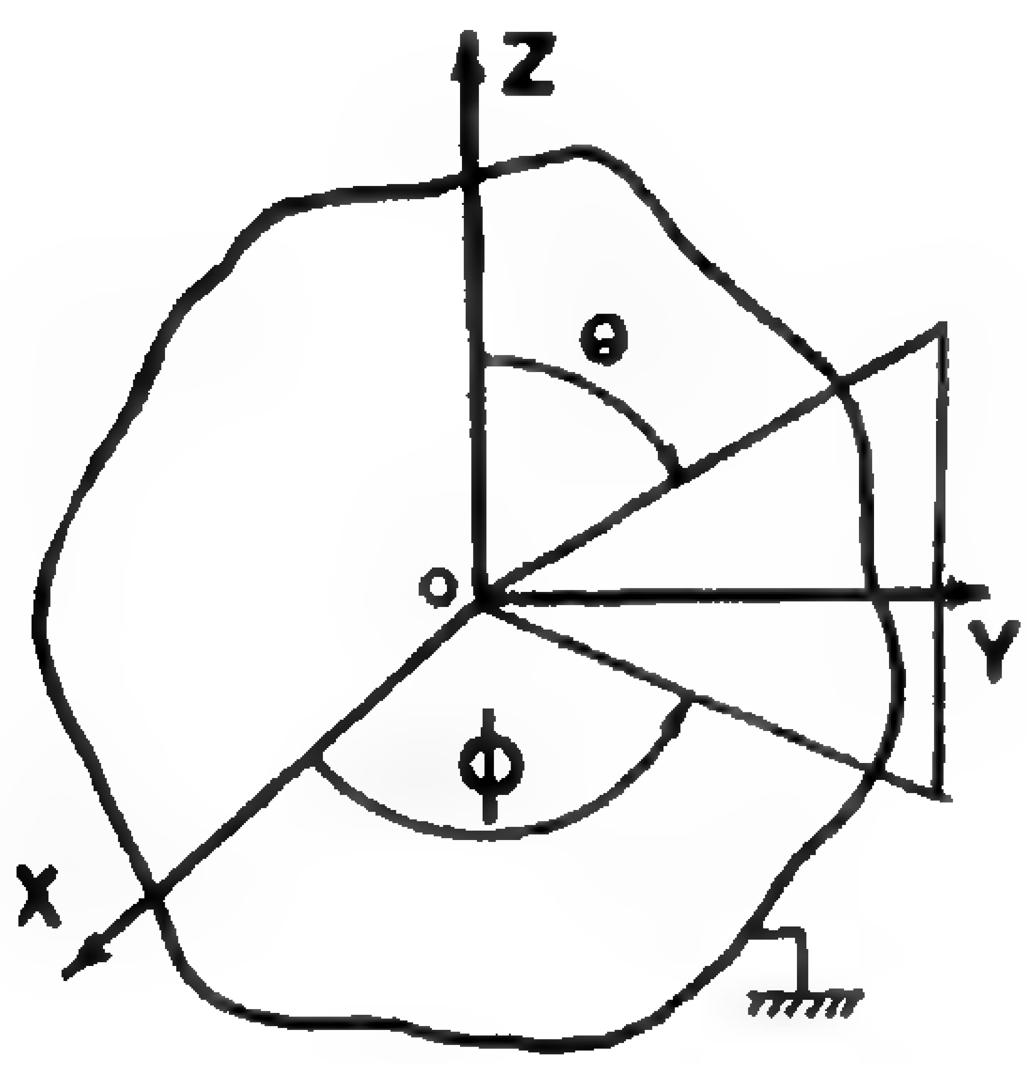


Fig 10.41.

To determine the coefficients A_n multiply both sides of (3) by $P_n(\cos\theta)$ and integrate with respect to $\cos\theta$ between -1 and 1.

$$A_n = r_o^n a_n A_o \qquad n > 1 \qquad (4)$$

where,

$$a_n = [2/(2n+1)] \int_{-1}^{1} f(\theta) P_n(\cos\theta)\, d\cos\theta$$

and,

$$A_0 = r_0 (1 + a_0), \quad \text{hence } A_n = r_0^{n+1}\, a_n \text{ to the first order.}$$

Thus (2) becomes,

$$V = r_0/r + \sum_0^{\infty} (r_0/r)^{n+1}\, a_n\, P_n (\cos \theta)$$

The total induced charge on the conductor surface is

$$Q' = -r_0 Q/r + Q \sum (r_0/r)^{n+1}\, a_n\, P_n (\cos \theta)$$

50. *The equation of a nearly spherical conducting surface is $r = a(1+2\,k \cos \theta + 15\, k \cos^3 \theta)$ where k is very small with respect to unity. If the conductor is kept at potential V_0, find an expression for the potential at any point and the surface charge density on the surface of the conductor.*

The equation of the conductor surface can be expressed in terms of Legendre polynomials

$$\cos \theta = P_1 \;, \qquad 5 \cos^3 \theta = 3\, P_1 + 2\, P_3$$

Thus the equation of the surface is,

$$r = a (1 + 11\, k\, P_1 + 6\, k\, P_3)$$

The potential outside the conductor satisfies Laplace's equation and can be written in the form,

$$V = \sum_0^{\infty} A_n\, r^{-n-1}\, P_n$$

The coefficients A_n can be determined from the boundary condition at the conductor surface,

$$V_o = A_0 r^{-1} + \sum_{1}^{\infty} A_n r^{-n-1} P_n$$

$$V_o = A_0 (1-11k\ P_1 - 6\ k\ P_3)\quad a + \sum_{1}^{\infty} A_n a^{-n-1} P_n$$

This gives

$$A_0 = V_0\ a\ ,\ A_1 = 11\ k\ a^2 V_0\ ,\ A_3 = 6\ k\ a^4\ V_0\ ,$$

$$A_n = 0 \text{ for other values of } n.$$

Hence

$$V = V_0\ [a/r + 11k\,(a^2/r^2)\ P_1 + 6k\,(a^4/r^4)\ P_3]$$

At the surface of the conductor $\partial/\partial r \simeq \partial/\partial n$, and the surface charge density is,

$$\sigma = -\epsilon_0\,(\partial V/\partial r)_{r=a}$$

$$\simeq (\epsilon_0\ V_0/a)\ (1 + 12k\ P_3)$$

$$= (\epsilon_0\ V_0/a)\ [1 + 6\,k\ (5\ cos^3\ \theta - 3\ cos\ \theta)]$$

$$= (\epsilon_0\ V_0/a)\ (1 - 18\,k\ cos\ \theta + 30\ k\ cos^3\ \theta)$$

where we have put $r^3 = a^3$ and $r^5 = a^5$.

51. *The equation of a nearly spherical conducting surface is $r = a\,[1+k\ P_n\ (cos\ \theta)]$. If this surface is earthed and a uniform electric field E_0 is acting along the $\theta = 0$ axis, find an expression for the surface charge density at any point on the surface. First order terms of k are only considered. Neglect terms containing $P_n\ (cos\ \theta)\ P_j\ (cos\ \theta)$, $j > 1$.*

If k is zero the potential at any point outside the conducting surface is, (see problem 10.17).

$$V_1 = -Er\cos\theta + E\,(a^3/r^2)\cos\theta$$
$$= -Er\,P_1(\cos\theta) + E\,(a^3/r^2)\,P_1(\cos\theta) \qquad (1)$$

If $k \neq o$, the potential at any point outside the conductor can thus be expressed in the form,

$$V = V_1 + \Sigma\, A_i\, r^{-i-1}\, P_i(\cos\theta) \qquad (2)$$

where A_i, $i = o, 1, 2, \ldots$ are small coefficients which can be determined from the boundary condition at the conductor surface,

$$V = 0 \text{ at } r = a\,[1 + k\,P_n(\cos\theta)] \qquad (3)$$

Substituting from (2) and neglecting terms containing $P_j(\cos\theta)\,P_n(\cos\theta)$, $j > 1$ we get that,

$$-3ka\,E\,P_1(\cos\theta)\,P_n(\cos\theta) + \sum_{0}^{\infty} A_i\,a^{-i-1}\,P_i(\cos\theta)$$
$$-\sum_{i=o}^{1} A_i\,a^{-i-1}\,(i+1)\,k\,P_i(\cos\theta)\,P_n(\cos\theta) = 0$$

Using the relation,

$$(2m+1)\,P_1(\cos\theta)\,P_m(\cos\theta) = (m+1)\,P_{m+1}(\cos\theta) + m\,P_{m-1}(\cos\theta)$$

we get,

$$-[(3k\,a\,E + 2A_1\,k/a^2)/(2n+1)]\,[(n+1)P_{n+1}(\cos\theta) + n\,P_{n-1}(\cos\theta)]$$

$$-A_o\,k\,a^{-1}\,P_n(\cos\theta) + \sum_{0}^{\infty} A_i\,a^{-i-1}\,P_i(\cos\theta) = 0$$

Since all the coefficients A_j, $j = 0, 1, 2, \ldots$ are small we can put $A_1\,k = A_o\,k \simeq 0$ so that we have,

$$-[3\,k\,a\,E/(2n+1)]\,[(n+1)\,P_{n+1}(\cos\theta) + n\,P_{n-1}(\cos\theta)]$$

$$+\sum_{0}^{\infty} A_i\,a^{-i-1}\,P_i(\cos\theta) = 0$$

Equating the coefficients of $P_i(\cos\theta)$, $i = 0, 1, 2, \ldots, \infty$ to zero we get, $A_i = 0$ for all i except $i = n+1$, $n-1$,

$$A_{n-1} = 3kn\, a^{n+1}\, E \,/\, (2n+1)$$

$$A_{n+1} = 3k\,(n+1)\, a^{n+3}\, E \,/\, (2n+1)$$

Hence we have that,

$$V = -Er\, P_1(\cos\theta) + E\,(a^3/r^2)\, P_1(\cos\theta)$$

$$+ [3ka^{n+1} E/(2n+1)]\, [n\, r^{-n}\, P_{n-1}(\cos\theta) + a^2\,(n+1)\, r^{-n-2}$$

$$P_{n+1}(\cos\theta)] \qquad (4)$$

The surface charge density at any point on the conducting surface is,

$$\sigma = \epsilon_0 E = -\epsilon_0 \nabla V \qquad \text{at } r = a[1+k\, P_n(\cos\theta)]$$

$$\simeq -\epsilon_0\, \partial V/\partial r$$

$$\simeq 3E\, P_1(\cos\theta) + [3En\,k \,/\, (2n+1)]\, [(n-2)P_{n-1}(\cos\theta)$$

$$+ (n+1)\, P_{n+1}(\cos\theta)]$$

Note that if the surface is a sphere, $k = 0$, and $\sigma = 3E\, P_1(\cos\theta)$ which can be checked from (1),

$$\sigma = -\epsilon_0 \nabla V_1 \qquad \text{at } r = a$$

$$= 3E\, P_1(\cos\theta)$$

9 - 9 مسائل إضافية

SUPPLEMENTARY PROBLEMS

1. Show that a solution of Laplace's equation in two dimensions is $V = [A \sin(nx) + B \cos(nx)]\,[C \cosh(ny) + D \sinh(ny)]$ when n is not equal to zero and $V = (A - Bx)(C + Dy)$ when n is zero.

   The potential in the $z = 0$ plane is given by $V = V_o \cos(kx)$ and is independent of y. Find an expression for the potential at all points.

2. A two-dimensional box has the sides $y = o, b$ at zero potential and the sides $x = a, -a$ at potential V_o. Find an expression for the potential at any point inside the box.

3. Find an expression for the potential on the interior of the section shown in Fig 10.9. The potential along the inner circular surface has the distribution shown and the outer circle is at zero potential.

4. The potential distribution on a circular section is,

$$V = V_o \quad \text{for} \quad |r| = a \;, \quad o < \theta < \pi$$

$$= -V_o \quad \text{for} \quad |r| = a \;, \quad \pi < \theta < 2\pi$$

   Show that the potential at any interior point is given by,

$$V = (4V_o/\pi) \sum_{0}^{\infty} (r/a)^{2n+1} \sin[(2n+1)\,\theta] \,/\, (2n+1)$$

5. If in the previous problem the potential on the upper plate is given by $V = V_o (a^2 - r^2)$, show that the potential at any point

inside the cylinder is,

$$V = 8a^2V_0 \sum_{1}^{\infty} [\sinh(a_i z/a) / a_i^3 J_1(a_i) \sinh (a_i h/a)] J_0 (a_i r/a)$$

where a_i is the i–th root of the equation $J_0 (x) = 0$.

6. An earthed conducting cylinder of length h and radius a is closed at both ends by two plates of radius $a-$. The lower plate is earthed while the upper is kept at potential V_0. Determine the potential at all points inside the cylinder.

7. A small magnet of moment $\mathbf{m} = m\,\mathbf{a}_z$ lies at the center of a spherical cavity of radius a which exists in an infinite medium of uniform permeability μ. Determine the potential at any point in the regions $o \leq r \leq a$, $r \geq a$.

8. A dielectric sphere of permittivity ϵ_1 is surrounded by a medium of dielectric constant ϵ_2 and acted upon by a uniform field $\mathbf{E} = E_0\, \mathbf{a}_z$. Find the potential inside and outside the sphere.

9. Two charges each of amount Q are placed at points A and B that are at a distance $2\,h$ apart. A uniform sphere of radius a, where $a < h$, and dielectric constant ϵ, has center at the mid-point of AB. Show that a repulsion

$$(Q^2/16\pi\, \epsilon_0\, h^2)\ [1 - 16(\epsilon_r - 1) \sum_{1}^{\infty} \{ n(2n+1)/[1+2n(1+\epsilon_r)] \} (a/h)^{4n+1}]$$

acts on each charge.

10. Two magnetic dipoles each of moment m are placed at a distance $2b$ apart with their axes in the same straight line and the same direction. Show that if a soft iron sphere of permeability μ and radius a $(<b)$ is placed with its center at the mid-point of the line joining the dipoles the force of attraction between them is increased by

$$\frac{8\,\mu_o\,(\mu-\mu_o)\,M^2}{4\pi\,b^4}\sum_{0}^{\infty}\frac{(n+1)^2\,(2n+1)\,(2n+3)}{2n\mu+\mu+2n\mu_o+2\mu_o}\,(a/b)^{4n+3}$$

11. An earthed conducting sphere of radius a is introduced into a permanent electric field whose potential at any point before the introduction of the sphere can be wirtten in terms of integral spherical harmonics with center at the center of the sphere in the form $V=\sum_{0}^{\infty} V_n$. Show that the surface density of electrification induced on the sphere is $-(\epsilon_0/4\pi\,a)\sum_{0}^{\infty}(2n+1)V_n$.

12. A soft iron shell of relative permeability μ is bounded by concentric spheres of radii a, b, $(b>a)$. A magnetic dipole of moment $\mathbf{m}$ is at the common center O. Show that the field outside the shell is identical with the field of a dipole $\mathbf{m}'$ at O in the absence of the shell, where,

$$\mathbf{m}' = \mathbf{m}\,[1+2\,(\mu-1)^2\,(b^3-a^3)\,/\,9\mu\,b^3]^{-1}$$

13. A conducting spherical shell of mass m and radius a floats on oil of permittivity ϵ_r. When it is uncharged, a fraction $f\,(<\tfrac{1}{2})$

of the volume of the sphere is submerged. Show that the charge Q which wil cause the sphere to be half-submerged is given by,

$$Q^2 \quad 4\pi\epsilon_0 (1-2f)(\epsilon_r+1)^2 \, m \, g \, a^2 \quad (\epsilon_r-1) f$$

where g is the gravitational acceleration.

14. A circular wire of radius a carries a steady current I, and a sphere of soft iron of uniform permeability μ is concentric with the wire. The radius of the sphere is less than a. Show that the magnetic field within the iron at a point on the axis a distance z from the center is directed along the axis, and has mangitude,

$$\frac{\mu_0 I}{2a}\left[\frac{3\mu_0}{2\mu_0+\mu} + \sum_{1}^{\infty} (-1)^n \frac{(2n+1)(4n+3)\mu_0}{(2n+2)\mu_0+(2n+1)\mu} \frac{1.3.5..(2n-1)}{2.4.6..2n} (z/a)^{2n}\right]$$

15. An electrostatic field is set up by a charge Q uniformly distributed round a circular ring which has spherical polar coordinates $r=c$, $\theta=\alpha$. Find the potential of the field at any point for which $r<c$. A dielectric sphere of permittivity ϵ_r is now placed with it's center at the origin of coordinates. If the radius a of the sphere is less than c, show that the potential of the modified field in the region $a \leq r \leq c$ is,

$$(Q/4\pi\epsilon_0) \sum_{0}^{\infty} c^{-n-1} \left\{ r^n - n(\epsilon_r-1)\, a^{2n+1} / (\epsilon_r n+n+1)\, r^{n+1} \right\} P_n(\cos\alpha)\, P_n(\cos\theta)$$

16. A conductor is bounded by $r = a\,(1+k\,P_n)$ where first order terms of k are only considered. Prove that the surface density on the conductor is approximately,

$(\epsilon_0 V_0/a)\ [1+(n-1)\ k\ P_n]$

where V_0 is the potential of the conductor.

17. Show that the magnetic scalar potential Φ due to a magnetic dipole of moment $m\ \mathbf{a}_z$ placed at a distance c from the origin of spherical polar coordinates is expressible in the form,

$$\Phi = (m/4\pi r^2) \sum_{1}^{\infty} n\ (c/r)^{n-1}\ P_n\ (\cos\theta) \qquad r>c$$

$$= -(m/4\pi c^2) \sum_{0}^{\infty} (n+1)\ (r/c)^n\ P_n\ (\cos\theta), \quad r<c$$

at the point $P\ (r, \theta, \phi)$.

A small magnet of moment m is placed at distance c $(>a)$ from the center of a soft iron sphere of radius a and uniform permeability μ, with its axis directed away from the center of the sphere. Find the force experienced by the dipole due to the magnetization of the soft iron, showing in particular that the magnitude of this force is,

$$[\mu_0\ m^2\ (\mu-\mu_0)\ /\ 4\pi c^4] \sum_{0}^{\infty} \frac{n\ (n+1)^2\ (n+2)}{\mu n + \mu_0\ (n+1)}\ (a/c)^{2n+1}$$

18. A solid dielectric hemisphere, of permittivity ϵ has its plane face, of which O is the center, in contact with an infinite conducting plane at zero potential. A ring center C of radius $c \sin a$ and carrying a charge Q is placed with its plane parallel to the conducting plane so that OC is normal to $\hat{n}$ and $OC = c \cos a$. Prove that the attractive force between the ring and the plane is increased by the presence of the dielectric by an amount,

$$[Q^2(\epsilon_r-1) / \pi \epsilon_0 c^2] \sum_1^{\infty} \{ n(2n-1) / [2n(\epsilon_r+1)-\epsilon_r] \} (a/c)^{4n-1}$$

$$P_{2n} (\cos \alpha) \, P_{2n-1} (\cos \alpha)$$

where a ($<c$) is the radius of the hemisphere.

19. A point charge Q is placed in the cavity of a spherical shell of material of dielectric constant ϵ at a distance c from O, its center. The radii of the inner and outer surfaces of the shell are b, a respectively ($b>c$) and the outer surface is coated with conducting material of zero potential. Show that the force acting on the point charge is directed away from O and is of magnitude,

$$(Q^2/4\pi \epsilon_0 \, b^2) \sum_1^{\infty} \frac{(\epsilon_r n+n+1) \, b^{2n+1}+(\epsilon_r-1)(n+1)a^{2n+1}}{n(\epsilon_r-1) \, b^{2n+1}+(\epsilon_r n+\epsilon_r+n) \, a^{2n+1}} \, n(c/b)^{2n-1}$$

20. A conductor of nearly spherical shape $r = a \, [1+k \, P_2 \, (\mu)]$ is placed in a uniform electric field E_0 parallel to the axis of symmetry. Show that if the total charge on the conductor is zero, the densities of the charge at the end of this axis are $\pm \, 3(5+6k) E_0/20\pi$.

21. A conductor has a nearly spherical shape $r = a \, [1 + kP_n \, (\mu)]$, where k is small. If this conductor is raised to a potential $V = 1$, find an expression for the potential at any point outside the conductor. First order terms of k are only considered. Show that when the conductor is earthed a point charge Q placed outside it at the point ($P \; r = h$, $\theta=0$) will induce on it a charge approximately equal to $-(Qa/h) \; [1+k \; (a/h)^n]$.

22. A nearly spherical conductor is bounded by the surface $r = a[1+k\cos\theta]$ where k is small. It is insulated and uncharged and placed in a field of potential $Er\cos\theta$. Show that, for first order terms of k, the potential of the disturbed field is,

$$V = -Er(1-a^3/r^3)\cos\theta + kE(a^4/r^3)(3\cos^2\theta - 1)$$

23. If U denotes the potential at a point P when a conductor is raised to unit potential, prove that if a charge Q is placed at P and the conductor earthed, then the charge which is induced on the conductor is $-U$Q.

The surface of a nearly spherical earthed conductor is given by $r = a+k\,P_2(\cos\theta)$ where k is small. Show that, neglecting squares and higher powers of k, the charge induced by a point charge Q placed at the point $(r=h,\ \theta=0,\ \phi=0)$ is $-$Q $(a/h + ka^2/h^3)$.

24. A grounded nearly spherical conductor is bounded by the surface $r = a\left[1+\sum\limits_{2}^{\infty} k_n P_n(\cos\theta)\right]$ is placed in a uniform field E_o which is parallel to the axis of symmetry of the conductor. Show that for first order terms of k_n, the potential is given by,

$$V = E_o a \Big\{ \left[(1+6k_2/5)(a/r)^2 - r/a\right] P_1 + 3\sum_{2}^{\infty}\left[nk_{n-1}/(2n-1)+(n+1)k_{n+1}/(2n+3)\right](a/r)^{n+1} P_n(\cos\theta)\Big\}$$

25. A nearly spherical conductor is bounded by the surface $r = a[1+k\,P_n(\mu)]$ where k is small and $n>2$. The conductor is in-

sulated and uncharged, and is placed in a uniform electrostatic field of intensity E; the axis of symmetry of the conductor is parallel to the field. Prove that, for first order terms of k, the potential at the point $P(r, \theta)$ outside the conductor is,

$$C - E\,(r - a^3/r^2)\,P_1(\mu) + [3kE/(2n+1)]\,\{(na^{n+1}/r^n)P_{n+1}(\mu)$$
$$+\,[(n+1)\,a^{n+3}/r^{n+2}]\,P_{n+1}(\mu)\}$$

where C is a constant.

If σ_1 and σ_2 are the surface densities of charge induced on the conductor at the two ends of its axis of symmetry, prove that, to the same approximation $\sigma_1 + \sigma_2$ is zero or $3n(2n-1)\,kE/2(2n+1)\,\pi$ according to as n is even or odd.

26. A conductor at a potential V whose surface is of the form $r = a\,[1 + k\,P_n(\mu)]$ is surrounded by a dielectric ϵ whose boundary is the surface $r = b\,[1 + \delta\,P_n(\mu)]$. and outside this dielectric is air show that the potential at any point in air is,

$$\frac{\epsilon_r\,abV}{(\epsilon_r - 1)a + b}\left\{\frac{1}{r} + \frac{(2n+1)\,k\,a^n b^{2n+1} + (\epsilon_r - 1)\,\delta b^n\,[nb^{2n+1} + (n+1)\,a^{2n+1}]\,P_n}{[(1 + n + n\epsilon_r)\,b^{2n+1} + (\epsilon_r - 1)(n+1)\,a^{2n+1}]\,r^{n+1}}\right\}$$

where only first order terms of k, and δ are considered.

27. A nearly spherical conducting surface $r = a + ak\cos^2\theta$, is charged with a charge Q. Prove that the surface charge density at any point on the surface is $(1 - k\sin^2\theta)\,Q/4\pi a^2$ where first order terms of k are only considered.

28. A conductor is bounded by the surface $r = a\,(1+k\cos^3\theta)$. If this conductor is raised to unit potential, find, considering only first order terms of k, the potential at any point and show that the surface charge density at any point on the conductor is,

$$\sigma = [1+2k\cos^3\theta - (6k/5)\cos\theta] \,/\, 4\pi a$$

29. Current flows through a medium of uniform conductivity σ between two nearly concentric spheres of radii a, b $(b > a)$ whose centers are a small distance δa apart. The potential difference between the electrodes is V_o. Prove that the current density at the outer electrode is

$$[\sigma a V_o / b\,(b-a)]\,[1+3a\,b^2\delta\cos\theta \,/\, (b^3-a^3)]$$

where θ is the angle between the line of centers and the radius vector to the point at which the current density is specified.

الفصل العاشر

حل المسائل الحدية الاستاتيكية باستخدام طريقتي الصور والتعاكس

SOLUTION OF THE BOUNDARY - VALUE STATIC PROBLEMS USING THE METHODS OF IMAGES AND INVERSION

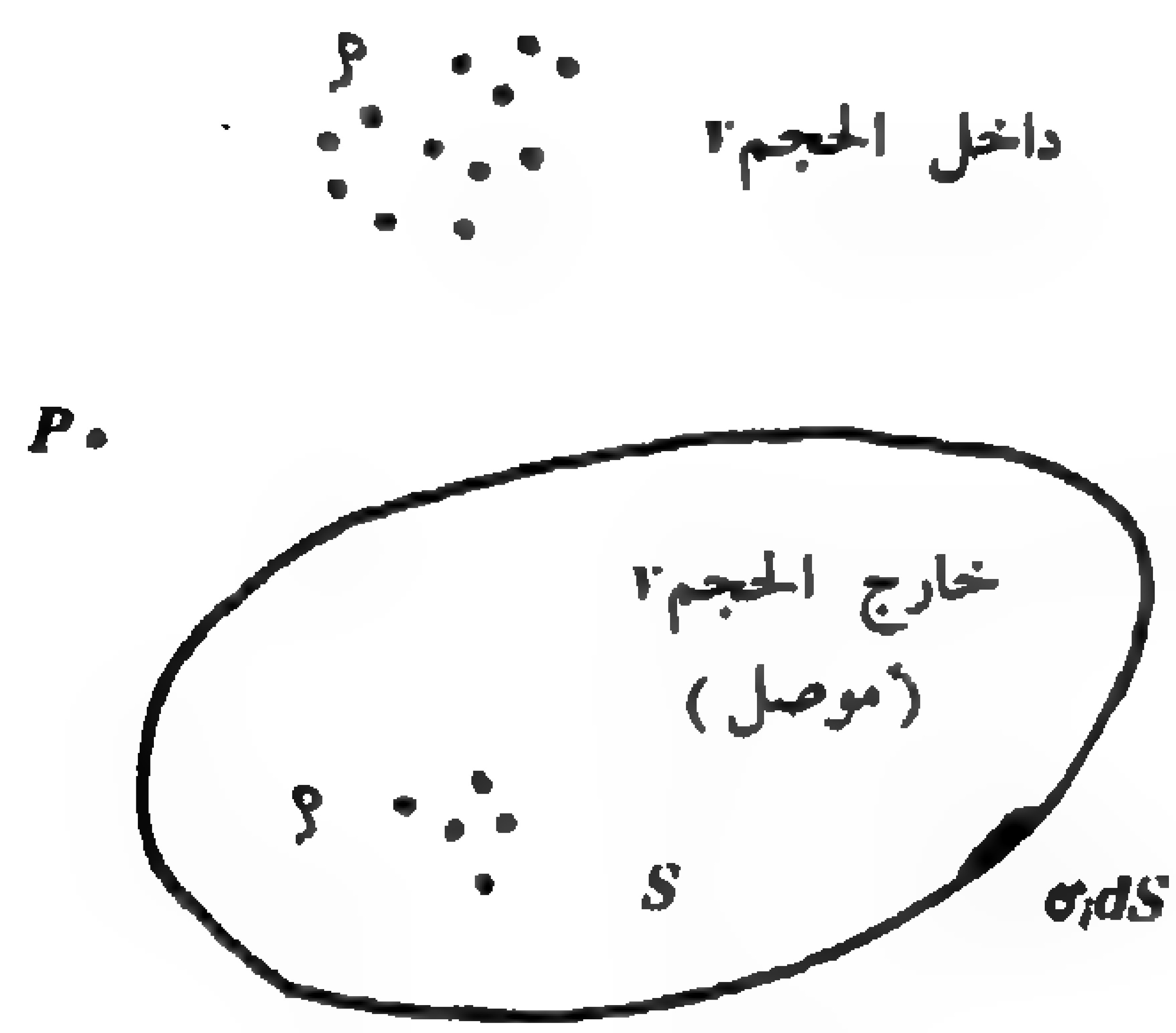
10 - 1 مقدمة :

تستخدم خاصية وحدوية الحل في النظرية الكهروستاتيكية لحل المسائل الحدية بطريقة الصور الكهربية . وتتلخص هذه الطريقة في استبدال مسألة ما تحتوي على شحنات وموصل أو أكثر بمسألة أخرى مكافئة تحتوي على شحنات كهربية فقط تسمى الشحنات الصوّرية وهي تعطي نفس المجال الكهروستاتيكي الذي تعطيه المسألة الأساسية وذلك عند استخدام العلاقات المعروفة لايجاد المجال أو الجهد الكهربي عند أي نقطة في وسط به شحنات كهربية . ويلاحظ عند تطبيق طريقة الصور أن عدد الموصلات المشحونة يكون صغيراً وأن أشكالها الهندسية بسيطة . وجدير بالذكر أن طريقة الصور يمكن تطبيقها على مسائل المغناطيسْية الساكنة . أما طريقة التعاكس فهي طريقة هندسية يمكن بواسطتها تحويل المسألة المعطاة إلى مسألة أخرى معْروف حلها أو أبسط من المسألة الأساسية . وتكون هذه الطريقة سهلة إذا أمكن معرفة انعكاس بعض الأشكال البسيطة بالنسبة لكرة مثلاً . أو دائرة أو مستقيم أو نقطة معلومة . وتكون المسألة الأساسية مكونة من مجموعة شحنات وموصلات ذات أشكال هندسية معروفة . أما المسألة المكافئة فإنها تحتوي على

مجموعة من الشحنات والموصلات التي أشكالها الهندسية هي تعاكس الأشكال الأساسية كما ذكرنا .

10 - 2 مفهوم نظرية الصور الكهربية :

افرض توزيع حجمي من الشحنات ρ داخل حجم معين v محدد من الخارج بكرة ذات نصف قطر لا نهائي ومن الداخل بسطح متساوي الجهد S يمثل سطح موصل كما هو مبين بشكل $(10 - a)$. الجهد الكهربي عند نقطة P



شكل (10 - a)
مفهوم طريقة الصور الكهربية

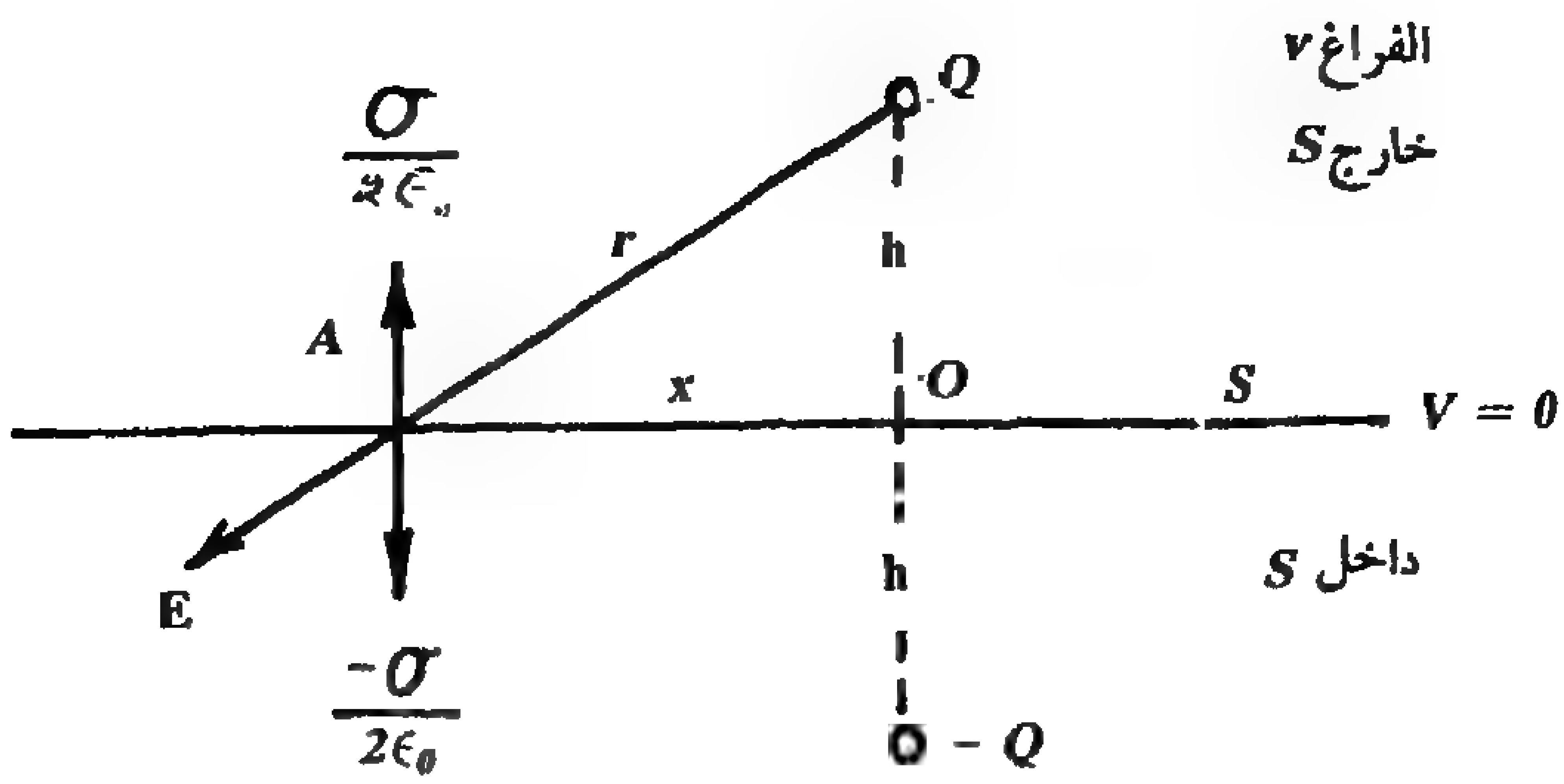
خارج الموصل هو مجموع جهدين أحدهما ينشأ من توزيع الشحن الحجمي في الفراغ والثاني ينشأ من توزيع الشحنات التأثيرية على S . ويمكن حل هذه المسألة أي ايجاد الجهد والمجال الكهربي عند أي نقطة داخل v إذا عرفنا كثافة الشحنات السطحية التأثيرية σ_i على S . ويمكن استبدال التوزيع السطحي الذي تمثله σ_i بشحنات تخيلية مكافئة ρ تقع خارج الحجم v بحيث تحدث نفس الجهد الذي تحدثه الشحنات σ_i عند النقطة P . وتسمى هذه الشحنات بالشحنات الصورية المكافئة *(Equicedlent Image Charges)* بالنسبة للسطح S .

ذكرنا من قبل أنه إذا كانت المركبة العمودية للجهد $\frac{\partial V}{\partial n}$ معروفة عند

كل نقط السطح S فإن المجال الكهربي عند أي نقطة امام هذا السطح يكون وحيد القيمة . وقد استنتجنا كذلك أن شرط وحدوّية الحل يمكن أن يتحقق كذلك إذا عرفنا المجال الكهربي المماس عند كل نقطة من نقط السطح S . وهذا يعطي إمكانية وجود مجال كهربي مماس للسطح S وبالتالي لا يشترط أن يكون S متساوي الجهد فقد يكون سطح عازل وعلى ذلك منظرية الصوّر لا تقتصر على الموصلات فقط . ويمكن فهم طريقة الصور الكهربية في المجالات الكهروستاتيكية باعتبار سطح مغلق S يقسم الفراغ إلى قسمين . قسم منها يعتبر هو داخل v والقسم الآخر خارجة . نعين المركبة العمودية أو المماسة للمجال الكهربي على السطح S نتيجة للشحنات الناتجة بالتأثير على هذا السطح (هذه الشحنات التأثيرية على S سببها هو مجموعة الشحنات الموجودة داخل الحجم v في الفراغ خارج S) . نوجد توزيع شحنات مناسب داخل S ليعطي على S نفس المجالات العمودية أو المماسة التي تسببها الشحنات التأثيرية . هذا التوزيع يعطي نفس الشحنات التأثيرية على S ويمثل الشحنات الصوّرية المكافئة . ويلاحظ أن هذا التوزيع هو توزيع افتراضي يقع دائماً في المنطقة التي تعتبر خارج الحجم v . ونستنتج من ذلك أن طريقة الصور الكهربية تحول مسألة كهروستاتيكية معينة تحتوي على شحنات كهربية فقط بدون موصلات . وتعتبر طريقة الصور مفيدة فقط إذا كانت المسألة المكافئة أبسط في حلها من المسألة الأساسية . مثال ذلك إذا كان السطح S هو مستوى لانهائي أو سطح كرة أو أسطوانة .

10 - 3 شحنة نقطية أمام مستوى موصل مؤرض لانهائي الامتداد :

نفرض أن شحنة Q تقع على مسافة h من مستوى S موصل لانهائي الامتداد ذات جهد صفر كما هو مبين بشكل $(10 - b)$. المستوى S يقسم الفراغ إلى قسمين أحدهما هو الفراغ v والآخر هو جسم الموصل . وباتباع



شكل (10 - b) شحنة نقطية امام مستوى موصل مؤرض

الخطوات التي ذكرناها يكون المجال الكهربي العمودي عند أي نقطة A على S يتجه للشحنة الأصلية Q هو :

$$E_n = E \cos \theta$$
$$= \frac{Qh}{4 \pi \epsilon_0 r^3} \qquad (1 - 10)$$

المركبة العمودية للمجال الكهربي على جانبي S تحقق الشرط الحدي لكثافة الفيض الكهربي العمودي على السطح الفاصل أي أن :

$$\epsilon_0 \left(\frac{\sigma}{2 \epsilon_0} - E_n \right) - \epsilon_0 \left(E_n + \frac{\sigma}{2\epsilon_0} \right) =$$

ومنها نستنتج أن :

$$\sigma = - 2 \epsilon_0 E_n$$

وبالتالي :

$$E_n = \frac{\sigma}{- 2 \epsilon_0} = \frac{Qh}{4 \pi \epsilon_0 r^3}$$

ومنها نستنتج أن المركبة العمودية نتيجة الشحنة التأثيرية على السطح

يمكن أن تنتج عن شحنة مقدارها Q – موضوعة على مسافة h تحت السطح S .

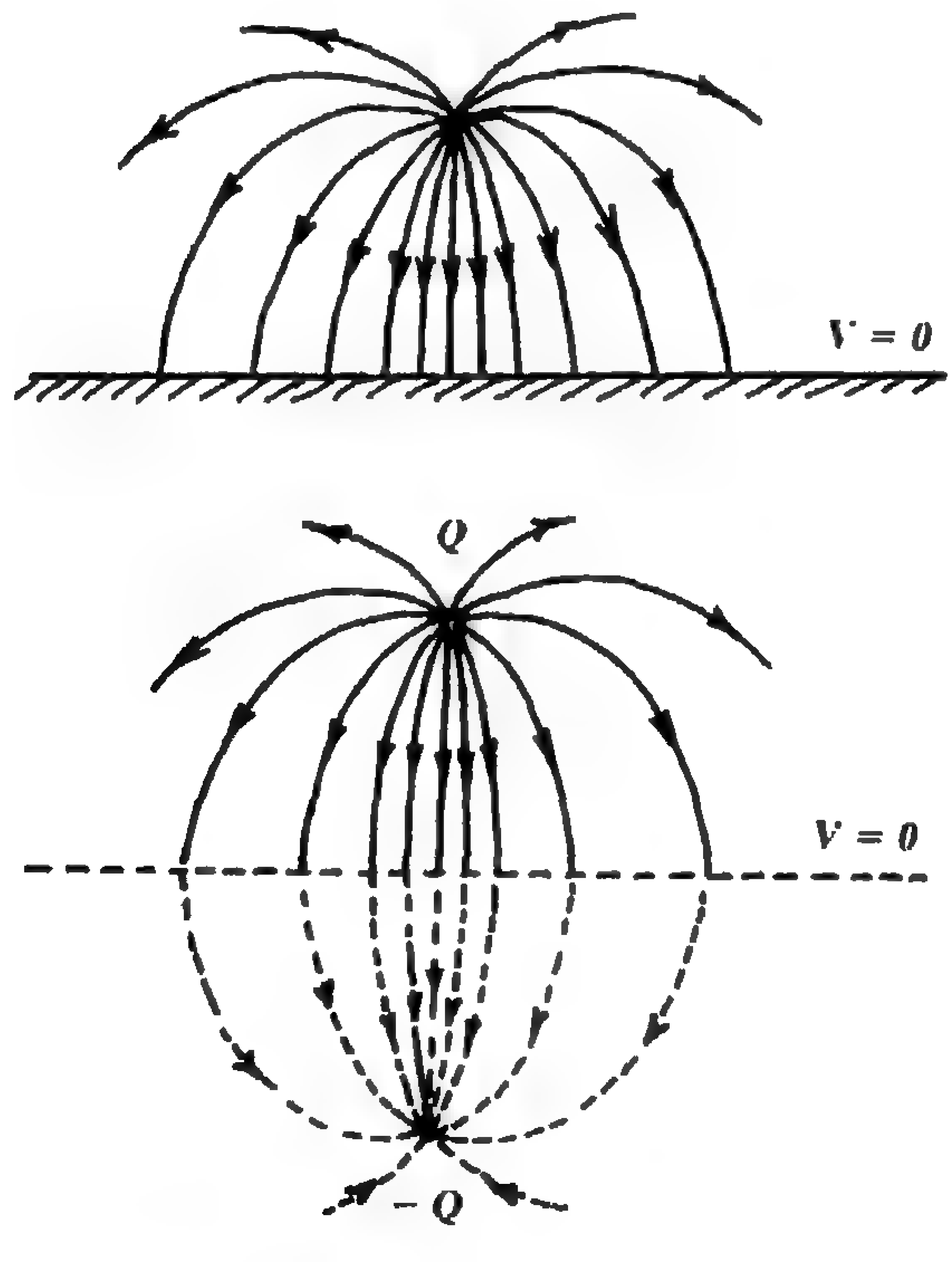
وتكون كثافة الشحن التأثيرية على S هي :

$$\sigma = -\frac{Qh}{2\pi r^3} \qquad (2 - 10)$$

ويمكن حساب الشحنة التأثيرية الكلية Q_i على S من المعادلة :

$$Q_i = \int^{\infty}_0 2\pi x \quad dx = -Q \qquad (3 - 10)$$

شكل $(c - 10)$ يبين خطوط القوى الكهربية في المسألة والمسألة المكافئة .

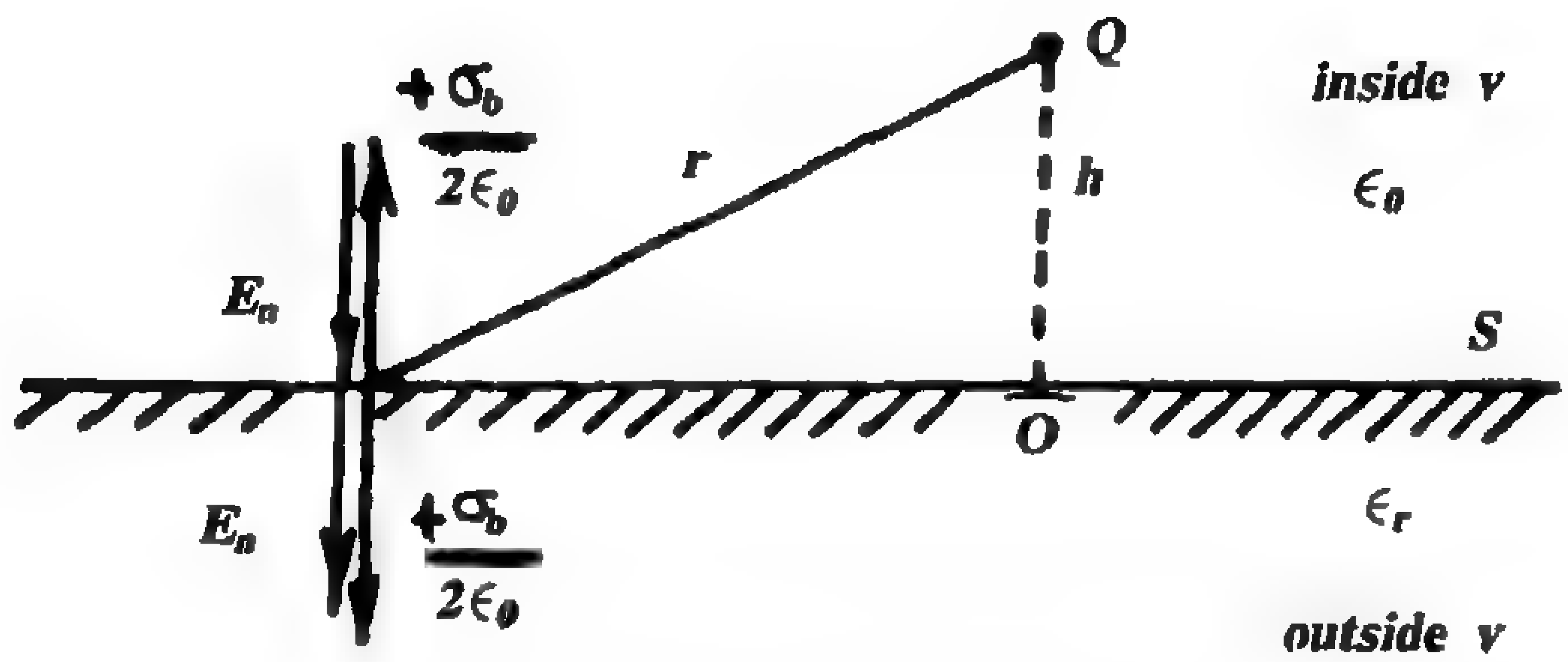


شكل (c - 10) خطوط القوى الكهربية في المسألة والمسألة المكافئة

10 - 4 شحنة نقطية أمام مستوى لانهائي يمثل السطح الفاصل لوسط عازل :

نفرض شحنة Q على مسافة h من السطح الفاصل لوسط عازل كما هو مبين بشكل $(d - 10)$. المجال الكهربي العمودي على السطح الفاصل S الناتج عن Q فقط هو :

$$E_n = \frac{Qh}{4 \pi \epsilon_0 r^3}$$



شكل $(d - 10)$ شحنة Q أمام سطح عازل

اعتبر أن σ_b هي كثافة الشحن المقيدة على سطح العازل نتيجة لوجود الشحنة Q . وبتطبيق الشرط الحدي على السطح الفاصل :

$$D_{n1} = D_{n2}$$

نجد أن :

$$\epsilon_0 \left(E_n - \frac{\sigma_b}{2\epsilon_0}\right) = \epsilon_0 \epsilon_r \left(E_n + \frac{\sigma_b}{2 \epsilon_0}\right)$$

ومنها نستنتج :

$$\sigma_b = - \frac{\epsilon_r - 1}{\epsilon_r + 1} 2 \epsilon_0 E_n \qquad (4 - 10)$$

ويكون المجال الكهربي العمودي نتيجة للشحنات المقيدة فقط هو :

$$\frac{\sigma_b}{2\epsilon_0} = - \frac{\epsilon_r - 1}{\epsilon_r + 1} \frac{Qh}{4\pi\epsilon_0 r^3}$$

وهذه العلاقة يمكن أن تنتج من شحنة مقدارها :

$$Q' = - \frac{\epsilon_r - 1}{\epsilon_r + 1} Q \qquad (5 - 10)$$

موضوعة على مسافة h أسفل السطح S . وتعطي الشحنتان Q و Q' الجهد والمجال الكهربيين عند أي نقطة أعلى السطح S في الفراغ . ويلاحظ أنه إذا كانت ϵ_r تؤول إلى ∞ فإن Q' تؤول إلى Q − وتصبح المسألة هي نفس المسألة السابقة .

ولتعيين النظام المكافىء عند إيجاد الجهد أو المجال الكهربي عند أي نقطة داخل العازل . اعتبر المجال الكهربي العمودي داخل العازل الناتج عن الشحنات المقيدة :

$$\frac{\sigma_b}{2\epsilon_0} = - \frac{\epsilon_r - 1}{\epsilon_r + 1} \frac{Qh}{4\pi\epsilon_0 r^3}$$

وهذا يمكن أن ينتج عن شحنة مقدارها Q' موضوعة مكان الشحنة الأصلية Q ويكون المجال الكهربي الكلي داخل العازل ناشىء عن :

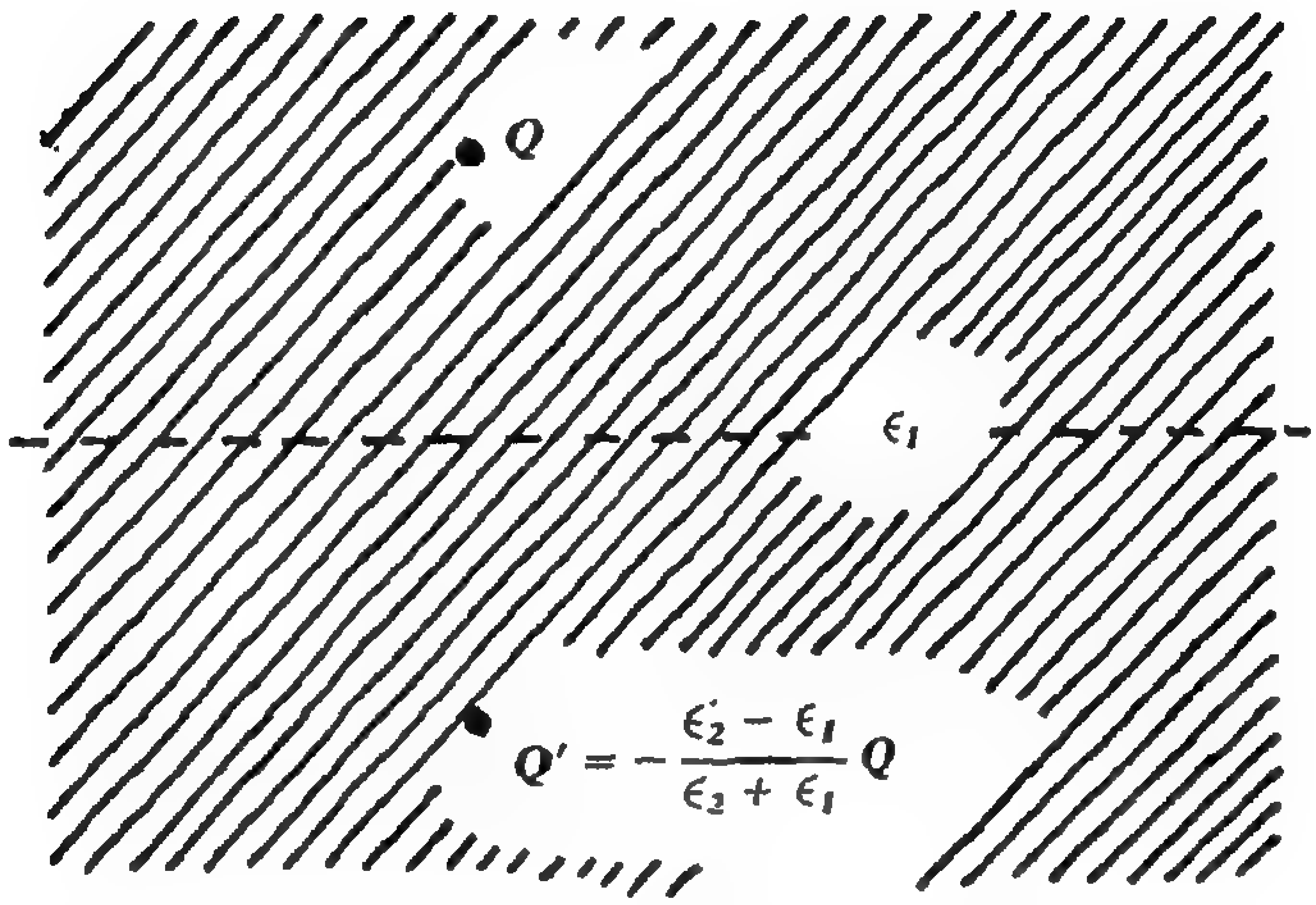
$$Q'' = Q + Q' = \frac{2}{\epsilon_r + 1} Q \qquad (6 - 10)$$

وهذه الشحنة تقوم مقام العازل والشحنة Q الأساسية معاً .

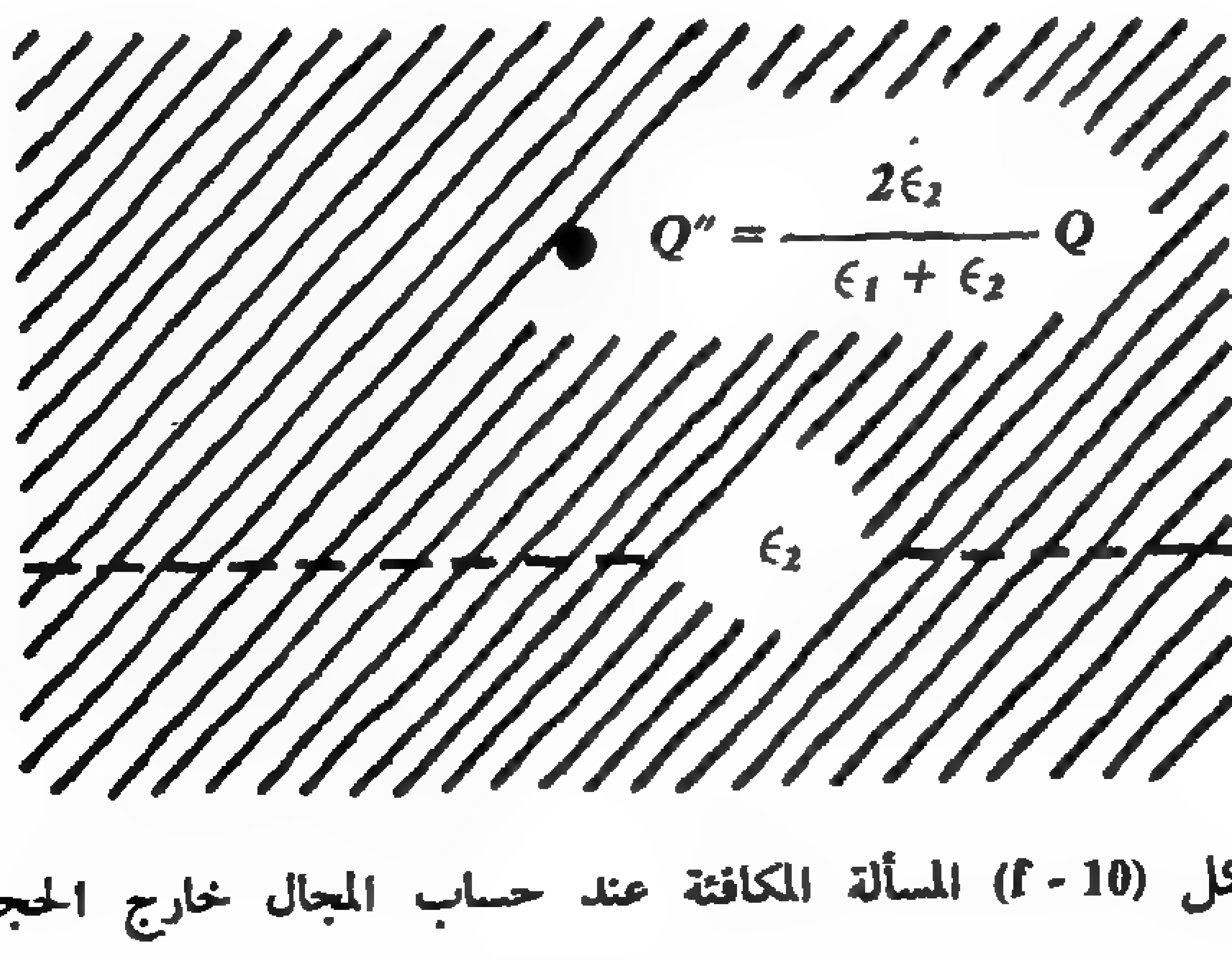
وإذا كان السطح S يفصل وسطين ذات سماحية ϵ_1 ، ϵ_2 فإن المجال عند أي نقطة في الوسط الأول يكون كما لو كان كل الفراغ مملوء بالعازل ϵ_1 وهناك شحنة أخرى Q' تقع في النقطة المقابلة للشحنة Q .

$$Q'' = \frac{2\, \epsilon_2}{\epsilon_1 + \epsilon_2}\, Q \qquad (8 - 10)$$

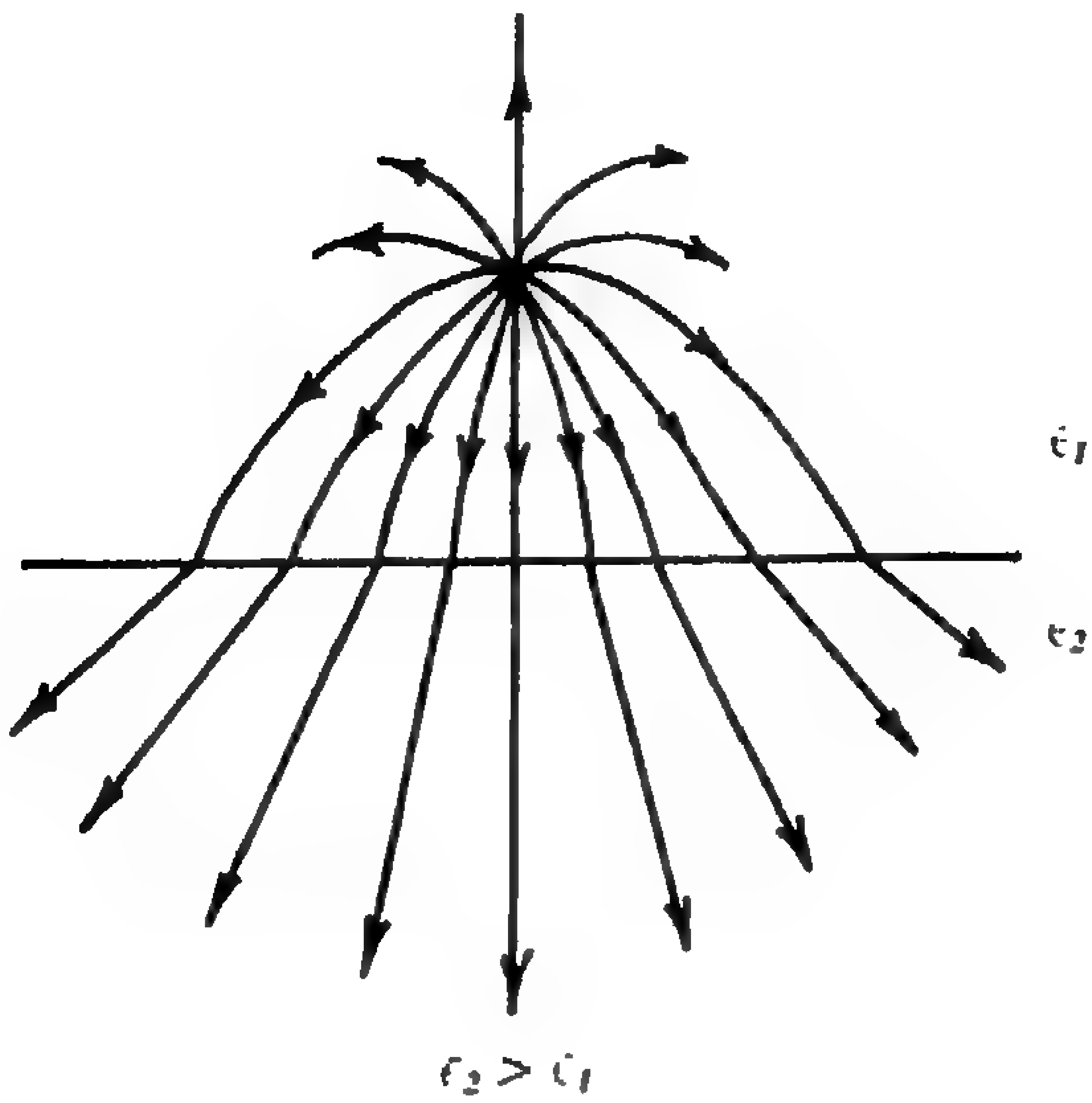
والفراغ كله ϵ_2 . الشكل $(f, e - 10)$ يبين المسألة في الحالتين وكذلك يعطي شكل $(g - 10)$ خطوط القوى الكهربية في الوسطين إذا كانت $\epsilon_1 > \epsilon_2$، وكذلك إذا كانت $\epsilon_2 > \epsilon_1$.

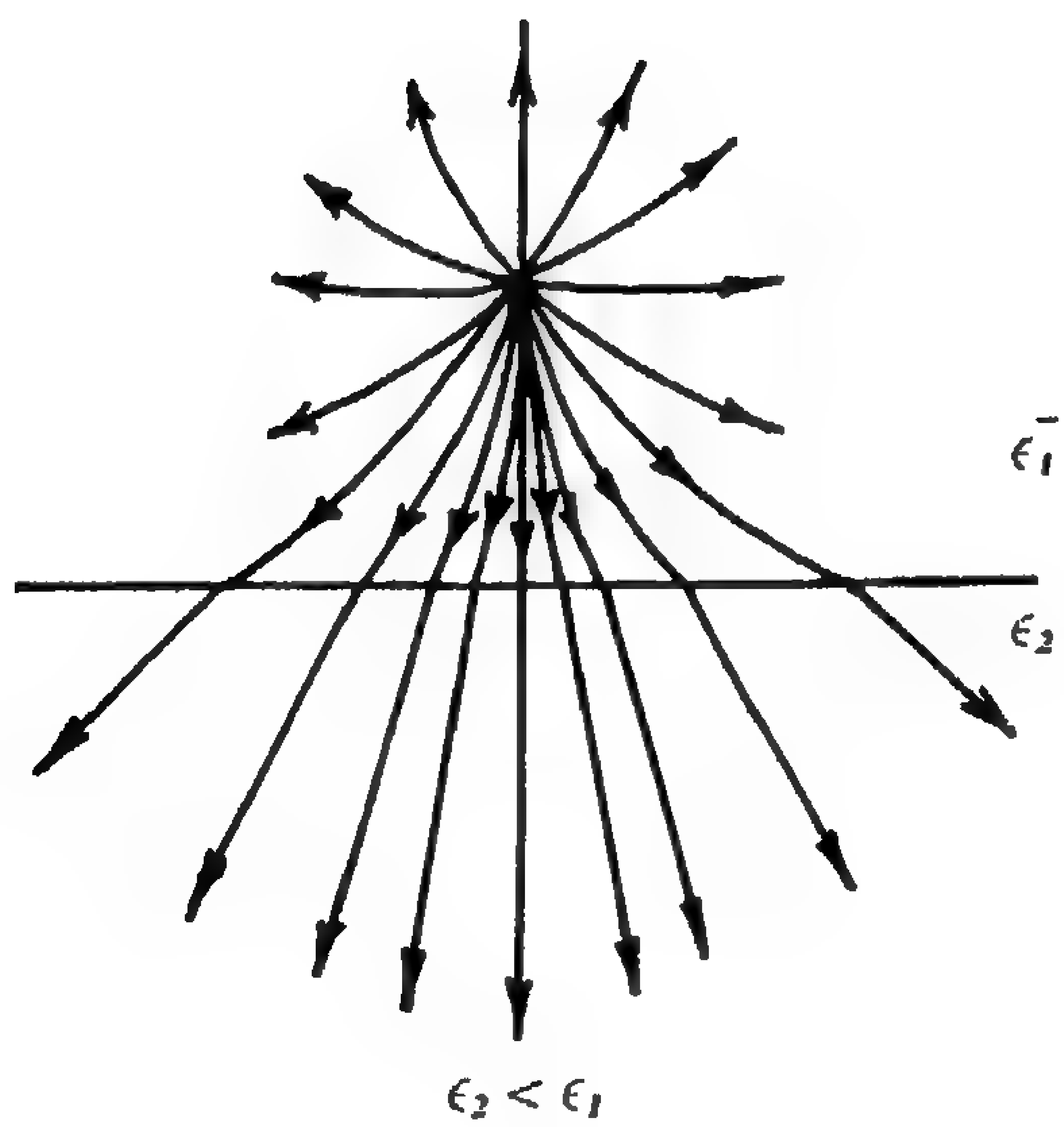


شكل (e - 10) المسألة المكافئة عند حساب المجال الكهربي داخل الحجم٢



شكل (f - 10) المسألة المكافئة عند حساب المجال خارج الحجم٢





شكل (10 - g)

خطوط القوى الكهربية من شحنة أمام سطح عازل

10 - 5 استخدام طريقة الصور في مسائل المغناطيسية الاستاتيكية :

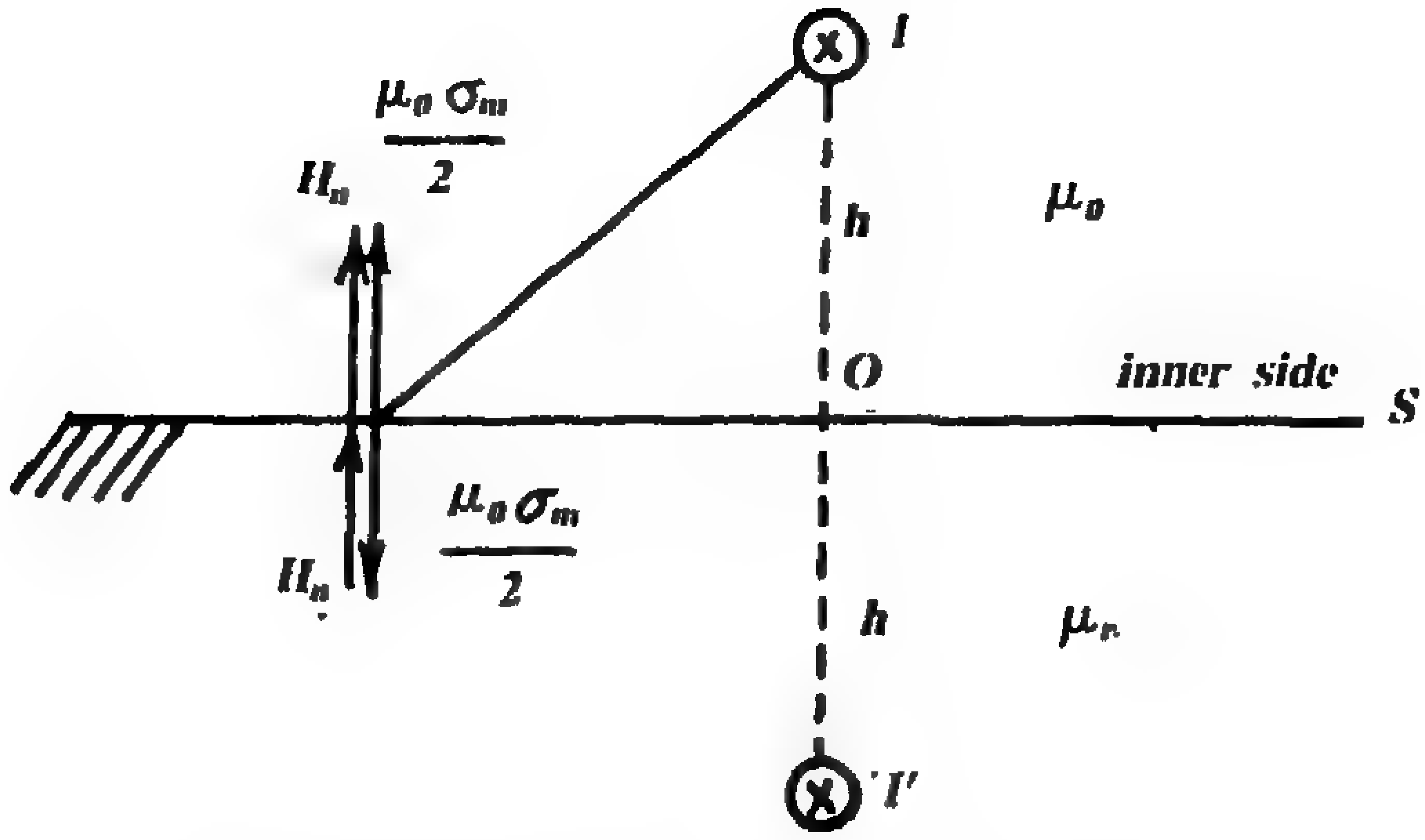
يمكن استخدام طريقة الصور في حل بعض مسائل المجالات المغناطيسية الساكنة . وكما ذكرنا في الفصل الثامن تنتج المجالات المغناطيسية الساكنة من التيارات الثابتة أو المغناطيسيات الدائمة . في الحالة الأولى يستبدل التيار بمجموعة من المغناطيسيات الصغيرة موزعة على سطح يحيط به التيار الثابت وفي الحالة الثانية يستبدل الجسم الممغنط بتوزيعين أحدهما حجمي والآخر سطحي للأقطاب المغناطيسية . وفي كلا الحالتين يمكن أن نحصل على المجال المغناطيسي H من دالة قياسية $\O_H$ وبذلك تصبح المسألة المغناطيسية مشابهة للمسألة الكهربية . نعتبر سطح مغلق S يقسم الفراغ إلى قسمين ثم نعين المركبة العمودية أو المماسة للمجال المغناطيسي H على السطح S الناتجة عن توزيع الأقطاب المغناطيسية عليه . ثم نوجد توزيع الأقطاب المكافىء خارج S ليعطي على S نفس المجال العمودي أو المماس . هذا التوزيع هو توزيع الأقطاب الصوّرية المطلوب .

10 - 6 تيار خطي I أمام سطح مغناطيسي مستوى لانهائي :

نفرض أن σ_m هي كثافة الأقطاب المغناطيسية الناشئة بالتأثير عند أي نقطة على السطح S . فإذا كانت H_n هي المركبة العمودية للمجال المغناطيسي الممغنط نتيجة التيار I فإن المركبة العمودية الكلية للمجال المغناطيسي المحصل الناشر من I والمغناطيسية التأثيرية على السطح S (شكل $10 - h$) تحقق العلاقة :

$$\mu_1 H_{n1} = \mu_2 H_{n2}$$

$$\mu_0 \left(\frac{1}{2} \sigma_m + H_n\right) = \mu_0 \mu_r \left(H_n - \frac{1}{2} \sigma_m\right)$$



شكل (h - 10) تيار خطي امام سطح عازل لانهائي

ومنها نستنتج أن :

$$\frac{1}{2}\sigma_m = \frac{\mu_r - 1}{\mu_r + 1} H_n \qquad (9 - 10)$$

ونلاحظ أن المجال المعطي بالمعادلة (9 - 10) ينتج عن تيار خطي لانهائي شدته I' :

$$I' = \frac{\mu_r - 1}{\mu_r + 1} I \qquad (10 - 10)$$

على مسافة h تحت السطح S كما هو مبين بشكل (h - 10) . وعلى ذلك فإن المجال المغناطيسي عند أي نقطة في الفراغ أمام السطح المغناطيسي ينشأ من تيارين أحدهما التيار الأساسي I والآخر صورة هذا التيار I' وهي موضوعة على مسافة $2h$ من التيار I والوسط كله الفراغ .

ولايجاد H عند أي نقطة داخل الوسط المغناطيسي نلاحظ أن المركبة العمودية للمجال H_n الموجودة داخل السطح S الناتجة عن σ_m هي :

$$\frac{1}{2}\sigma_m = \frac{\mu_r - 1}{\mu_r + 1} H_n$$

وهذا المجال ينشأ عن تيار خطي مقداره .

$$- I' = - \frac{\mu_r - 1}{\mu_r + 1} I \qquad (11 - 10)$$

على مسافة h خارج S . وبناء على ذلك فإن المجال المغناطيسي داخل المادة الممغنطة ناشىء عن التيار المحصل :

$$I'' = I - \frac{\mu_r - 1}{\mu_r + 1} I = \frac{2}{\mu_r + 1} I \qquad (12 - 10)$$

وهو في نفس اتجاه التيار I . وموضوع في وسط لانهائي ذات انفاذية نسبية μ_r . شكل $(i - 10)$. وبذلك تكون خطوط الفيض المغناطيسي داخل الوسط الممغنط هي دوائر مركزها التيار I_2 .

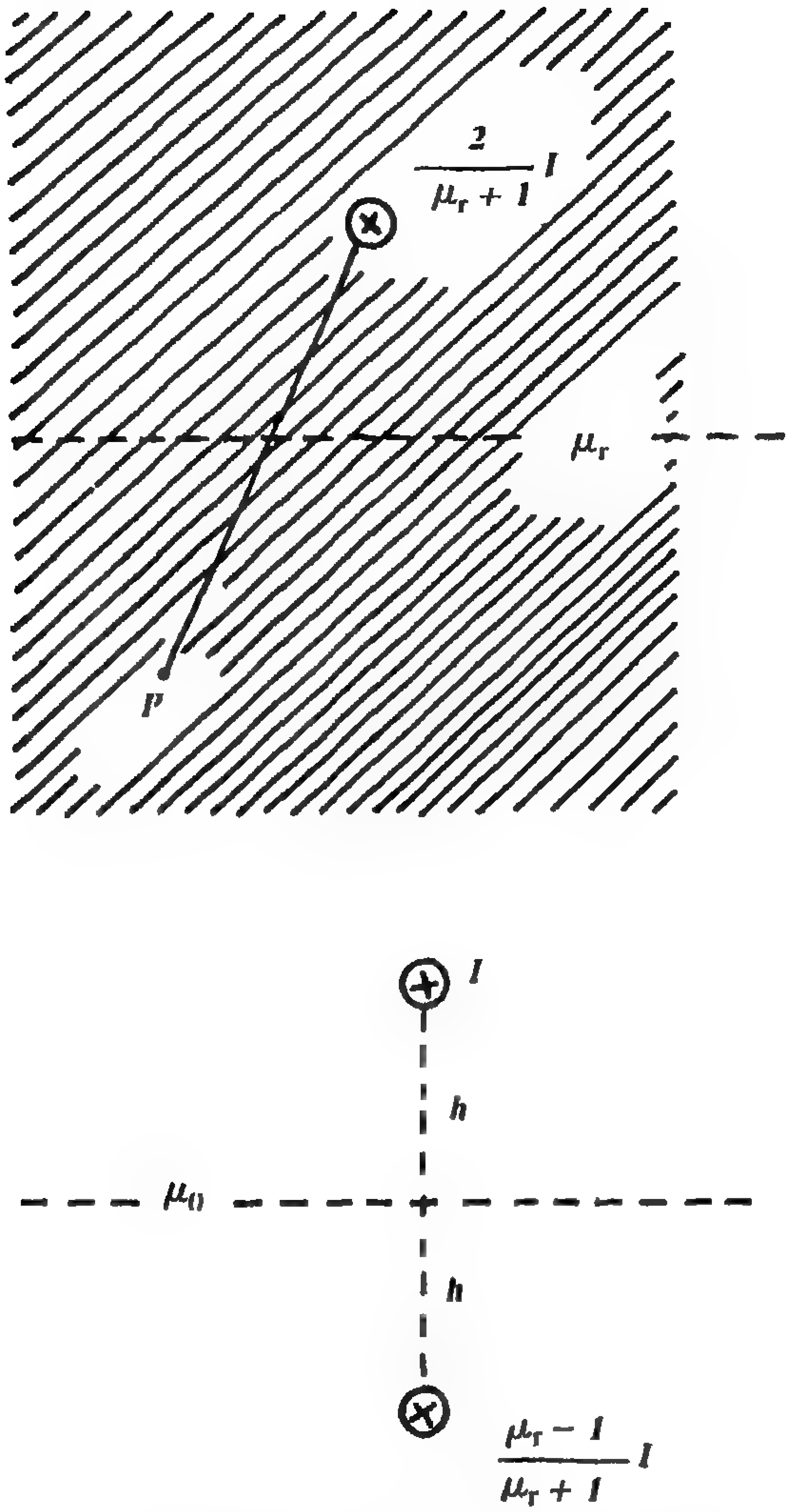
إذا اعتبرنا الحالة التي فيها تؤول μ_r إلى اللانهاية نجد أن التيار I' يؤول إلى I ويكون المجال في الفراغ ناشىء عن تيارين خطيين I ، I . في هذه الحالة يلاحظ أن خطوط القوى المغناطيسية عند أي نقطة على S تكون عمودية عليه . ويؤول التيار I'' إلى الصفر . كثافة الفيض المغناطيسي في الحديد هي :

$$B_I = \frac{\mu_r I_2}{2 \pi r} = \frac{\mu_0 \mu_r}{\mu_r + 1} \cdot \frac{I}{\pi r}$$

وعندما تؤول μ_r إلى اللانهاية :

$$B_I = \frac{\mu_0 I}{\pi r}$$

ويكون المجال المغناطيسي منعدماً $(H_I = \frac{B_I}{\mu_r} \rightarrow 0)$ ويبين شكل $(i - 10)$ خطوط المجال B داخل وخارج المادة الممغنطة ذات النفاذية اللانهائية .

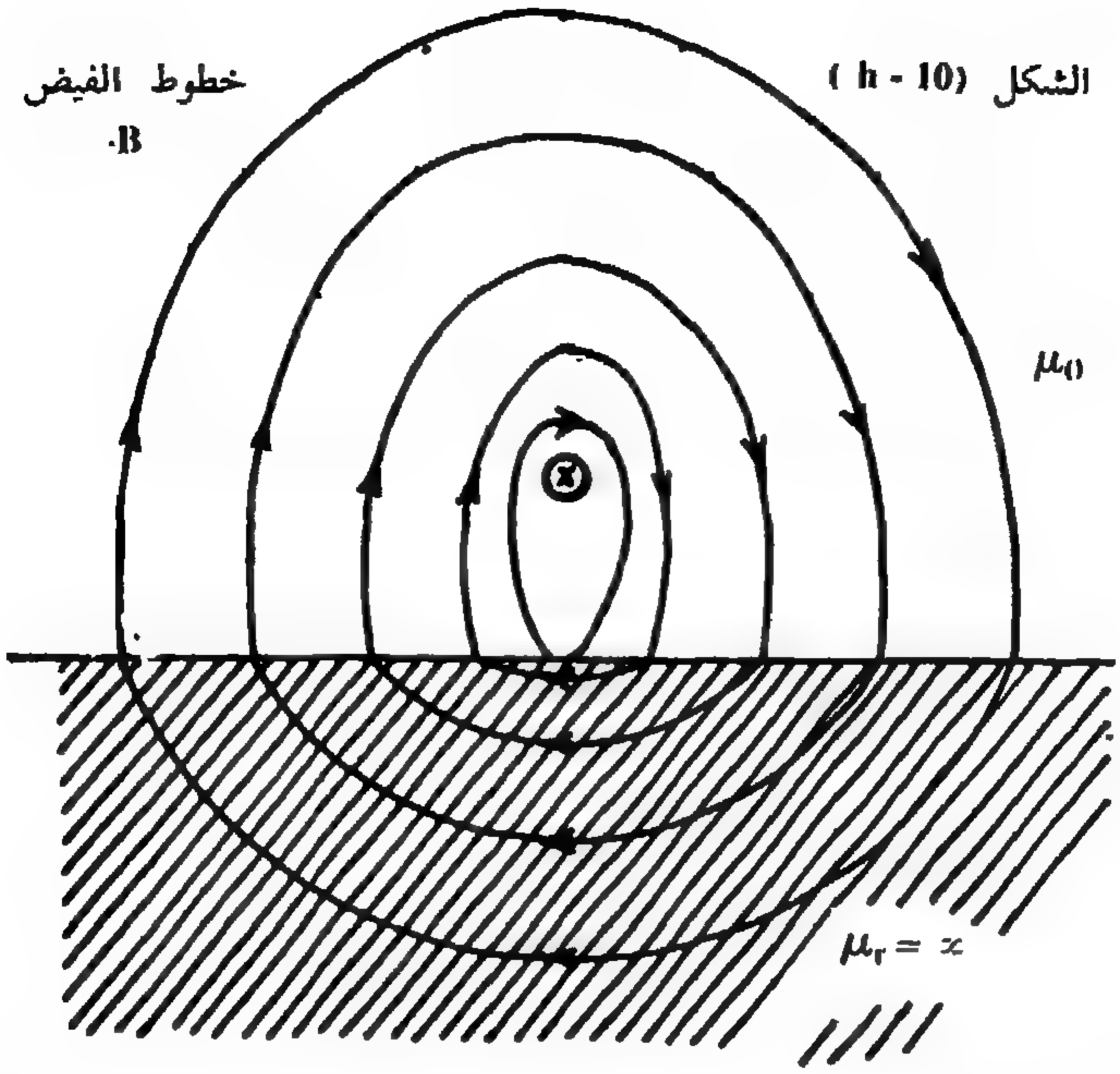


شكل (10 - i) المسألة المكافئة لشكل (10 - h)

وإذا افترضنا أن التيار الخطي I يقع داخل الوسط المغناطيسي يمكن أن نبرهن أن المجال المغناطيسي داخل هذا الوسط يحدد بتيار I وصورة مقدارها :

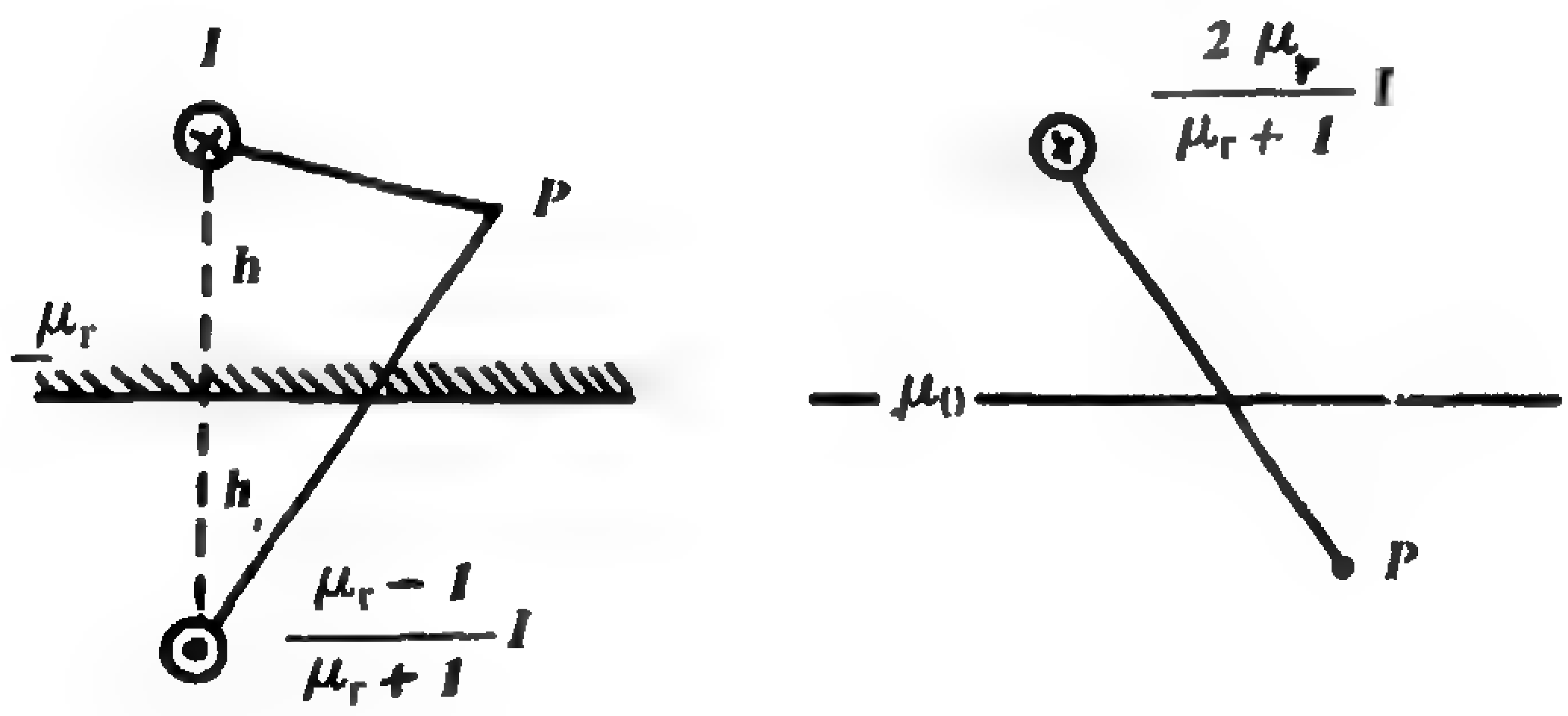
$$I' = -\frac{\mu_r - 1}{\mu_r + 1} I \qquad (13 - 10)$$

الشكل (h - 10)



شكل (j - 10) شكل خطوط الفيض المغناطيسي B حول تيار امام وسط لانهائي النفاذية

على مسافة h من السطح وكلاهما موجودان في وسط لانهائي الانفاذية النسبية له μ_r . شكل $(k - 10)$.



شكل (k - 10) المسألة المكافئة لتيار I يقع داخل الوسط μ_r

اما المجال في منطقة الفراغ فينشأ من تيار خطي واحد معطي بالعلاقة :

$$I'' = \frac{2\mu_r}{\mu_r + 1} I \qquad (14 - 10)$$

له نفس اتجاه I وفي مكانه وكل الوسط فراغ وعلى ذلك تكون خطوط المجال الكهربي دوائر في الفراغ مركزها I_2 . ويمكن اختبار الحالة التي فيها μ_r تؤول إلى ∞ . فيصبح التيار $I' = -I$ والمجال في الحديد يكون ناشىء عن I ، $-I$.

10 - 7 طريقة التعاكس : (Inversion Method) :

طريقة التعاكس هي طريقة هندسية يمكن أن تستخدم لحل بعض مسائل الكهربية الساكنة وهي تعتمد على تحويل المسألة المعطاة إلى مسألة أخرى معروف حلها أو أسهل من المسألة الأساسية . وتعطي المسألة رقم 51 من المسائل المحلولة النظريات الهندسية المتعلقة بطريقة التعاكس ويلي ذلك المسائل من 52 إلى 58 التي توضح باسهاب استخدام طريقة التعاكس لحل المسائل الكهروستاتيكية .

10 - 8 مسائل محلولة

SOLVED PROBLEMS

I. THE METHOD OF IMAGES

1. *A point charge Q is placed in front of an infinite conducting earthed plate. Prove that the lines of force from the point charge, making an angle ϕ or less with the normal from the charge to the plate, end on a circular region on the plane whose radius subtends an angle θ at Q where $\sin(\frac{1}{2}\phi) = \sqrt{2}\sin(\frac{1}{2}\theta)$.*

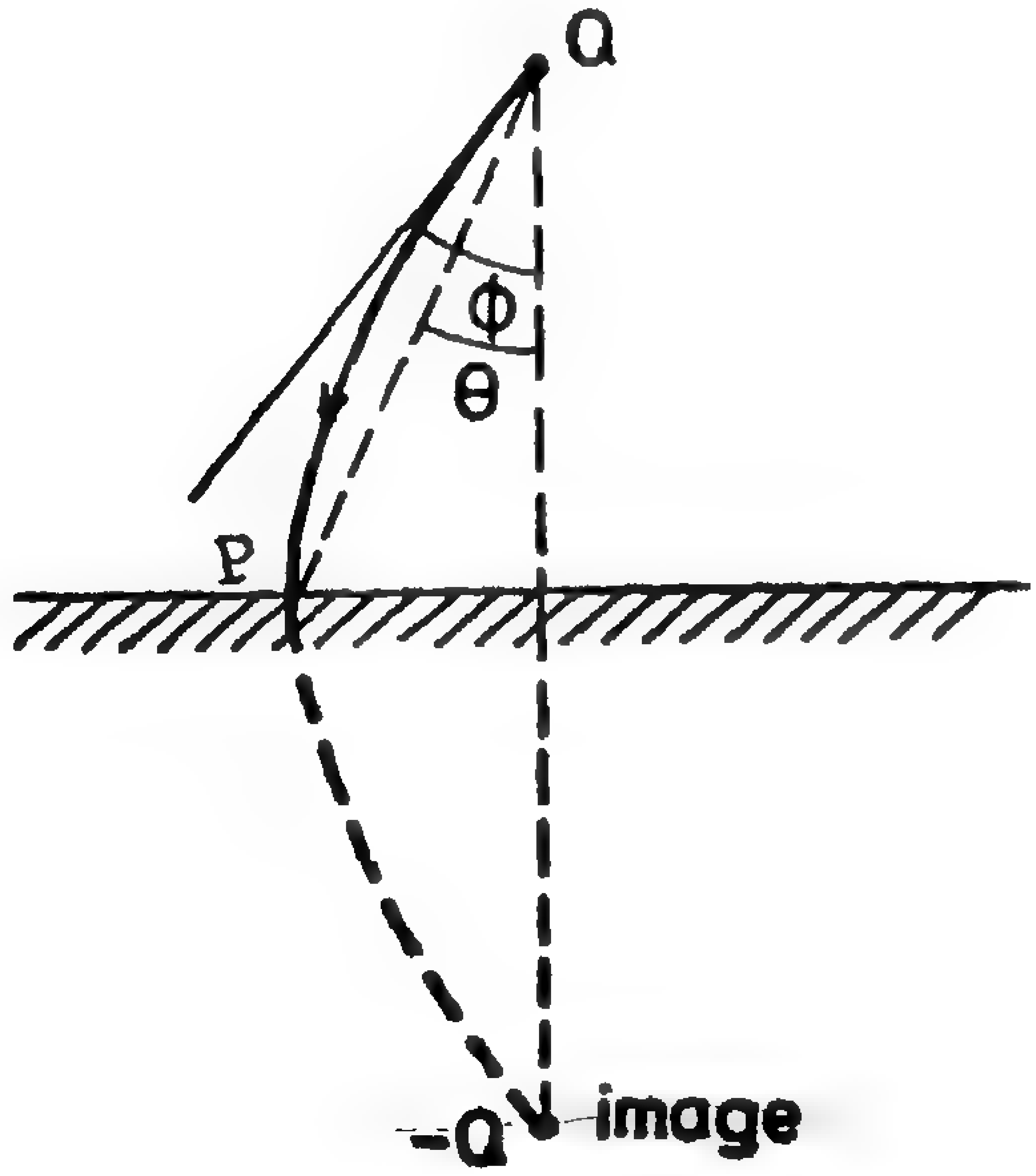


Fig. 9.1.

The charge Q has an image $Q' = -Q$ on the opposite side of the earthed plate as shown in (Fig. 9.1). The equation of any of the lines of force starting from Q is, (see Problem 2.15).

$$Q \cos \theta_1 - Q \cos \theta_2 = const. \tag{1}$$

At the point occupied by Q, $\theta_1 = \phi$, and $\theta_2 = \pi$, while at the point occupied by P, $\theta_1 = \theta$, and $\theta_2 = \pi - \theta$. Thus equation (1) gives,

$$Q \cos \phi - Q \cos \pi = Q \cos \theta - Q \cos (\pi - \theta)$$

This gives,

$$\cos \phi + 1.0 = 2 \cos \theta,$$

Thus,

$$\sin (\tfrac{1}{2} \phi) = \sqrt{2} \sin (\tfrac{1}{2} \theta)$$

2. *Two point charges ±Q, distance 2a apart, are placed in front of an infinite conducting plate, each at distance a from the plane. Show that the force required to maintain either charge in position is at an angle 45° to the plate, and is less than it would be if the plane were absent in the ratio 0.914.*

As shown in Fig. 9.2 the charges +Q and — Q have images — Q and + Q respectively. The charge Q at point *A* experiences three forces as shown.

$$F_1 = Q^2/32 \pi \epsilon_0 a^2, \quad F_2 = F_3 = Q^2/16 \pi \epsilon_0 a^2$$

The net force on the charge at *A* has a horizontal component

$$F_x = F_1 \cos 45^\circ - F_3 = [(2^{3/2} - 1) / 8 \sqrt{2}] \, Q^2/8 \pi \epsilon_0 a^2$$

and the vertical component

$$F_y = F_1 \cos 45^\circ - F_2 = [(2^{3/2} - 1) / 8 \sqrt{2}] \, Q^2/8 \pi \epsilon_0 a^2$$

The total force on Q makes an angle θ with the plane given by

$$\theta \quad \arctan (F_y/F_x) \quad 45^\circ$$

If the plane does not exist the force on Q is that due to — Q only

$$F = Q^2/16 \pi \epsilon_0 a^2$$

The ratio between the two forces is

$$(F_x^2 + F_y^2)^{1/2} / F = (2^{3/2} - 1) / 2 = 0.911$$

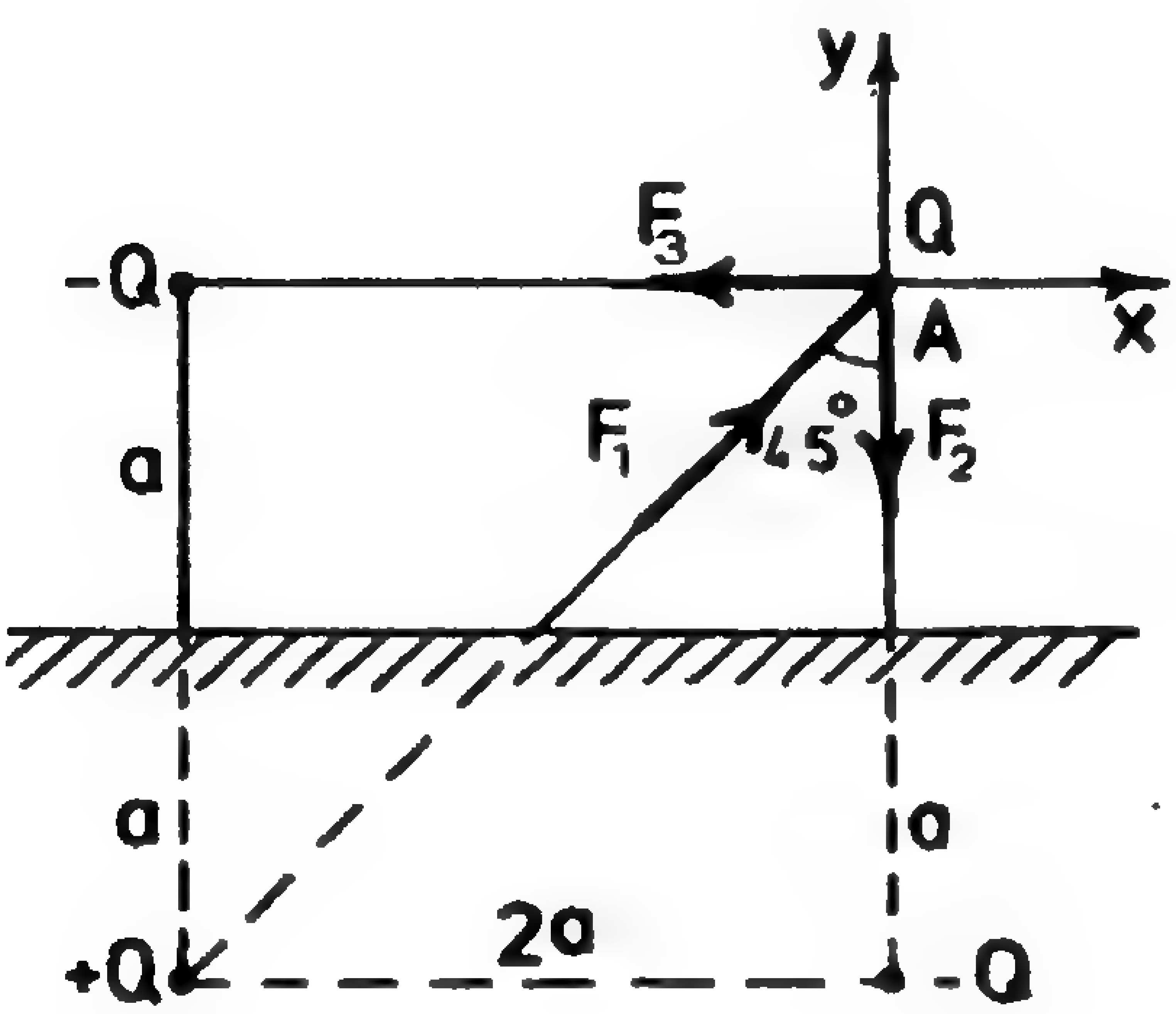


Fig. 9.2.

3. *Two particles each of mass m are connected by an insulating rod of length 2a. The rod is supported by parallel threads and hang in a horizontal position in front of an infinite earthed conducting vertical plane, Fig. 9.3. The particles are charged each with ½ Q and are in equilibrium at distance a from the conducting plane. Show that the inclination θ of the strings to the vertical is given by* $\tan \theta = [1 + 1/(2^{3/2})] Q^2 / 64 \pi \epsilon_0 m g a^2$.

As shown in Fig. 9.3 each of the charges Q/2 has an image — Q/2. The two image charges, together with the third charge exert

forces on the fourth charge. Due to the symmetry of the problem the force between the real charges will not affect the angle θ. The force on Q/2 in the direction normal to the plane is,

$$F = Q^2/(64\pi\epsilon_0 a^2) + [Q^2/(128\pi\epsilon_0 a^2)] \cos 45^\circ$$
$$= [1+1/(2^{3/2})]\, Q^2/64\pi\epsilon_0 a^2 \qquad (1)$$

Considering the equilibrium of this charge,

$$F = T \sin\theta \qquad (2)$$

$$mg = T \cos\theta \qquad (3)$$

Substituting for F from (1) and then dividing (2) by (3) we get the required result.

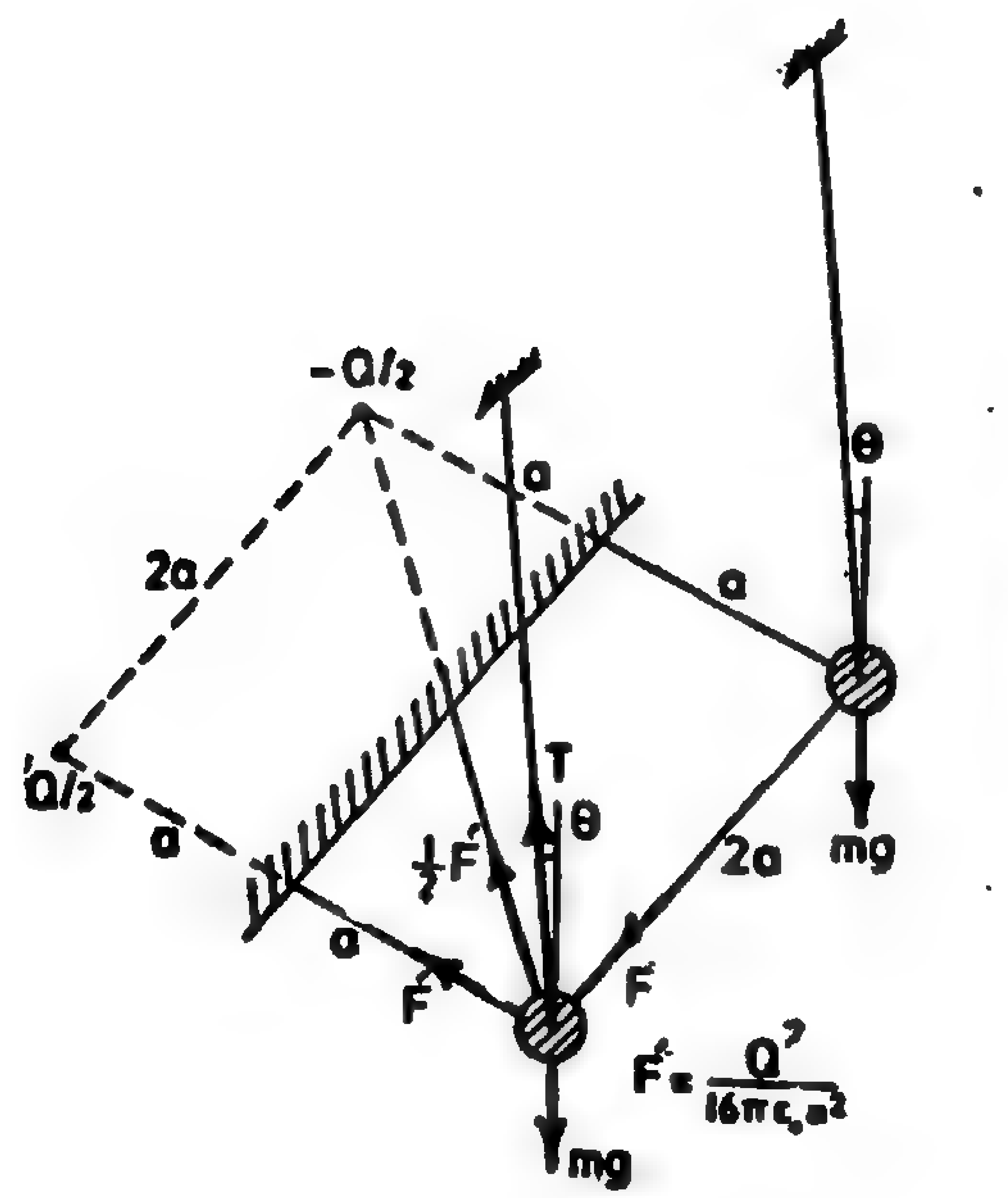


Fig. 9.3.

4. *A point charge Q is located at a distance a in front of an infinite conducting plane. If the plane is kept at zero potential, determine the surface charge density at any point on the plane.*

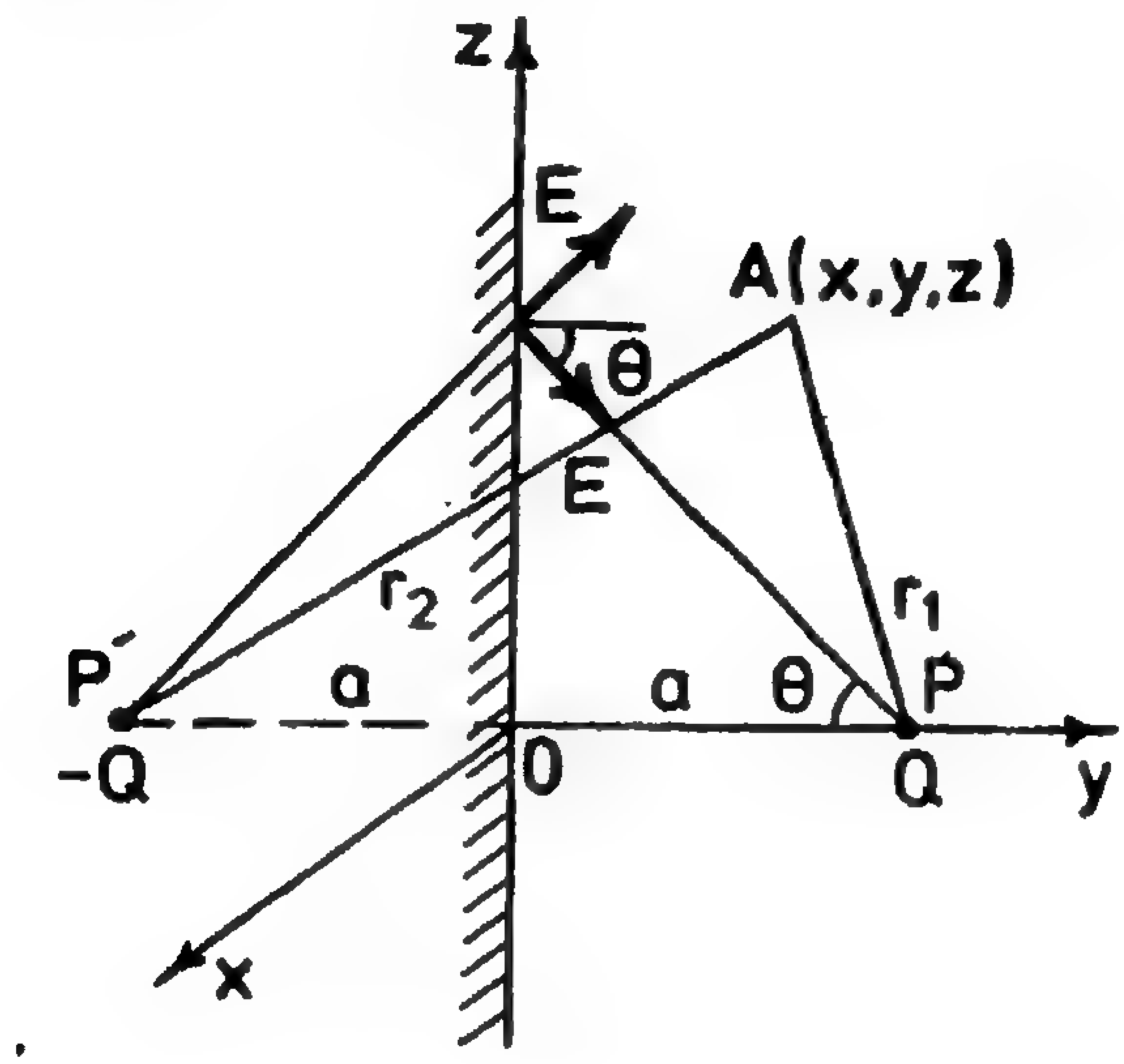


Fig. 9.4.

As shown in Fig. 9.4 the charge Q at point P has an image — Q at the point P' on the other side of the plane at $y = -a$. The potential at a general point A (x, y, z) in the region $y \geq o$ is,

$$V = (Q/4\pi\epsilon_0)(1/r_1 - 1/r_2) \tag{1}$$

where,

$$r_1 = [x^2 + (y-a)^2 + z^2]^{1/2}$$

$$r_2 = [x^2 + (y+a)^2 + z^2]^{1/2}$$

The surface charge density at any point on the conducting plane is

$$= D_n = -\epsilon_0 \, \partial V/\partial y \ \text{ at } y = o \tag{2}$$

Substituting from (1) in (2) we get,

$$\sigma = -a\, Q/2\pi\, r^3 \tag{3}$$

where r is the distance between any point on the plane and the point P. Note that we have symmetry about the OP axis.

Note that the surface charge density σ can be obtained directly from the normal component of the electric as shown in Fig. 9.4,

$$E_n = -2\, E \cos\theta = -2\, (Q/4\pi\epsilon_0 r^2)\, (a/r)$$

$$= -a\, Q/2\pi\epsilon_0 r^3$$

$$\sigma = \epsilon_0 E_n = -Qa/2\pi\, r^3$$

The surface charge distribution is shown in Fig. 9.5. It is inversely proportional to the cube of the distance from Q.

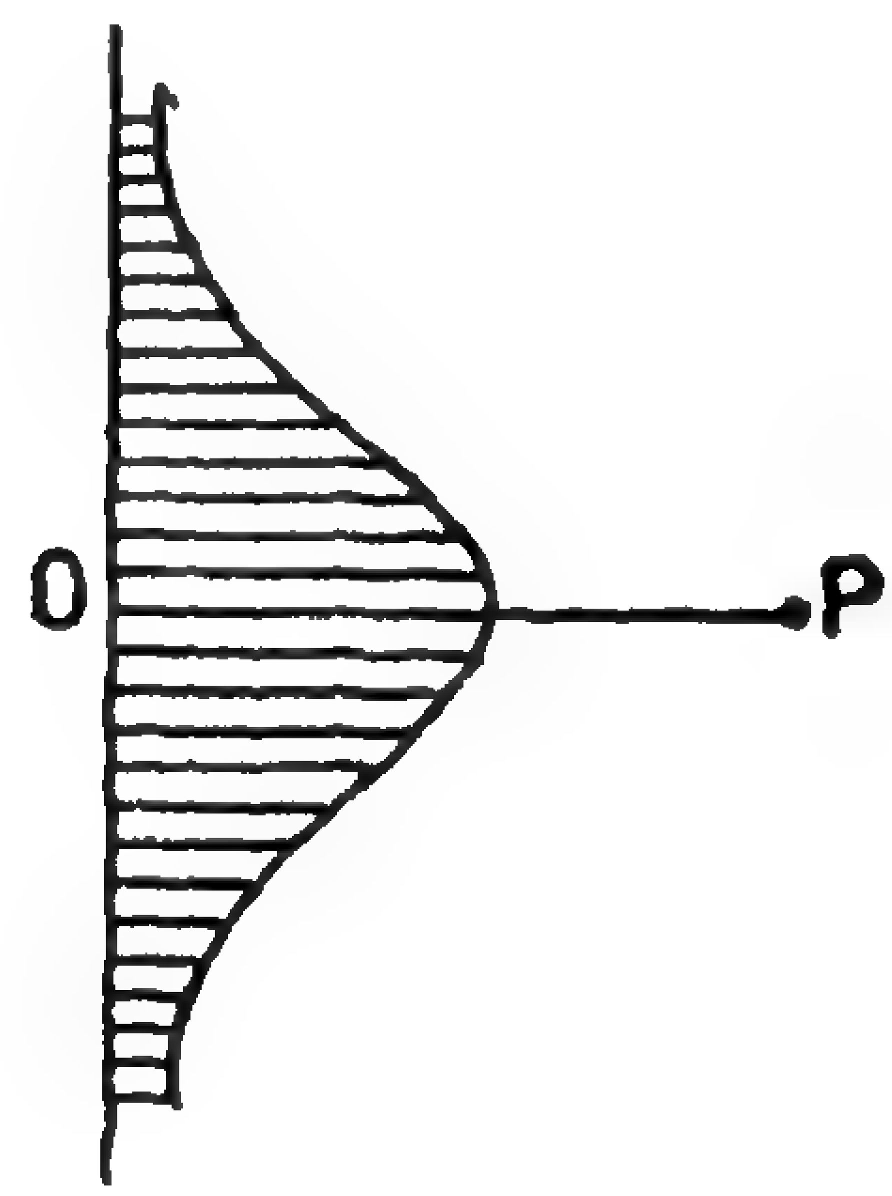


Fig.9 .5.

5. *An electron is at a distance x in vacuum from an infinite conducting plane at zero potential. Plot the potential energy of the electron as a function of x. If there is a uniform electric field in the*

region outside the plane and terminating at the plane, plot the energy of the electron as a function of distance and find the distance from the plane at which this energy is minimum.

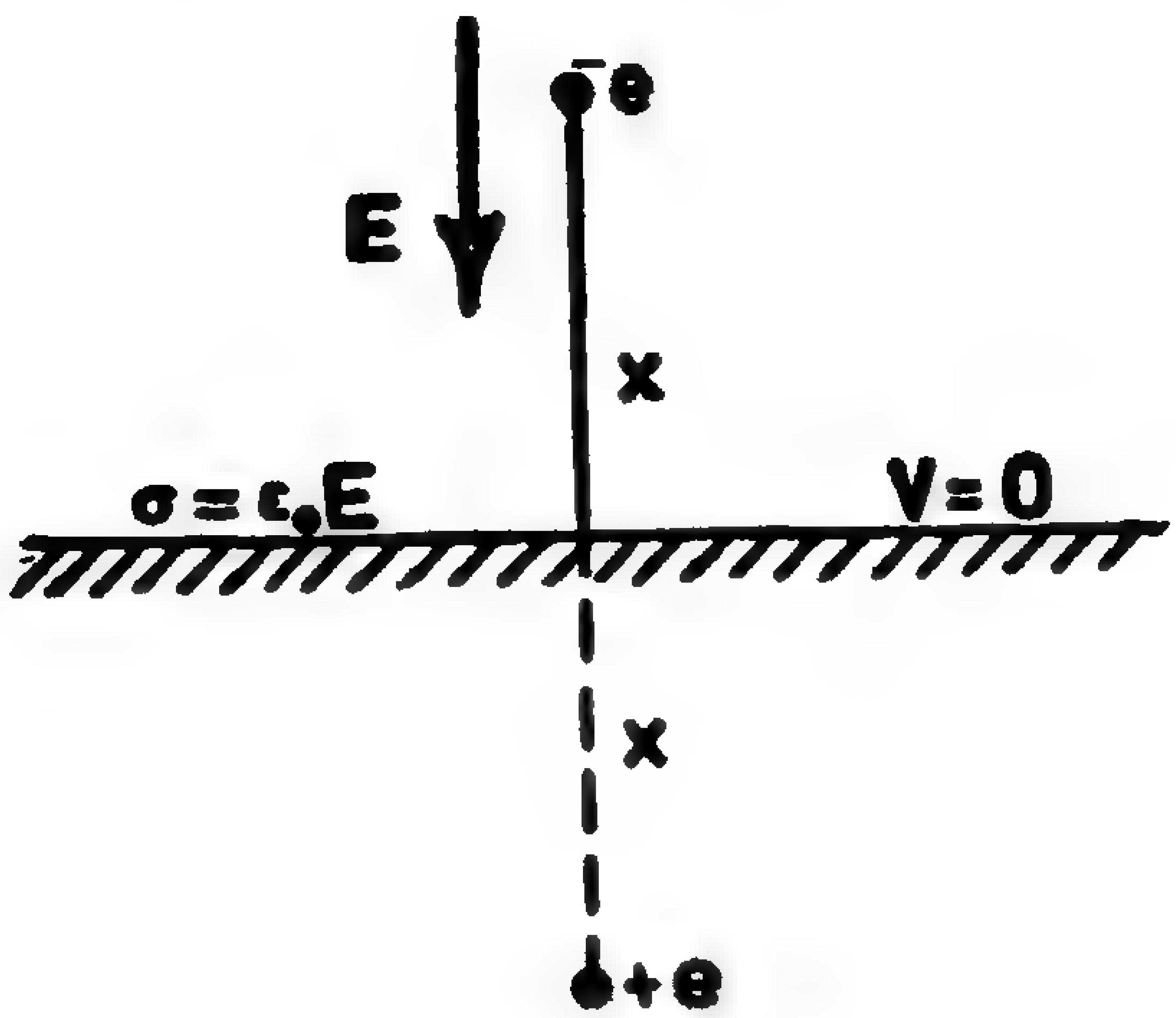


Fig. 9.6.

As shown in Fig. 9.6. the charge on the electron, $-e$, has an image $+e$ below the conducting plane. The potential at the position of the electron due to the image alone is,

$$V = e/(8\pi\epsilon_0 x) \tag{1}$$

The potential energy of the electron is thus

$$U = -eV = -e^2/(8\pi\epsilon_0 x) \tag{2}$$

If there is a uniform field E perpendicular to the plane, the potential due to this field is obtained from the relation,

$$-E = -\partial V/\partial x\,,$$

that is, $V = Ex$, so that the total potential due to E and $-e$ is,

$$V = Ex + e/(8\pi\epsilon_0 x) \tag{3}$$

The energy of the electron becomes

$$U = -eV = -eEx - e^2/(8\pi\epsilon_0 x) \tag{4}$$

For minimum energy of the electron,

$$\partial U/\partial x = 0 \qquad -eE + e^2/(8\pi\epsilon_0 x^2)$$

This gives, $x = x_0 = (e/8\pi\epsilon_0 E)^{1/2}$

The potential energy of the electron is plotted in Fig. 9.7.

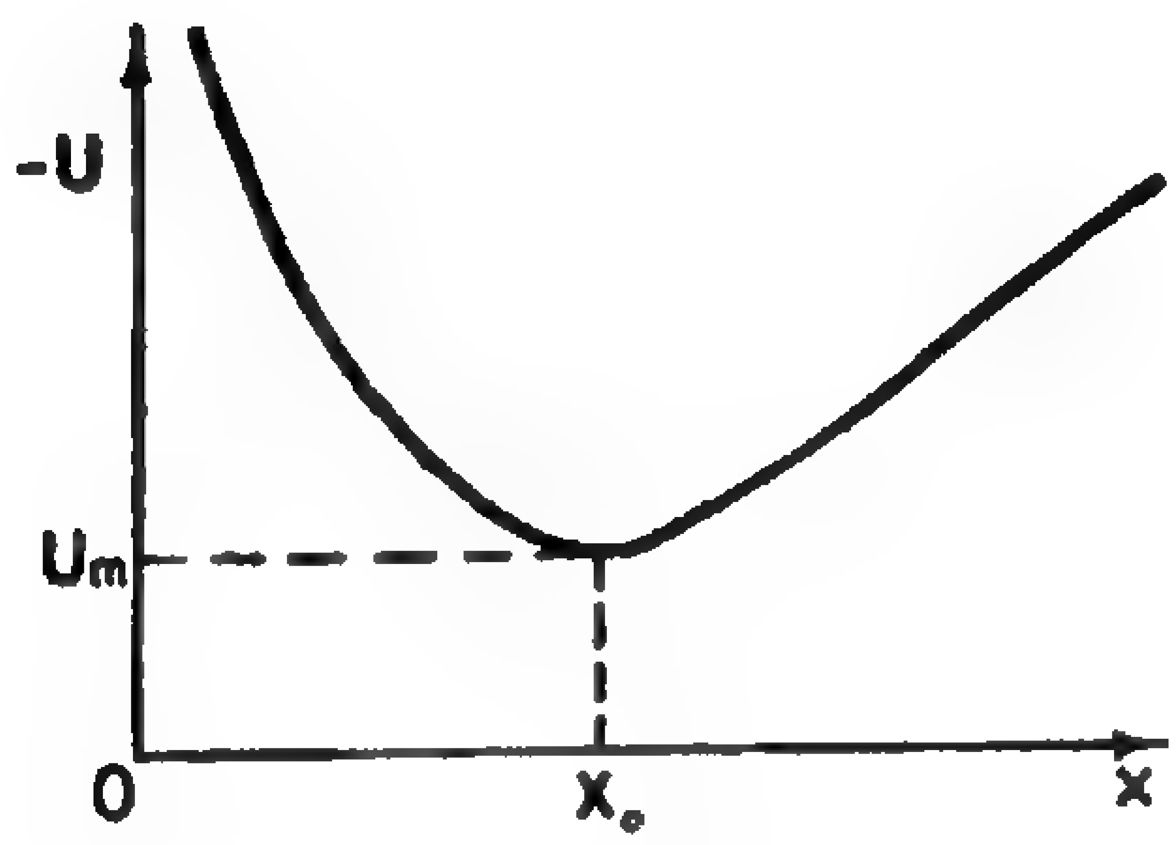


Fig. 9.7.

6. *A sphere of radius a has its center at distance c from an infinite conducting plane at zero potential. Show that its capacity is approximately* $4\pi\epsilon_0 a(1+a/2c)$ *where* a/c *is regarded small.*

The image of the charged sphere center A is another sphere at A' the image point of A with a charge—Q (Fig. 9.8). The potential at point A due to the two charged spheres is, (c is very large compared to a).

$$V_A = Q/4\pi\epsilon_0 a - Q/8\pi\epsilon_0 c = (Q/4\pi\epsilon_0)(1/a - 1/2c)$$

and that at A' is,

$$V_{A}' = -(Q/4\pi\epsilon_0)(1/a - 1/2c)$$

The capacity between the two spheres at A and A' is

$$C_1 = Q/(V_A - V'_A)$$

$$= 2\pi\epsilon_0 a/(1-a/2c)$$

$$\simeq 2\pi\epsilon_0 a\,(1+a/2c) \tag{1}$$

The capacitance between the sphere A and ground is double that of (1),

$$C = 2C_1 = 4\pi\epsilon_0 a\,(1+a/2c)$$

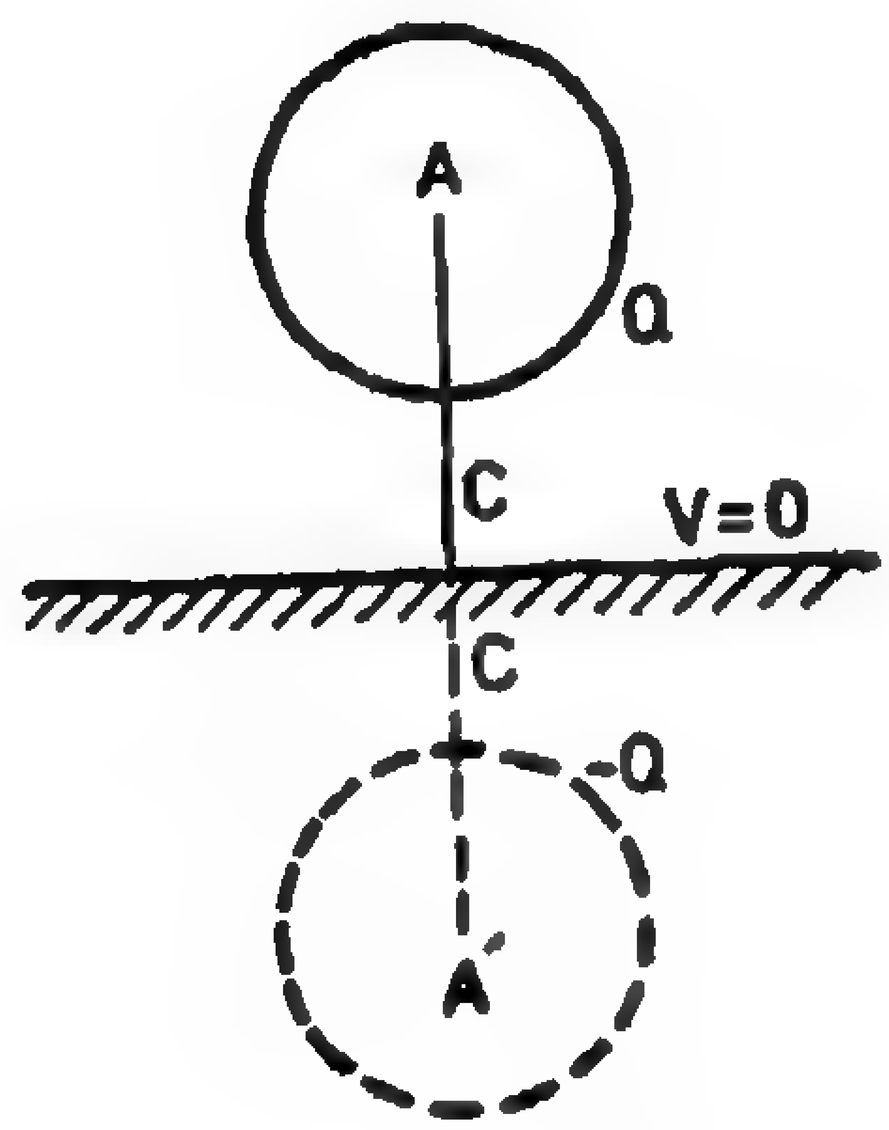


Fig. 9.8

7. *A point charge Q is placed at distances a and b from two earthed semi infinite planes intersecting at right angles. Find expressions for the surface charge densities on the nearst point of each plane.*

As shown in Fig. 9.9. the point charge Q at C has images $-$Q at D, Q at E, and $-$ Q at F. The surface charge densities at the nearst points to C are obtained from the normal field components on the two planes at A and B.

$$E_{nA} = -Q/4\pi\epsilon_0 b^2 - Q/4\pi\epsilon_0 b^2 + 2Q\sin\theta/4\pi\epsilon_0(b^2+4a^2)$$

$$= (Q/2\pi\epsilon_0)\,[b/(b^2+4a^2)^{3/2} - 1/b^2] \quad (1)$$

Similiarly,

$$E_{nB} = (Q/2\pi\epsilon_0)\,[a/(a^2+4b^2)^{3/2} - 1/a^2] \quad (2)$$

The surface charge densities at A and B are,

$$\sigma_A = \epsilon_0 E_{nA} = -(Q/2\pi)\,[1/b^2 - b/(b^2+4a^2)^{3/2}]$$

and,

$$\sigma_B = \epsilon_0 E_{nB} = -(Q/2\pi)\,[1/a^2 - a/(a^2+4b^2)^{3/2}]$$

respectively.

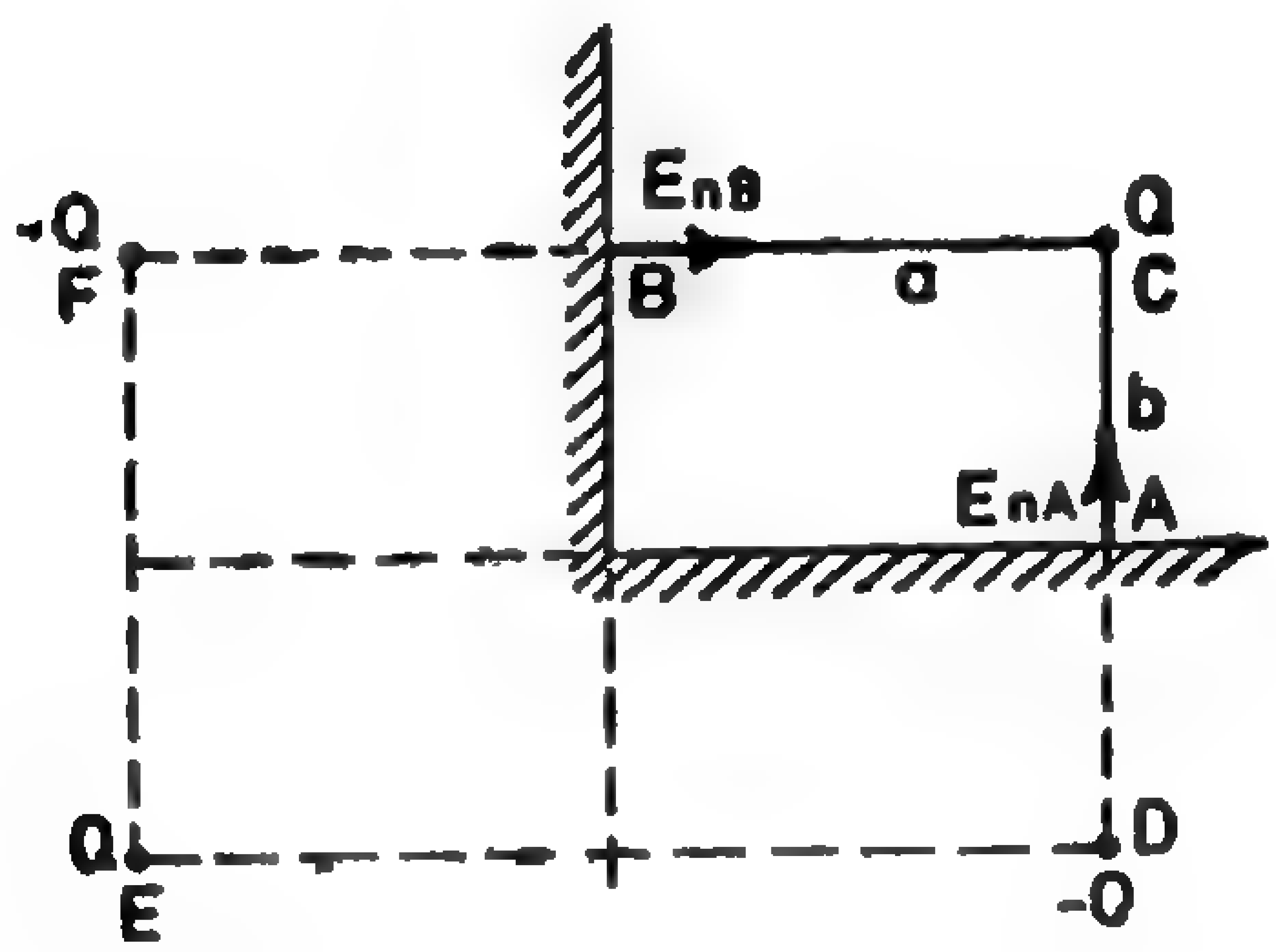


Fig. 9 9.

8. *A point charge Q is placed at a distance h from an earthed conducting plane. Find the electric field at any point.*

A thundercloud may be regarded as an eleectric dipole with vertical axis. Find an expression for the electric field at a point on the ground from which the elevation of the cloud is θ_0.

As explained in problem 9.4 the potential at a general point (x, y, z) (see Fig. 9.4) is,

$$V = (Q/4\pi\epsilon_0)(1/r_1 - 1/r_2) \tag{1}$$

with
$$r_1 = [x^2 + y^2 + (z-h)^2]^{1/2}$$
$$r_2 = [x^2 + y^2 + (z+h)^2]^{1/2}$$

Thus the electric field at any point is

$$\mathbf{E} = -\nabla V = (Q/4\pi\epsilon_0)\{(x/r_1^3 - x/r_2^3)\,\mathbf{a}_x + (y/r_1^3 - y/r_2^3)\,\mathbf{a}_y + [(z-h)/r_1^3 - (z+h)/r_2^3]\,\mathbf{a}_z\}$$

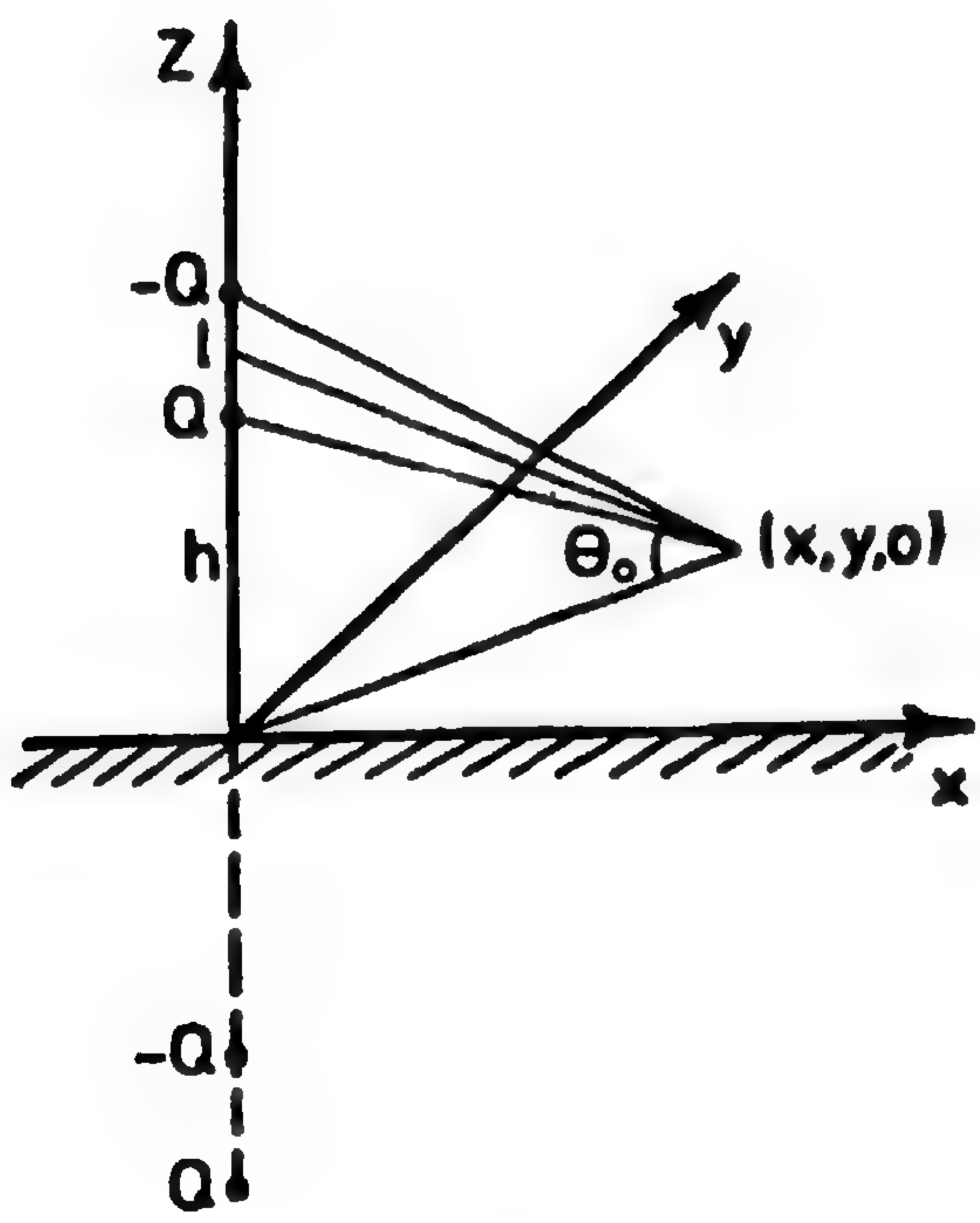


Fig. 9.10.

The electric field at any point on the conducting plane is obtained by putting $z = o$.

$$\mathbf{E}_1 = -Qh/2\pi\epsilon_0 (x^2 + y^2 + h^2)^{3/2} \ \mathbf{a}_z$$

If a second charge is placed at a small distance l, ($l = dh$) as shown in Fig. 9.10, the field due to the second charge $-$ Q is,

$$\mathbf{E}_2 = -(\mathbf{E}_1 + d\,\mathbf{E}_1) = -\mathbf{E}_1 - (\partial E_1/\partial h)\, dh$$

$$= -\mathbf{E}_1 - (\partial \mathbf{E}_1/\partial h)\, l$$

Thus the total field at any point on the conducting plane is

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 = -(\partial \mathbf{E}_1/\partial h)\, l$$

Substituting for E_1 we get

$$\mathbf{E} = [Ql/2\pi\epsilon_0]\,[3h^2/(x^2+y^2+h^2)^{5/2} - 1/(x^2+y^2+h^2)^{3/2}]\ \mathbf{a}_z$$

$$= (p/2\pi\epsilon_0 h^3)\,(3\sin^5\theta_0 - \sin^3\theta_0)$$

which is the required result.

9. *An infinite line charge of strength λ C/m is placed parallel to an infinite conducting plate at zero potential. Find an expression for the induced charge density on the conducting plate.*

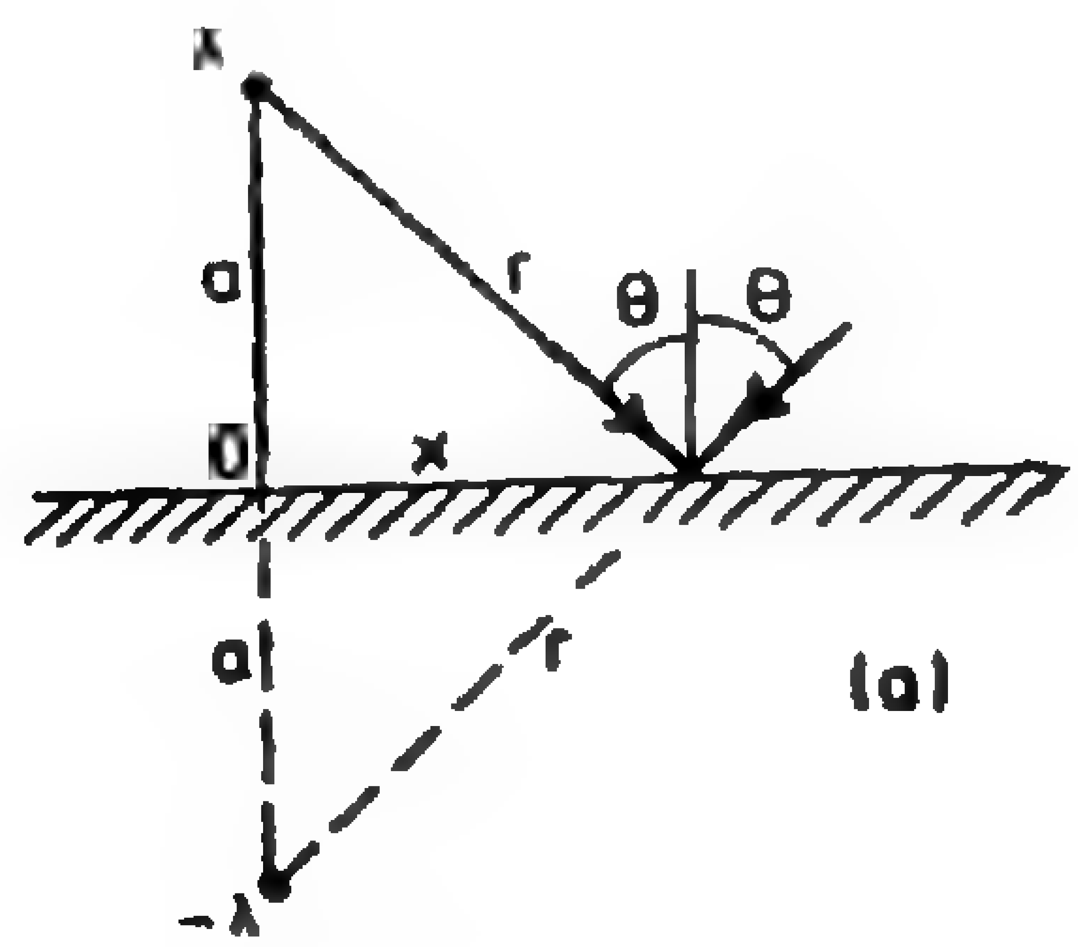


Fig. 9.11. a

The line charge of strength λ C/m has an image of strength $-\lambda$ C/m on the other side of the plate (Fig. 9.11.a). The normal component of the displacement density D_n at point P is,

$$D_n = -2 (\lambda/2\pi r) \cos \theta$$

$$= -\lambda a/\pi (x^2 + a^2)$$

The charge density at point P is thus,

$$\sigma = D_n = -\lambda a/\pi (x^2 + a^2)$$

This density function is sketched in Fig. 9.11.b. The charge is concentrated about the $x = 0$ line and decreases rapidly with the increase of x.

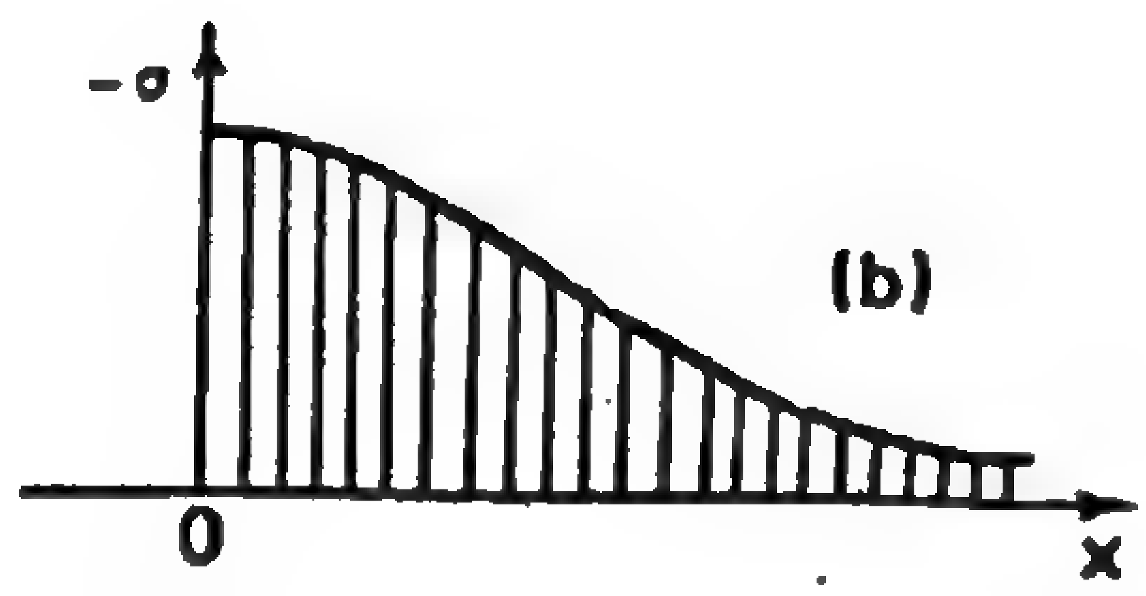


Fig. 9.11. b

10. *Two infinite line charges of strength λ_1 and λ_2 C/m are placed parallel to a grounded infinite plate as shown in Fig. 9.12. Find an expression for surface charge density at any point on the plate. If $d = o$, $b = 1$ m, $a = 2$ m, $c = 2m$, and $\lambda_1 = -2\lambda_2$, find the position of the line of no electrification.*

Similar to the previous problem, the surface charge densities due to the lines λ_1 and λ_2 at point P are,

$$\sigma_1 = -\lambda_1 a/\pi [a^2 + (y-d)^2] \qquad (1)$$

$$\sigma_2 = -\lambda_1 b/\pi [b^2 + (d+c-y)^2] \qquad (2)$$

The total surface charge density at P is,

$$\sigma = \sigma_1 + \sigma_2 \tag{3}$$

The line of no electrification is obtained by putting $\sigma = o$ in (3); we get,

$$\lambda_2 b \left[a^2 + (y-d)^2\right] + \lambda_1 a \left[b^2 + (d+c-y)^2\right] = 0$$

with $\lambda_1 = -2\lambda_2$, $d = 0$, $b = 1$, $a = 2$, and $c = 2$

we get,

$$3y^2 - 16y + 16 = 0$$

This gives $y = 1.33$, and 4. Thus we have two lines of no electrification parallel to the two line charges. These lines are at $y = 1.33$, and 4.

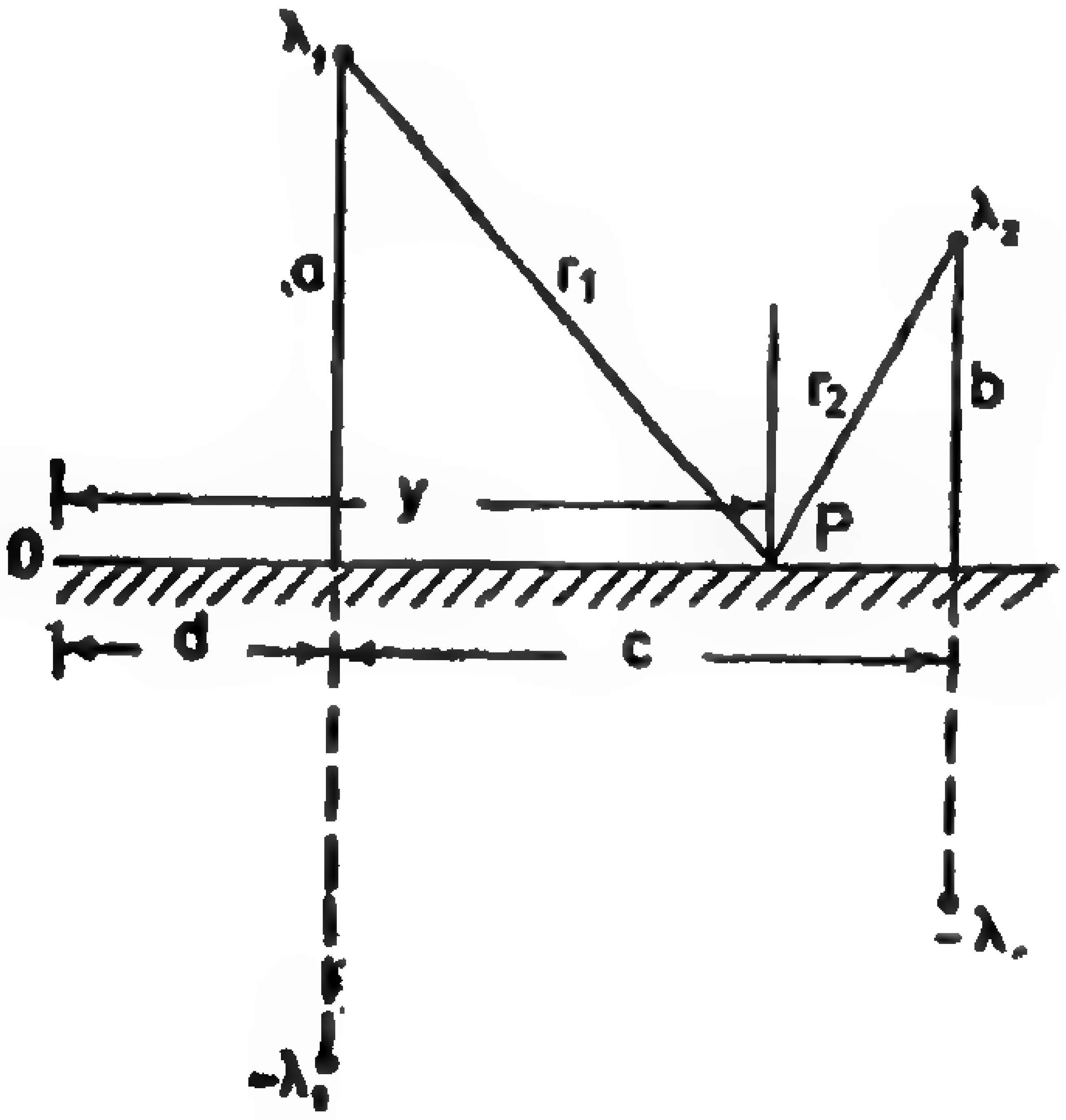


Fig. 9.12

11. *A charge Q is located within a spherical conducting shell of radius b at a point at distance a, (b>a) from the center of the sphere.*

Calculate the potential at any point inside the sphere and the surface charge density at any point on the sphere. Consider the cases

(*i*) *the spherical shell is earthed.*

(*ii*) *the spherical shell is insulated and uncharged.*

(*iii*) *the spherical shell is insulated and carries a charge q.*

(*iv*) *the spherical shell is maintained at constant potential V_o.*

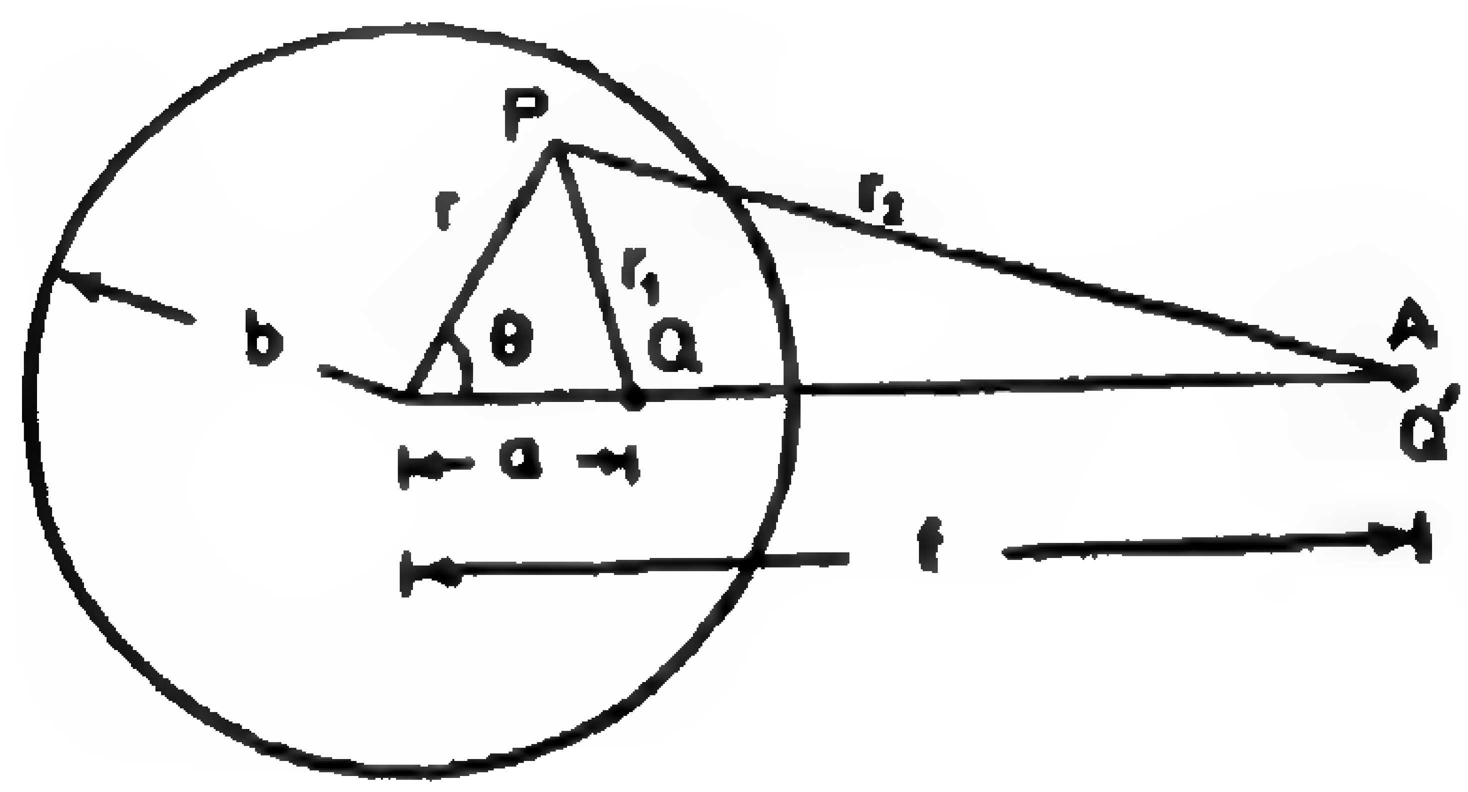


Fig. 9. 13

(i) In the method of images the conducting spherical shell is replaced by a charge Q′ located along the line OA at distance f from the center as shown in Fig. 9.13. The value of Q′ and f will be such that the sphere is at zero potential. The potential at any point P inside the sphere is

$$V(r, \theta) = (1/4\pi\epsilon_o)\,(Q/r_1 + Q'/r_2) \qquad (1)$$

where,

$$r_1^2 = r^2 + a^2 - 2a\,r\cos\theta$$

$$r_2^2 = r^2 + f^2 - 2r\,f\cos\theta$$

But $V(b,\theta)$ is zero for any θ so that

$$Q/r_1 + Q'/r_2 = 0$$

This gives,

$$Q/(b^2+a^2-2ab\cos\theta)^{1/2} = -Q'/(b^2+f^2-2bf\cos\theta)^{1/2}$$

squaring and rearranging we get,

$$[2fb(Q/-Q')^2-2ab]\cos\theta+[(a^2+b^2)-(b^2+f^2)(Q/-Q')^2] = 0$$

Since this is valid for all θ we must have,

$$2fb(Q/-Q')^2 - 2ab = 0 \qquad (2)$$

$$(b^2+f^2)(Q/-Q')^2 - (a^2+b^2) = 0 \qquad (3)$$

These two equations give

$$(Q/-Q')^2 = a/f \qquad (4)$$

Substituting from (4) in (3) gives,

$$(b^2+f^2)(a/f) = a^2+b^2$$

That is,

$$af^2-(a^2+b^2)f+ab^2 = 0$$

This gives the distance of the image charge from the center of the sphere,

$$f = [(a^2+b^2) \pm (a^2-b^2)]/2a = a,\ b^2/a$$

Now since a gives a trivial solution

$$f = b^2/a \qquad (5)$$

Which is the geometrical inverse point to A.

Therefore from (4) the image charge is

$$Q' = -(b/a)\,Q \qquad (6)$$

The potential at point P is thus given by,

$$V(r,\theta) = (Q/4\pi\epsilon_0)\left\{1/(r^2+a^2-2ar\cos\theta)^{1/2} - (b/a)/[r^2+b^4/a^2 - 2(b^2/a)r\cos\theta]^{1/2}\right\} \tag{7}$$

Equation (7) gives the potential for the imaginary situation created by the elimination of the sphere. For our problem there is equivalence between the real problem and the imaginary solution only for points inside the sphere. This is illustrated in Fig. 9.14.

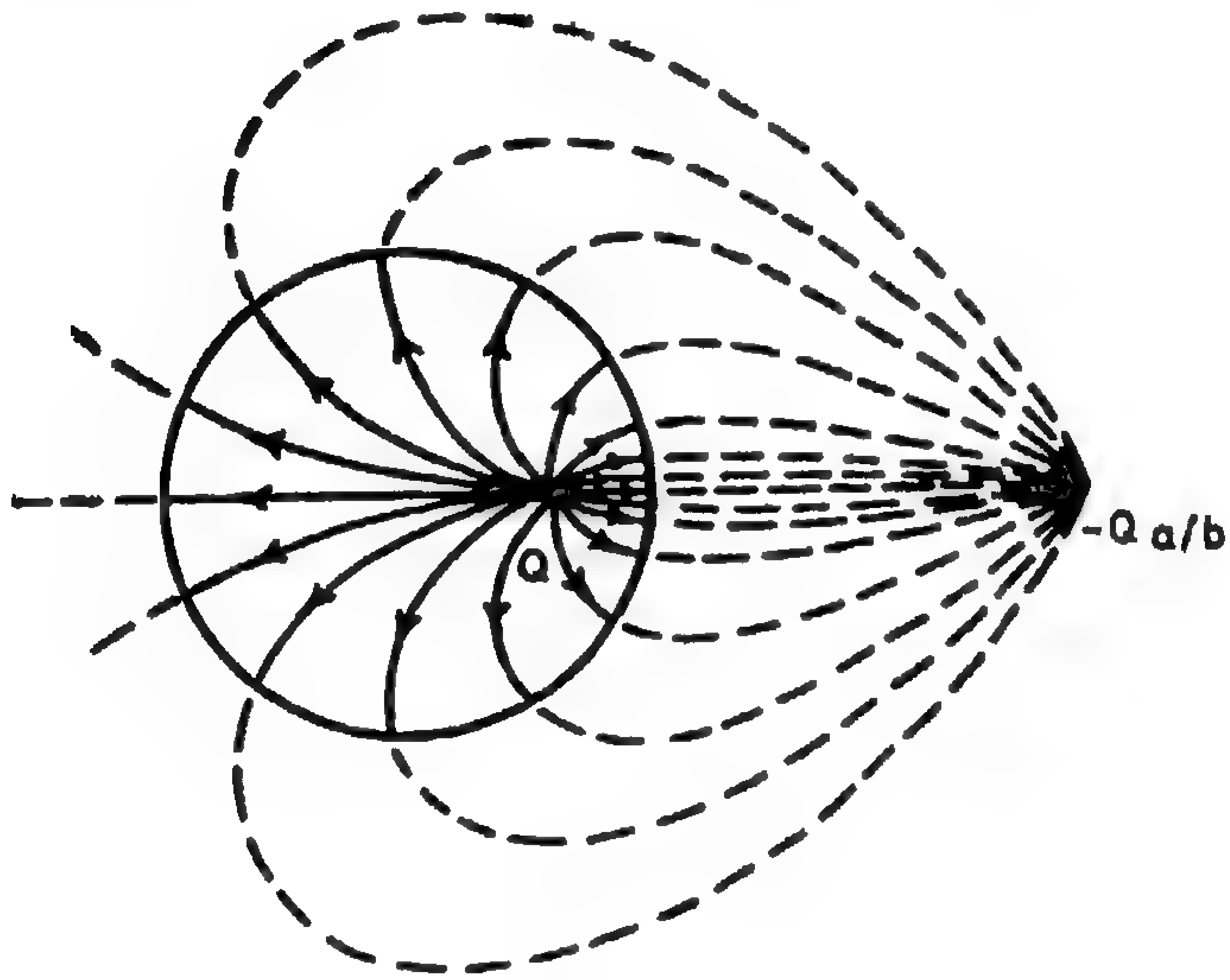


Fig. 9.14.

The electric field at point P is

$$\mathbf{E} = -(\partial V/\partial r)\,\mathbf{a}_r - (\partial V/r\partial\theta)\,\mathbf{a}_\theta$$

Substituting from (7) we get

$$E_r = (Q/4\pi\epsilon_0)\left\{(r-a\cos\theta)/(r^2+a^2-2ar\cos\theta)^{3/2} - (b/a)\,[r^2-(b^2/a)\cos\theta]/[r^2+b^4/a^2-2(b^2/a)\,r\cos\theta]^{3/2}\right\}$$

$$E_\theta = (Q/4\pi\epsilon_0)\{a/(r^2+a^2-2ar\cos\theta)^{3/2} - (b^2/a)/[r^2+(b^4/a^2)$$
$$-2(b^2/a)r\cos\theta]^{3/2}\}\sin\theta$$

For $r = b$, we have

$V(b,\theta) = 0$, $E_\theta(b,\theta) = 0$,

$$E_b(b,\theta) = (Q/4\pi\epsilon_0 b)\,[(b^2-a^2)/(a^2+b^2-2ab\cos\theta)^{3/2}] \quad (8)$$

in the radial direction as indicated in Fig. 9.14. The surface charge induced on the inner surface of the sphere is,

$$D_{b+} - D_{b-} = \epsilon_0(E_{b+} - E_{b-}) = \sigma_b$$

Since $E_{b+} = 0$ we get

$$\sigma_b = -(Q/4\pi b)\,[(b^2-a^2)/(a^2+b^2-2ab\cos\theta)^{3/2}] \quad (9)$$

The total charge induced on the surface of the sphere is

$$Q_s = \int \sigma_s\, dS$$

$$= \int_0^{\pi}\int_0^{2\pi} \sigma_b\; b^2 \sin\psi\, d\psi\, d\theta$$

$$= -[b(b^2-a^2)Q/2\pi]\int_0^{2\pi} d\theta/(a^2+b^2-2ab\cos\theta)^{3/2}$$

However

$$\int_0^{2\pi} d\theta/(a^2+b^2-2ab\cos\theta)^{3/2} = 2\pi/[b(b^2-a^2)],\; b > a$$

so that we have

$$Q_s = -Q$$

that is when the sphere is earthed the total induced charge on the inner surface is $-Q$, if the charge Q is inside the conducting sphere.

For case (ii) since the sphere is insulated and uncharged, the total charge on the inner spherical surface must be zero. Thus a charge + Q must be inserted at the center, which guarantees that the spherical surface remains equipotential with potential

$$V_o = Q/4\pi \epsilon_o b$$

The electric field at its internal surface is

$$\mathbf{E}_b = \{ Q/4\pi \epsilon_o b^2 + (Q/4\pi \epsilon_o b) [(b^2-a^2) / (a^2+b^2-2a\, b \cos\theta)^{3/2})] \} \mathbf{a}_r \quad (10)$$

and the charge density on the internal surface is

$$\sigma_{o-} = -(Q/4 \pi b) [1/b + (b^2-a^2) / (a^2+b^2-2a\, b \cos\theta)^{3/2}] \quad (11)$$

The cases (iii) and(iv) can be treated in a similar manner so that case (iii) is similar to (ii) but the potential is increased by the term $q/4\pi \epsilon_o b$ and the inner surface density is increased by $q/4\pi b^2$, while the electric field intensity (10) is increased by $q/4\pi \epsilon_o r^2$, $r = b$. This case is justified by the addition of a charge q at the center of the sphere so that the total charge at the center is Q+q.

The case (iv) is similar to case (iii) if the charge q of case (iii) is taken such that

$$V_o = q/4\pi \epsilon_o b$$

i.e. $$q = 4\pi \epsilon_o b \; V_o.$$

The potential and the electric field outside the spherical conducting shell can be obtained as follows :

In case (i) the potential of the shell is zero so that the charge induced on the outer surface is zero and consequently the electric potential and electric field vanish outside the shell.

In case (ii) since we have a zero total charge on the spherical shell, we conclude that the charge induced on the outer spherical surface is Q. Therefore for points external to the shell the sphere is substituted by a point charge Q placed at center, then

$$V(r) = Q/4\pi\,\epsilon_0\,r \qquad , \qquad r > b \tag{12}$$

and $$\mathbf{E}(r) = (Q/4\pi\,\epsilon_0\,r^2)\,\mathbf{a}_r \qquad , \qquad r > b \tag{13}$$

The charge distribution on the outer surface of the shell is uniform, and given by,

$$\sigma_{b+} = Q\,/\,4\pi\,r^2 \qquad , \qquad r > b \tag{14}$$

The cases (iii) and (iv) are similar to case (ii) but with Q replaced by $Q+q$ and $4\pi\,\epsilon_0\,b V_0$ in (12)—(14) respectively. The lines of force inside and outside the shell are shown in Fig. 9.15.

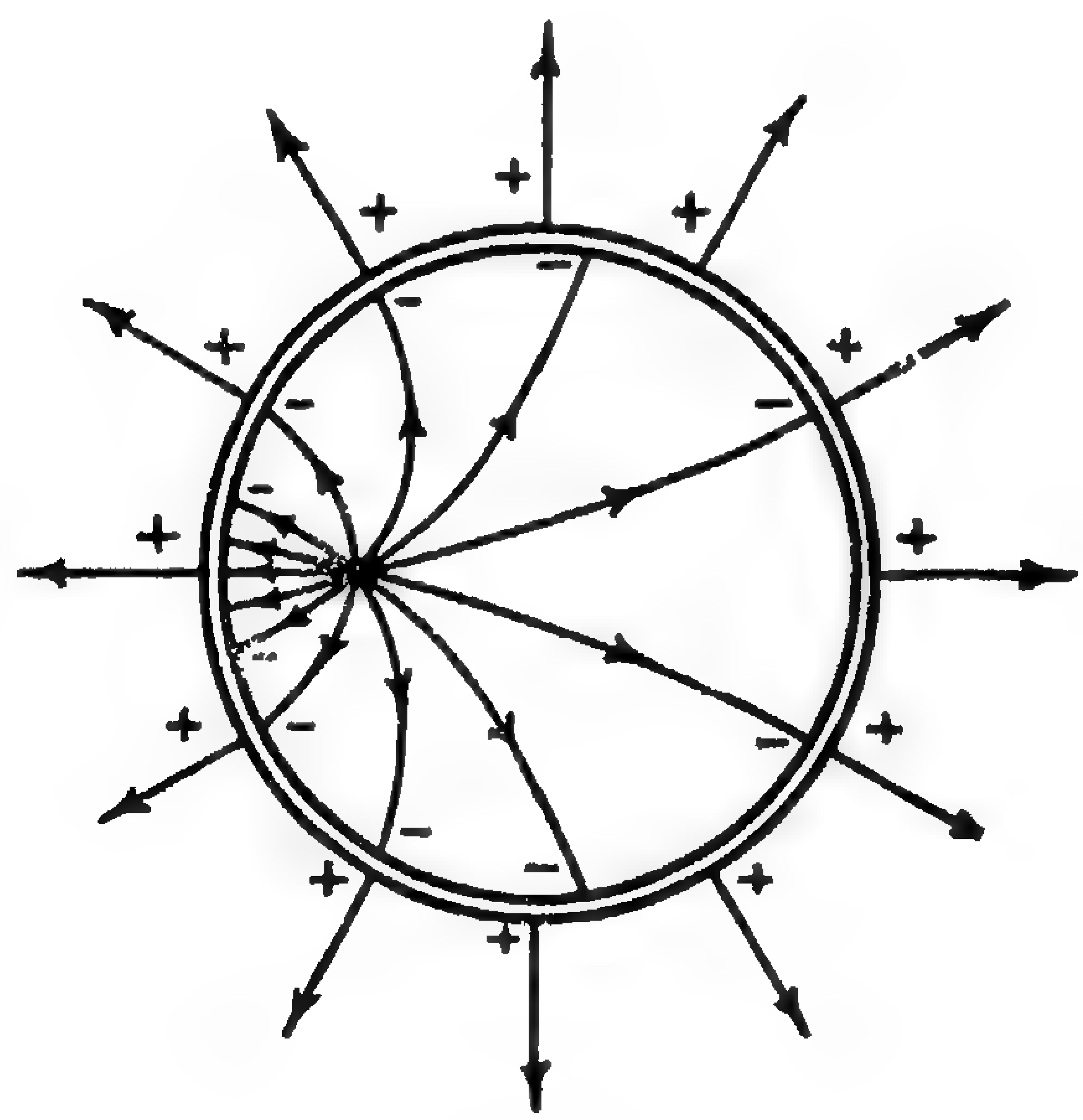


Fig. 9.15.

12. *An electric dipole of moment* **p** *is placed at the center of a spherical conducting shell of radius a. Find an expression for the induced charge density on the internal and external surfaces of the shell. Consider (i) the shell grounded, and (ii) the shell insulated. Sketch the lines of force in both cases. Also find an expression for the electric field inside the shell due to the induced charges only.*

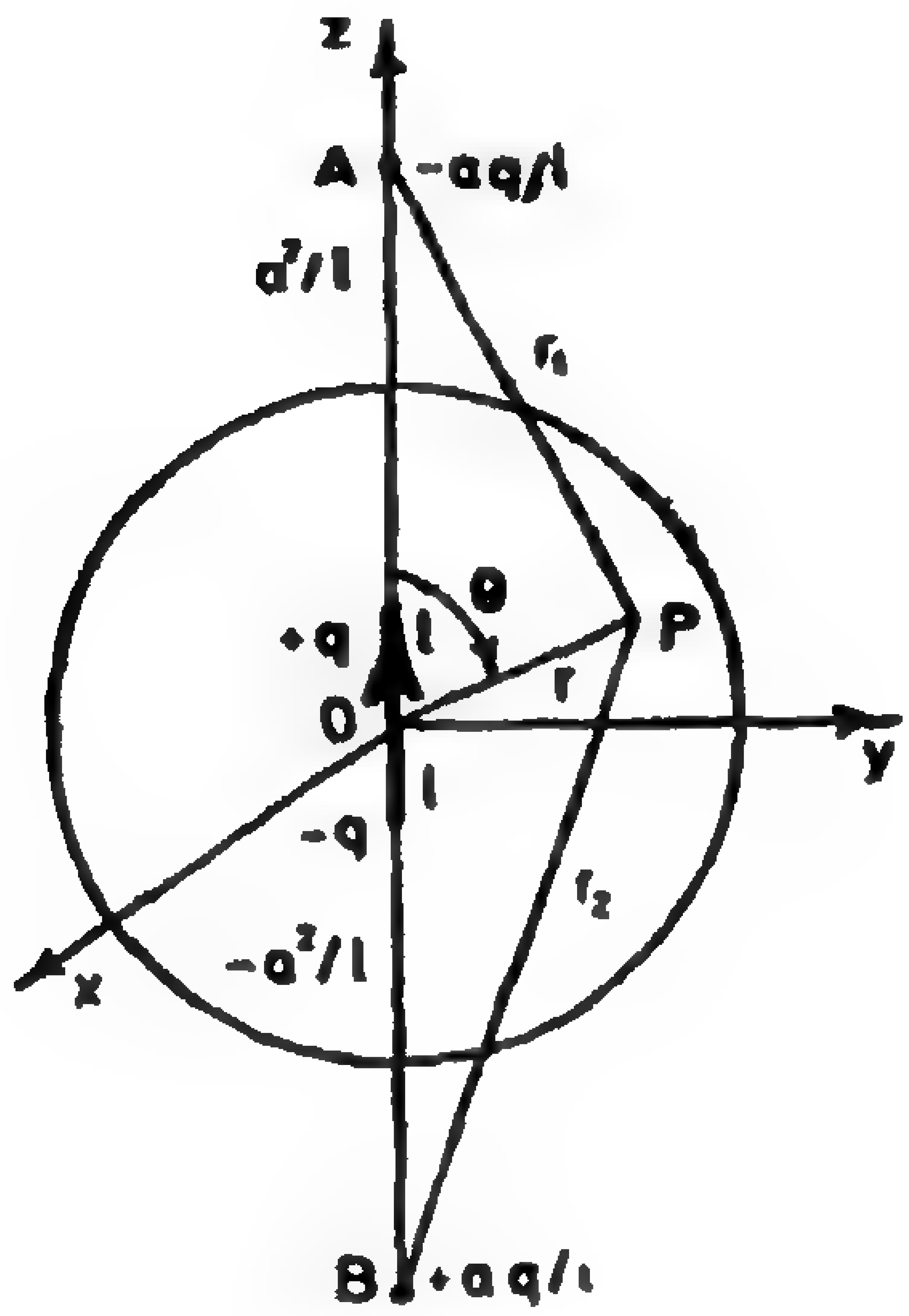


Fig. 9.16.

As shown in Fig. 9.16 the dipole is located along the z-axis; it consists of two charges $+q$, and $-q$ at distance $2l$ from each other where l is very small. The charge $+q$ will have an image $-aq/l$ at distance a^2/l from the center while the charge $-q$ has an image $+aq/l$ at a distance $-a^2/l$ along the z-axis. When the sphere is earthed the potential at an internal point $P(r, \theta, \phi)$ is due to a dipole of moment p along the z-axis and the two image charges,

$$V_p = p \cos\theta / 4\pi \epsilon_0 r^2 + aq / 4\pi \epsilon_0 l r_2 - aq / 4\pi \epsilon_0 l r_1 \quad (1)$$

where

$$r_1 = [(a^2/l)^2 + r^2 - 2\,a^2\,r\cos\theta/l]^{1/2} \tag{2}$$

$$r_2 = [(a^2/l)^2 + r^2 + 2a^2 r\cos\theta/l]^{1/2} \tag{3}$$

since l is very small r^2 can be neglected with respect to a^2/l so that

$$r_1 \simeq [(a^2/l)^2 - 2a^2\,r\cos\theta/l]^{1/2} \tag{4}$$

$$r_2 \simeq [(a^2/l)^2 + 2a^2\,r\cos\theta/l]^{1/2} \tag{5}$$

and

$$1/r_1 = (l/a^2)\,[1 - (2r\,l/a^2)\cos\theta]^{-1/2}$$

$$\simeq l/a^2 + (r\,l^2/a^4)\cos\theta$$

Similarly

$$1/r_2 \simeq l/a^2 - (r\,l^2/a^4)\cos\theta$$

Hence (1) becomes,

$$V_P = p\cos\theta/4\pi\,\epsilon_0\,r^2 + (a\,q/4\pi\,\epsilon_0\,l)\,(-2r\,l^2\cos\theta/a^4)$$

$$= (p\cos\theta/4\pi\,\epsilon_0)\,[1/r^2 - r/a^3] \tag{6}$$

It is clear that at $r = a$, $V_P = 0$.

The electric field at P can be obtained from

$$\mathbf{E} = -(\partial V/\partial r)\,\mathbf{a}_r - (\partial V/r\,\partial\theta)\,\mathbf{a}_\theta$$

$$= (p\cos\theta/4\pi\epsilon_0)\,(2/r^3 + 1/a^3)\,\mathbf{a}_r$$

$$+ (p\sin\theta/4\pi\,\epsilon_0)\,(1/r^3 - 1/a^3)\,\mathbf{a}_\theta \tag{7}$$

At $r = a$ the electric field has only a radial component

$$E_r = 3\,p\cos\theta/4\pi\,\epsilon_0\,a^3 \tag{8}$$

so that the surface charge density on the sphere

$$\sigma_s = -\,\epsilon_0\,E_r = -\,3\,p\cos\theta/4\,\pi\,\epsilon_0\,a^3 \tag{9}$$

The case when the shell is insultated will be the same as the grounded shell since in both cases the total charge on the internal surface (and hence the external) is zero. The lines of force are sketched in Fig. 9.17.

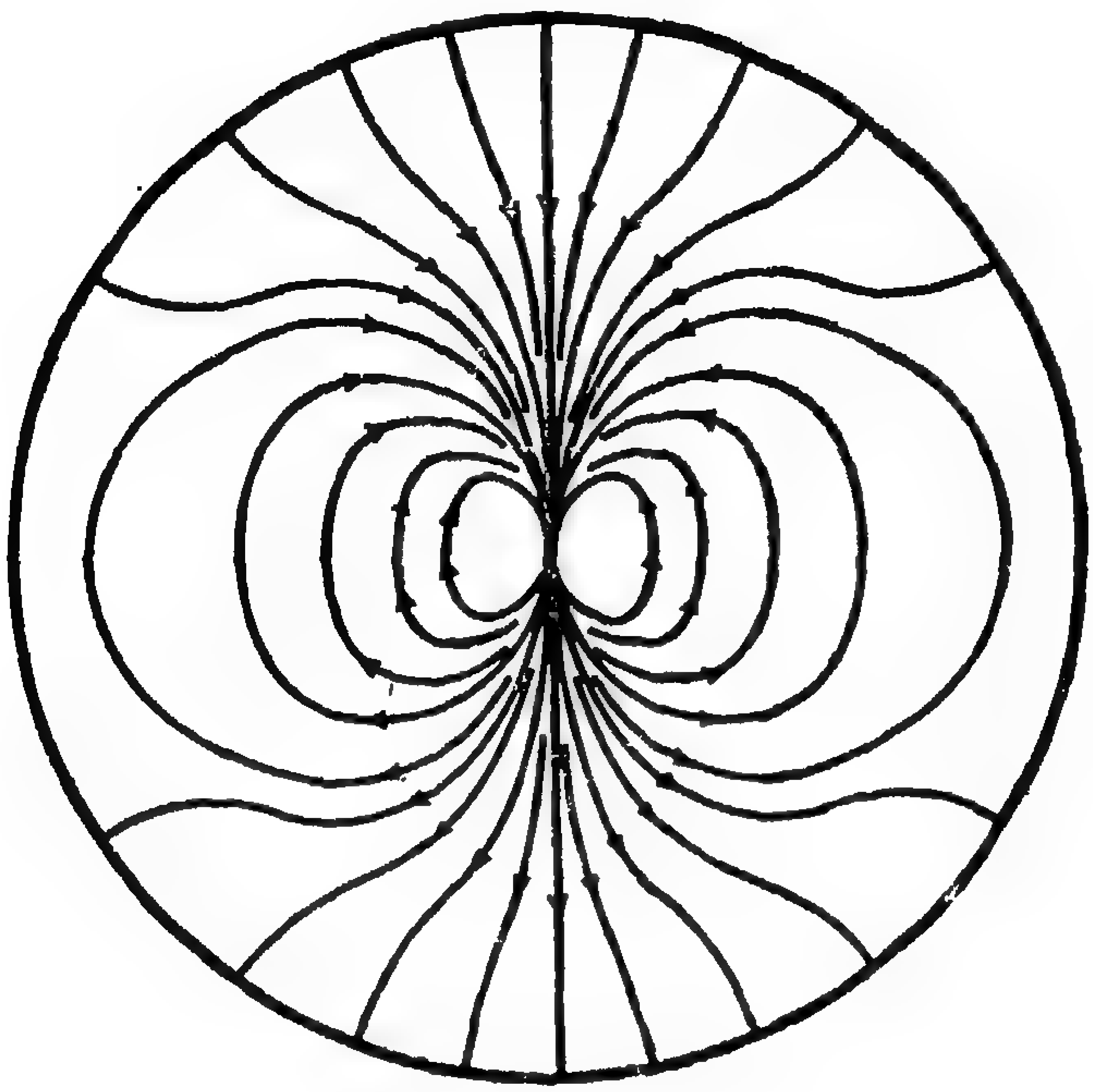


Fig. 9.17.

The electric field inside the shell due to the induced charges is the same as that due to the two image charges $-a\,q/l$ at $z = a^2/l$ and $a\,q/l$ at $z = -a^2/l$ in the limiting case when l tends to zero and q tends to infinity such that $p = 2\,l\,q$ is constant.

At any point P inside the shell the electric field due to the image charges only is,

$$\mathbf{E} = -(a\,q/4\pi\,\epsilon_0\,r_1^2\,l)\,\mathbf{a}_{r1} + (aq/4\pi\,\epsilon_0\,r_2^2\,l)\,\mathbf{a}_{r2} \tag{10}$$

where $\mathbf{a}_{r_1}$ and $\mathbf{a}_{r_2}$ are unit vectors in the directions from A to P and B to P respectively.

As l tends to a very small value

$$\mathbf{a}_{r_1} = -\mathbf{a}_{r_2} = -\mathbf{a}_s$$

and $\quad r_1 = r_2 = a^2/l$

Hence (10) becomes

$$\mathbf{E} = [aq/4\pi\epsilon_0\,(a^4/l) + aq\,/\,4\pi\epsilon_0\,(a^4/l)]\,\mathbf{a}_s$$

$$= (2\,lq/4\pi\epsilon_0\,a^3)\,\mathbf{a}_s = (p/4\,\pi\,\epsilon_0\,a^3)\,\mathbf{a}_s$$

This shows that the field inside the shell due to the induced charges is uniform and parallel to the dipole axis.

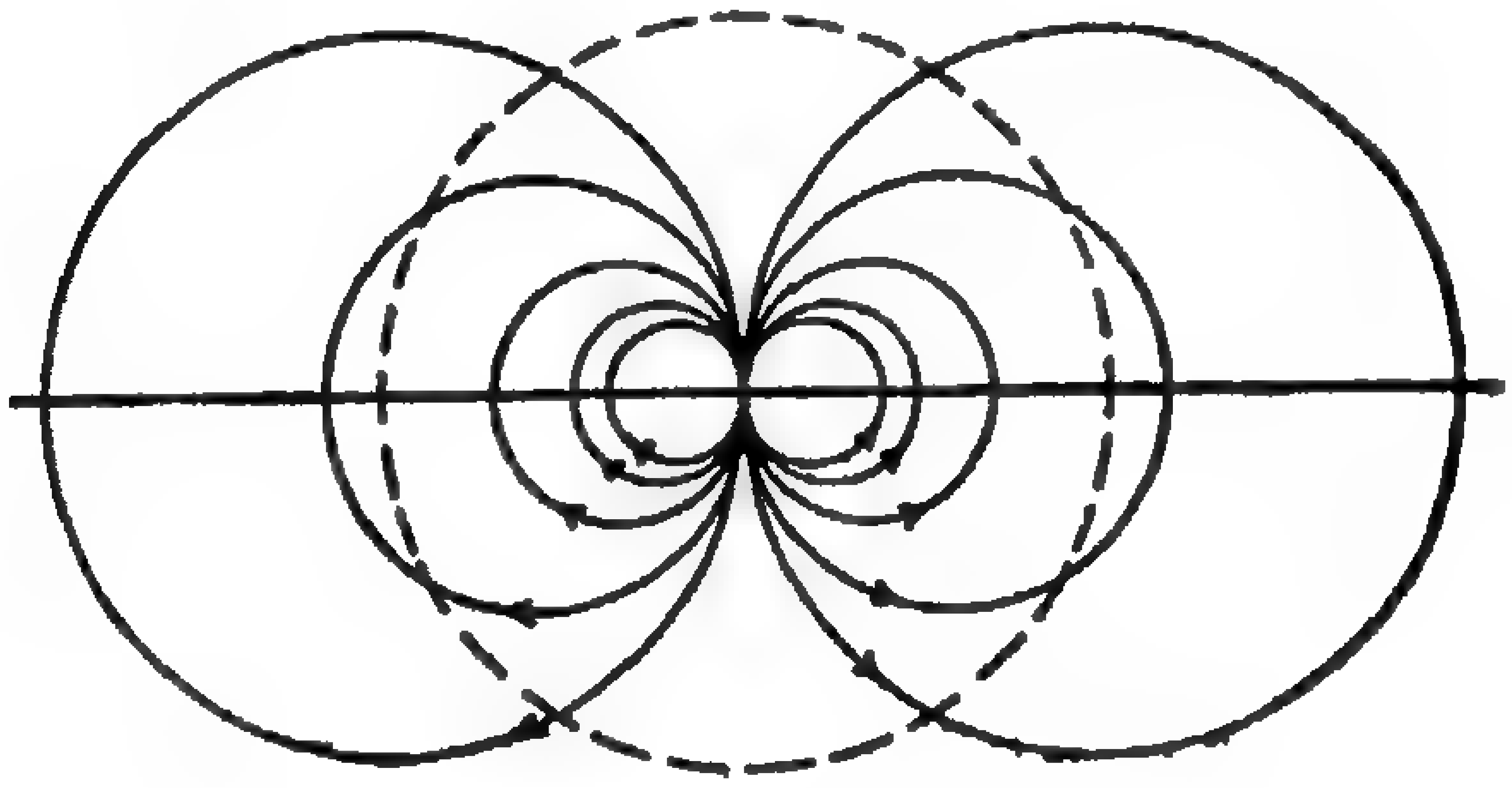


Fig. 9.18.

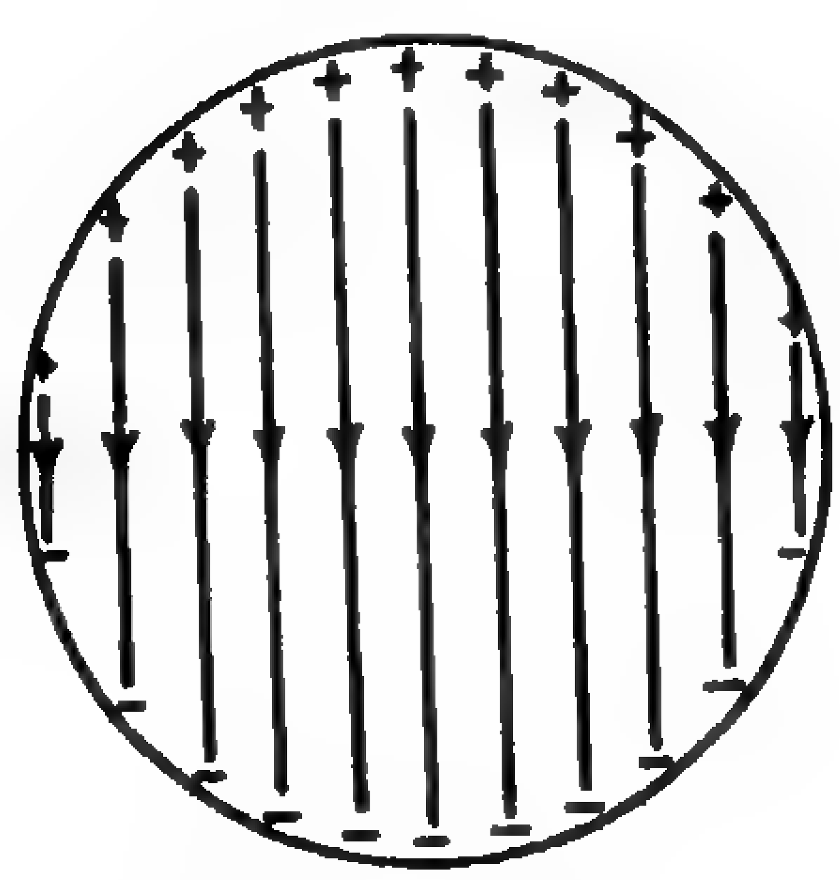


Fig. 9.19.

Note that the lines of force drawn in Fig. 9.17 may be interpreted as the sum of the lines of force due to the dipole when isolated in air (Fig. 9.18) plus the parallel lines due to the induced surface charges (Fig. 9.19).

13. *A point charge* Q *is located at a distance b from the center of a conducting sphere of radius a,* $b>a$. *Calculate the potential at any point outside the sphere, and the surface charge density at any point on the sphere. Consider the cases,*

(*i*) *The sphere is earthed.*

(*ii*) *The sphere is insulated and uncharged.*

(*iii*) *The sphere is insulated and carries a charge q.*

(*iv*) *The sphere is maintained at constant potential* V_0.

In each of the four cases find the force on the charge Q.

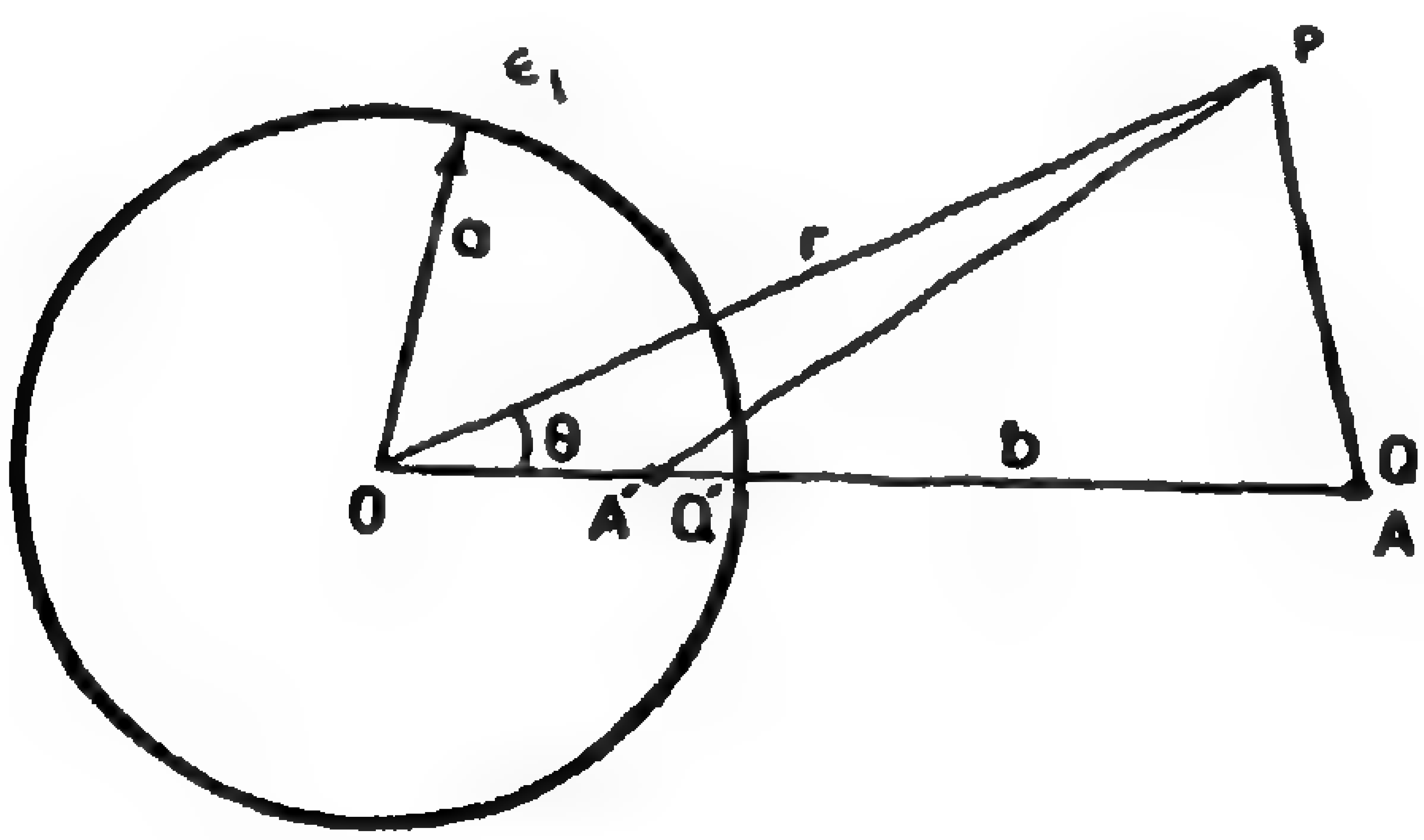


Fig. 9.20.

(i) The sphere is earthed.

The charge Q at A will have an image Q′ at point A' as shown in Fig. 9.20. The potential at point P outside the sphere is

$$V(r,\theta) = (1/4\pi\epsilon_o)(Q/AP - Q'/A'P) \quad (1)$$

where

$$(AP)^2 = (r^2 + b^2 - 2\,b\,r\cos\theta)$$

$$(A'P)^2 = (r^2 + f^2 - 2\,r\,f\cos\theta)$$

and, $\quad OA' = f.$

At $r = a$, V is zero so that we have,

$$Q/(a^2+b^2-2\,a\,b\cos\theta)^{1/2} = -Q'/(a^2+f^2-2\,a\,f\cos\theta)^{1/2}$$

By a procedure similar to problem 9.11., we find that,

$$Q' = -Q\,a/b \quad , \quad f = a^2/b \quad (2)$$

Thus the potential at a general point P is,

$$V(r,\theta) = (Q/4\pi\epsilon_o)\{1/(r^2+b^2-2\,b\,r\cos\theta)^{1/2} - (a/b)/[r^2+(a^4/b^2) - 2\,(a^2r/b)\cos\theta]^{1/2}\} \quad (3)$$

Now the surface density of the induced charges on the sphere is given by

$$\sigma = \epsilon_o E_n \quad (4)$$

where E_n is the field intensity normal to the surface of the sphere, that is

$$\sigma = -\epsilon_o(\partial V/\partial r) \quad \text{at } r = a \quad (5)$$

Substituting for V from (3) we get,

$$\sigma = -Q(b^2-a^2)/4\pi a(a^2+b^2-2\,a\,b\cos\theta)^{3/2} \quad (6)$$

The force on the charge Q is equal to the force between Q and its image $-Qa/b$. By Coulomb's law,

$$F = ba\,Q^2/4\pi\epsilon_o(b^2-a^2)^2 \quad (7)$$

(ii) The sphere is uncharged and insulated.

In this case the total charge induced on the sphere must be zero; therefore an additional charge Q a/b must be inserted at the center of the sphere. Thus to the potential of (3) we add a potential due to a charge Q a/b uniformly distributed over the spherical surface. This gives,

$$V(r,\theta) = (Q/4\pi\epsilon_0)\{1/(r^2+b^2-2br\cos\theta)^{1/2} - (a/b)/[r^2+(a^4/b^2) - (2a^2r/b)\cos\theta]^{1/2} + a/br\} \quad (8)$$

In this case the surface charge density is

$$\sigma = -Q(b^2-a^2)/4\pi a(a^2+b^2-2ab\cos\theta)^{3/2} + Q/4\pi ab \quad (9)$$

The force on the charge Q is that due to the charge $-Qa/b$ at A' and Qa/b at center

$$F = (Q^2/4\pi\epsilon_0)\,[a/b^3 - ab/(b^2-a^2)^2] \quad (10)$$

The potential of the sphere can be obtained by putting $r = a$ in (8),

$$V(a,\theta) = Q/4\pi\epsilon_0 b \quad (11)$$

This is equivalent to the potential of the image charge Q a/b at $r=a$.

(iii) The sphere is insulated and carries a charge q.

This is similar to (ii) but we add the potential due to q which is uniformly distributed over the sphere. At an external point to the sphere the potential is that of (8) plus $q/4\pi\epsilon_0 r$, the surface charge density is that of (9) plus $q/4\pi a^2$, and the force is that of (10) plus $qQ/4\pi\epsilon_0 b^2$.

(iv) The sphere is kept at potential V_0.

This potential may be referred to a charge Q′ such that

$$V_0 = Q'/4\pi\epsilon_0 a$$

that is $Q' = 4\pi\epsilon_0 a V_0$ located at the center of the sphere. The potential at an external point is that of (3) plus $Q'/4\pi\epsilon_0 r = aV_0/r$, the charge density of the induced charges is that of (6) plus $Q'/4\pi a^2 = \epsilon_0 V_0/a$, and the force on Q is that of (7) plus $Q'Q/4\pi\epsilon_0 b^2 = aV_0Q/b^2$.

Figs. 9.21 & 22 represent the variation of the surface induced charge for the earthed sphere

$$\sigma_A = \sigma(\theta=0) = -Q(b+a)/4\pi a(b-a)^2$$
$$\sigma_{B,D} = \sigma(\theta=\pi/2, 3\pi/2) = -Q(b^2-a^2)/4\pi a(a^2+b^2)^{3/2}$$
$$\sigma_C = \sigma(\theta=\pi) = -Q(b-a)/4\pi a(b+a)^2$$

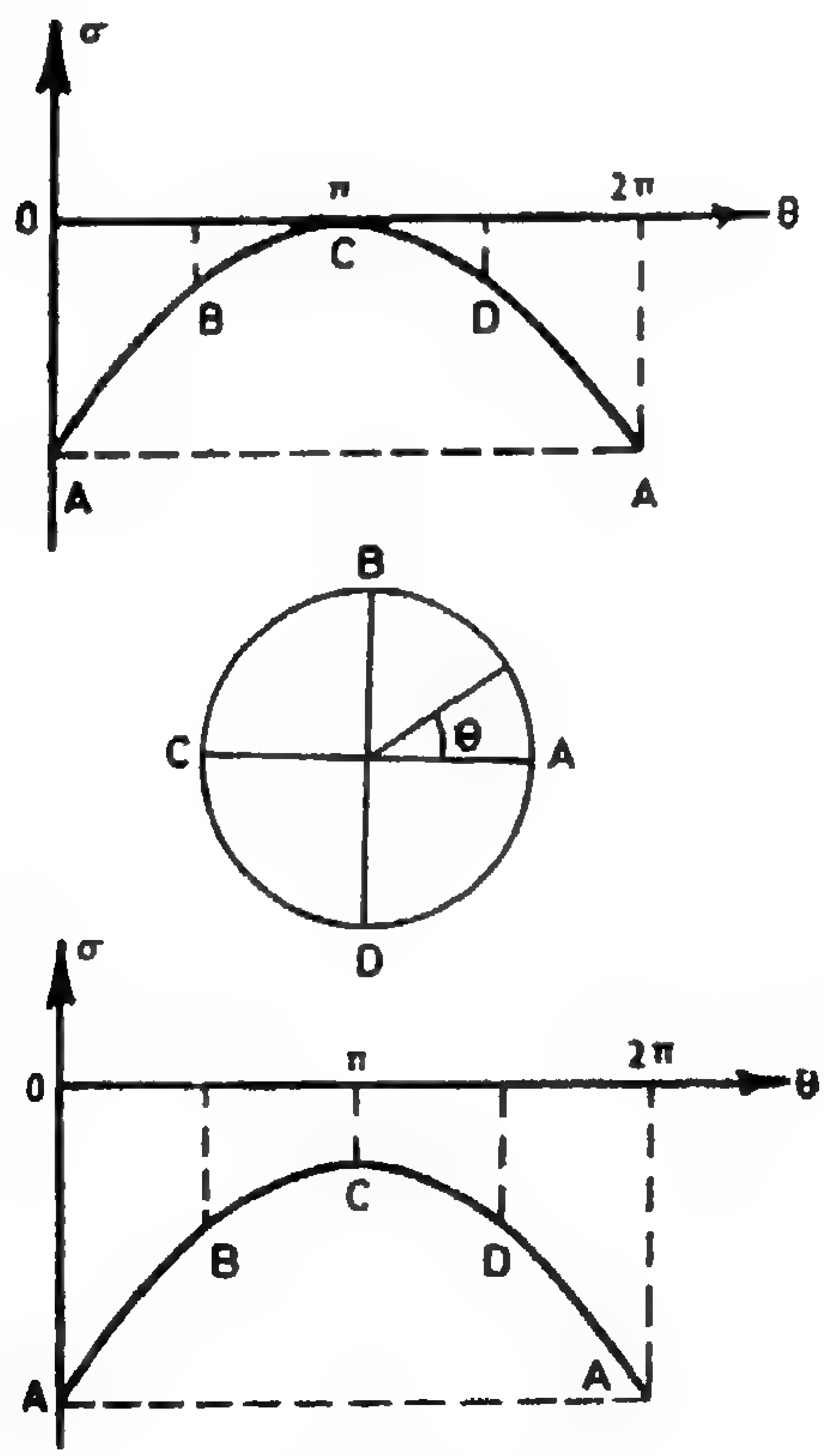


Fig. 9.21,22,23.

14. *A point charge Q is located at a distance b from the center of an insulated spherical conductor of radius a which carries a charge q. Prove that the surface density at the point of the sphere most remote from Q will be zero if*

$$q = -Q\, a^2\, (3b+a) \,/\, b\, (b+a)^2.$$

From (iii) of problem 13 the surface charge density at a general point on the sphere is ,

$$\sigma = -Q\,(b^2-a^2) \,/\, 4\pi a\,(a^2+b^2-2\,a\,b\cos\theta)^{3/2} + Q/4\pi ab + q/4\pi a^2$$

At the point most remote from Q, $\theta = \pi$ and $\sigma = 0$, that is,

$$0 = -Q\,(b^2-a^2) \,/\, 4\pi a(a+b)^3 + Q/4\pi ab + q/4\pi a^2$$

After simple manipulation we get

$$q = -Q\, a^2\, (3b+a) \,/\, b\,(b+a)^2$$

Fig. 9.23. shows the surface charge distribution on the surface of the sphere as a function of θ.

15. *A charge Q is placed at a distance b from the center of an insulated uncharged conducting sphere of radius a, b>a. Show that the part of the sphere which is positively charged is separated from that which is negatively charged by a circle at a distance from Q given by* $r = [b\,(b^2-a^2)]^{1/3}$.

From problem 13 the surface charge density at a general point on the conducting sphere is

$$\sigma = -Q\,(b^2-a^2) \,/\, 4\pi a\,(a^2+b^2-2\,a\,b\cos\theta)^{3/2} + Q/4\pi ab \qquad (1)$$

The curve of no electrification on the surface of the sphere is that for which $\sigma = 0$. Thus (1) gives

$$(a^2+b^2-2\,a\,b\cos\theta)^{3/2} = b\,(b^2-a^2) \qquad (2)$$

that is $\theta = \arccos\{[a^2+b^2-(b^3-ba^2)^{2/3}] \,/\, 2\,ab\}$

Hence the curve of no electrification is a circle normal to the line joining the point charge Q to the center of the sphere. If r is the distance between this circle and the charge Q, (2) gives

$$r^3 = b\,(b^2-a^2)$$

Thus we get,

$$r = [b\,(b^2-a^2)]^{1/3}.$$

16. *The center of an insulated uncharged conducting sphere of radius a is at the middle point of the straight line joining two equal point charges which are at distance $2b$ apart. If a/b is small show that the force on either charge would be increased in the ratio $1+2\cdot4\,(a/b)^3 : 1$, if the sphere is removed.*

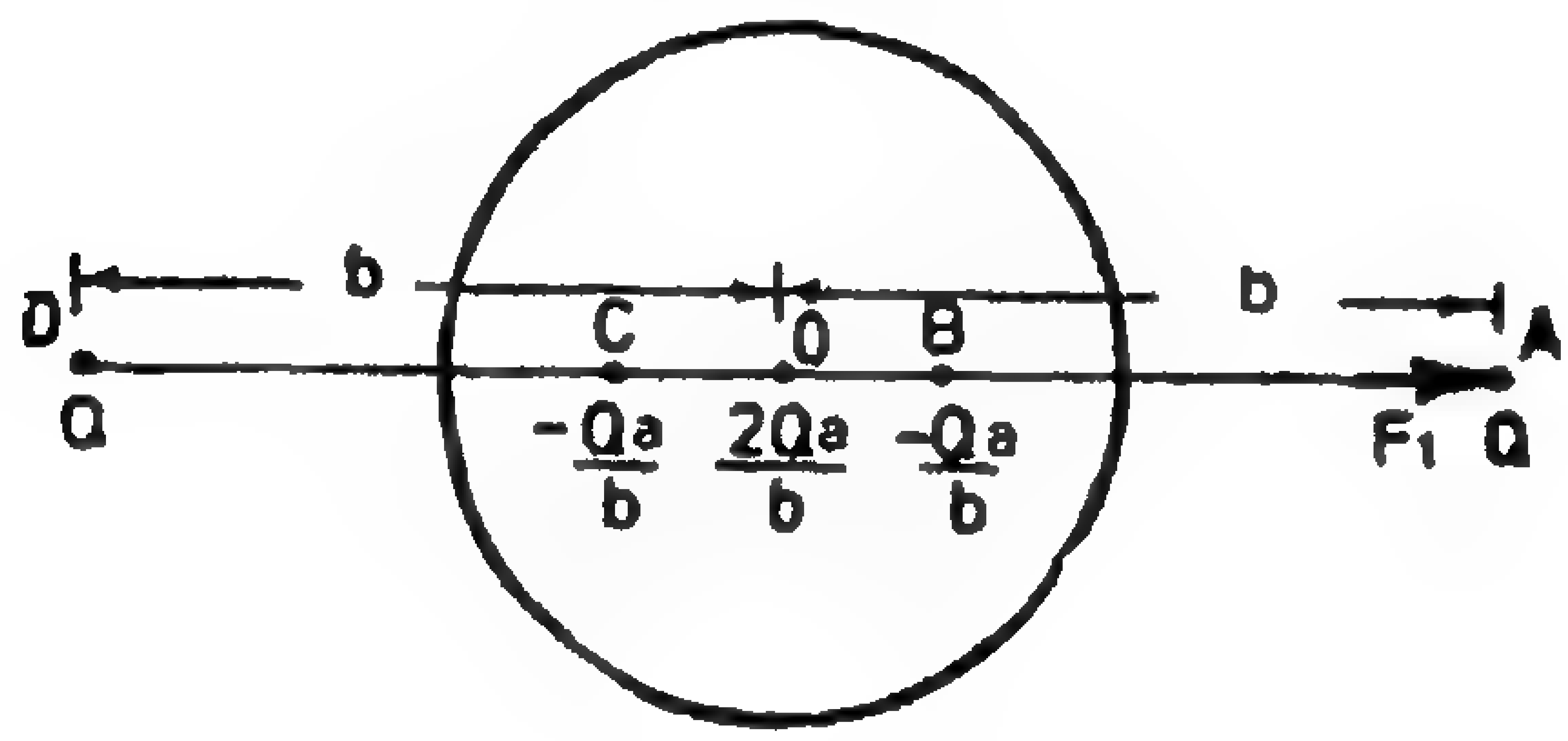


Fig. 9.24.

Since the sphere is insulated and uncharged, we have in addition to the image charges $-$ Q a/b, at B and C, two other charges Qa/b at the origin (see problem 13). The image system is shown in Fig. 9.24. The force on the charge Q at A will be that due to the four charges Q, $-$Q a/b, 2Qa/b, and $-$Qa/b at points D, C, O and B respectively,

$$F = Q^2/16\pi\epsilon_0 b^2 + 2Q^2a/4\pi\epsilon_0 b^3 - Q^2a/4\pi\epsilon_0 b\,(b-a^2/b)^2$$
$$- Q^2a/4\pi\epsilon_0 (b+a^2/b)^2$$
$$= (Q^2/4\pi\epsilon_0)\,[1/4b^2+2a/b^3 - (a/b^3)\,(1-a^2/b^2)^{-2}$$
$$- (a/b^3)\,(1+a^2/b^2)^{-2}]$$

Expanding the bracketed terms and considering only the second order we get,

$$F \simeq (Q^2/4\pi\epsilon_0)\,(1/4\,b^2 - 6\,a^5/b^7)$$

If the sphere is removed, the force on Q is

$$F' = Q^2 / 16\pi\epsilon_0\,b^2$$

The ratio of F' to F is

$$F'/F = 1 / (1 - 24\,a^5/b^5)$$

$$\simeq 1 + 24\,(a/b)^5$$

17. *Show that the capacitance of two conducting spheres of equal radii a in contact is equal to that of a single sphere of radius (2 log 2)a*

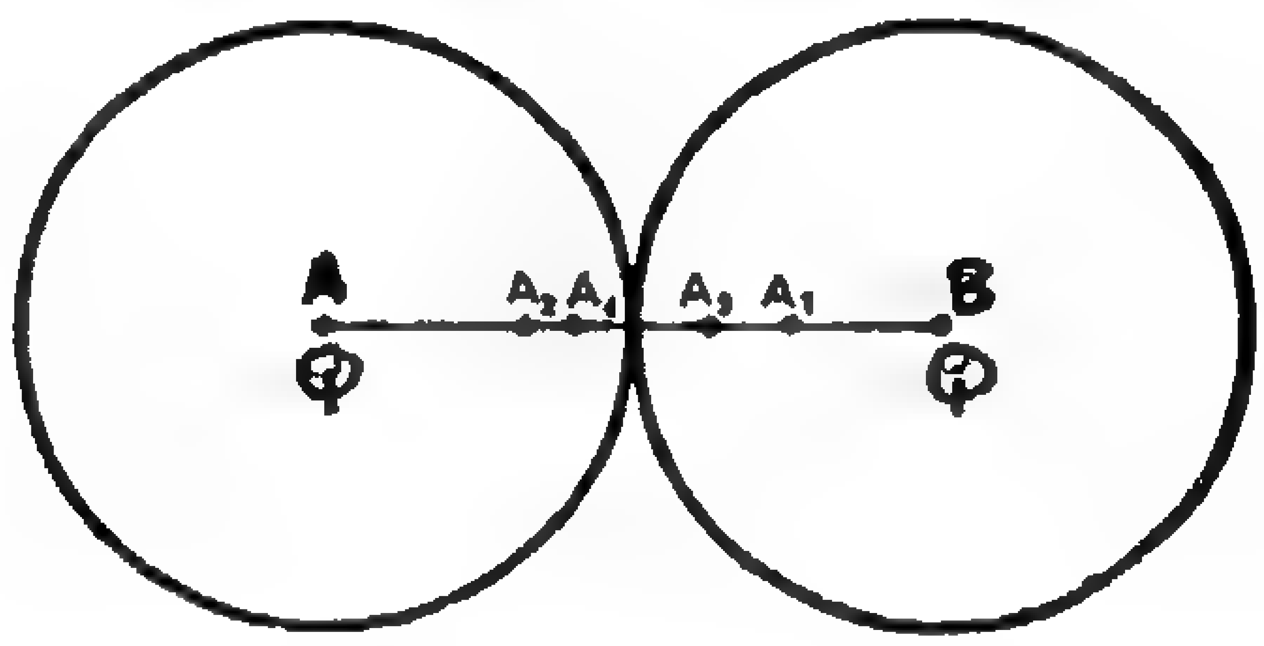


Fig. 9.25.

Let the conductor formed by the two spheres (Fig. 9.25), be raised to a potential V_0. If sphere 2 is absent, the potential V_0 is equivalent to a charge $Q = 4\pi\epsilon_0 a V_0$ at the center A. This is exactly the same situation of sphere 2 in the absence of 1. In the presence of the two spheres in contact, the equivalent charges Q at A and B do not satisfy the constant potential condition, and we need search for a system of image charges. The charge Q at A has an image $Q_1 = -Qa/2a = -Q/2$ at point A_1 where $AA_1 = 3a/2$. The charge Q_1 at A_1 has an image $Q_2 = -Q_1 a/(3a/2) = Q/3$ at A_2 where $AA_2 = a^2/(3a/2) = 2a/3$ ... etc.

Thus the charges within sphere 1 due to Q at A only are,

$$Q, Q_2 = Q/3, Q_4 = Q/5, Q_6 = Q/7, \ldots, Q_{2i} = Q/(2i+1),$$

with, $\quad AA_2 = 2a/3, AA_4 = 4a/5, AA_6 = 6a/7, \ldots, AA_{2i} = 2ia/(2i+1)$

The charges within sphere 2 due to Q at A only are;

$$Q_1 = -Q/2, Q_3 = -Q/4, Q_5 = -Q/6, \ldots, Q_{2j+1} = -Q/(2j+2)$$

with, $\quad AA_1 = 3a/2, \; AA_3 = 5a/4, \; AA_5 = 7a/6, \ldots,$
$\quad AA_{2j+1} = (2j+3)\, a/(2j+2)$

A similar set of charges exists due to the charge Q at B. Thus the total charge within the conductor formed by the two spheres in contact is

$$q_t = 2Q\,[1 - 1/2 + 1/3 - 1/4 + 1/5 - 1/6 + \ldots] = 2Q \log 2$$

By definition the capacitance of the conductor is,

$$C = q_t/V_0 = 2\,Q \log 2 / (Q/4\pi\epsilon_0 a) = 8\pi\epsilon_0 a \log 2$$

Since the capacitance of a single sphere of radius a is $4\pi\epsilon_0 a$, the required result follows immediately.

6. *A line charge of strength λ C/m is placed such that its mid point is at distance b from the center of an earthed conducting sphere of radius a. Show that the total charge induced on the sphere is,*

$$Q = -2a\,\lambda \log \{ [l + (l^2 + b^2)^{1/2}] / b \}$$

where l is the length of the line charge.

Consider an elementary charge $\lambda\, dx$ as shown in Fig. 9.26. This elementary charge has an image— $(a/OA)\,\lambda\, dx$. The total charge on the sphere is obtained by integration,

$$Q = \int_{-l}^{l} -(a\lambda/OA)\, dx$$

$$- a \lambda \int_{-l}^{l} dx \;/\; (a^2 + x^2)^{1/2}$$

This integration can be performed by assuming $x = a \sinh \theta$,

$$Q = -2 a \lambda \operatorname{arcsinh} (l/b)$$
$$= -2 a \lambda \log \{ [l + (l^2+b^2)^{1/2}] \;/\; b \}$$

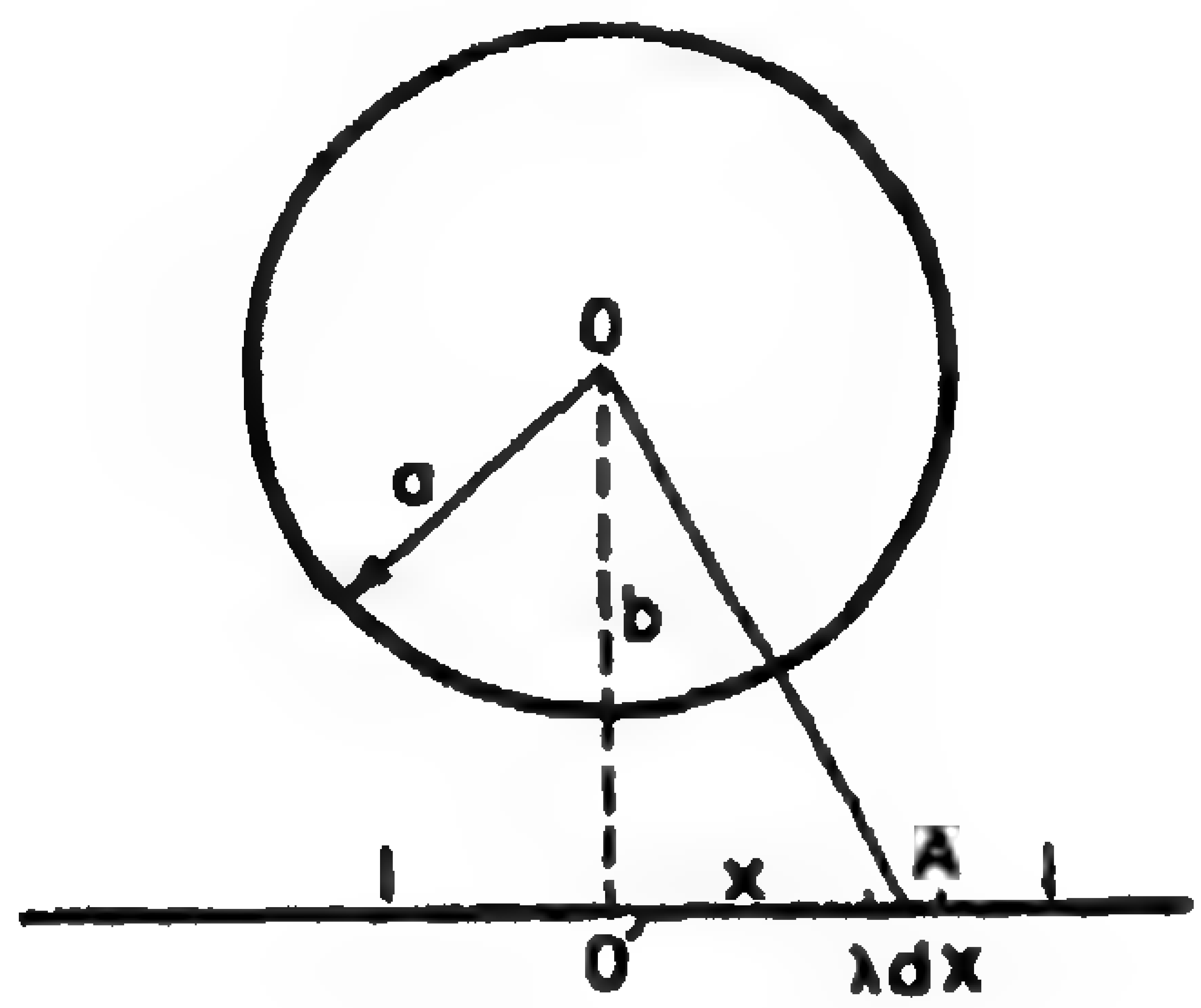


Fig. 9.26.

19. *The electrostatic field in space is expressed as that due to N point charges Q_i, $i = 1, 2, \ldots, N$ Show that the charge induced on any part of a conducting surface in the field can be expressed in terms of the solid angles which this part subtends at the N point charges.*

As shown in Fig. 9.27 the field at point P due to the charge Q_i is,

$$\mathbf{E} = (Q_i / 4\pi \epsilon_o r^2)\, \mathbf{a}_r$$

The normal component to the elementary area dA is,

$$\mathbf{E}_n = (\mathbf{E}.\mathbf{n})\, \mathbf{n} = (Q_i / 4\pi \epsilon_o r^2)\, (\mathbf{a}_r.\mathbf{n})\, \mathbf{n}$$
$$= (Q_i \cos \theta / 4\pi \epsilon_o r^2)\, \mathbf{n}$$

where **n** is the unit normal to dA at point P. The charge on dA due to Q_i is,

$$\Delta Q_i = \sigma_i \, dA = \epsilon_o E_n \, dA$$

$$= (Q_i / 4\pi)(\cos\theta \, dA/r^2)$$

Now $\cos\theta \, dA/r^2$ is the solid angle $d\Omega_i$ subtended at Q_i by the area dA. Thus,

$$\Delta Q_i = (Q_i / 4\pi) \, d\,\Omega_i$$

If dA subtends elementary solid angles $d\,\Omega_1, d\,\Omega_2, \ldots, d\,\Omega_N$ at the point charges $Q_1, Q_2, \ldots, Q_N$, the total surface density on $d\,A$ is,

$$\Delta Q = \Sigma \Delta Q_i = (1/4\pi)(Q_1 d\Omega_1 + Q_2 d\Omega_2 + \ldots + Q_N d\Omega_N)$$

By integration we obtain the total induced charge on the surface A,

$$Q = (1/4\,\pi) \sum_{1}^{N} Q_i \, \Omega_i$$

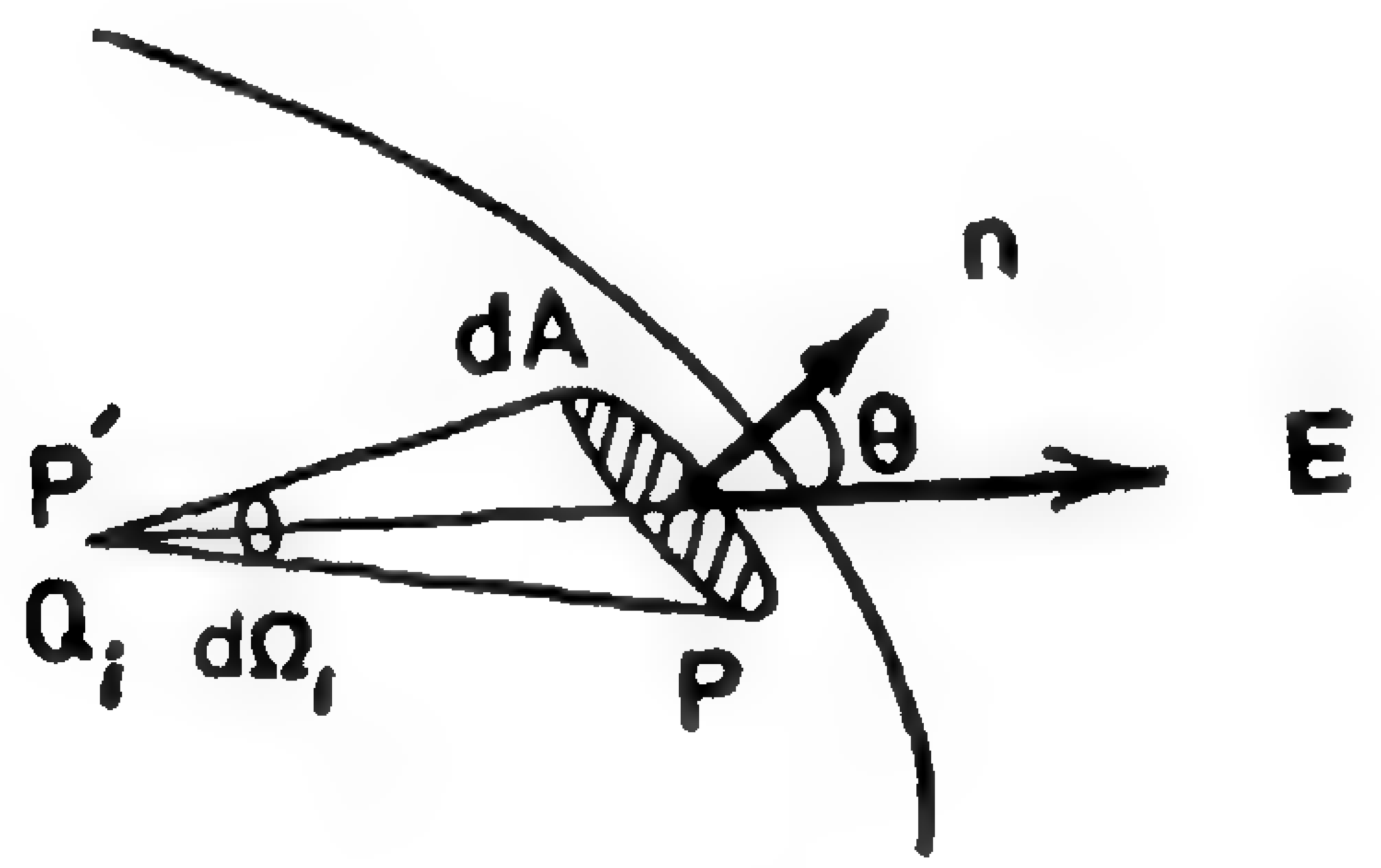


Fig. 9.27.

20. *A point charge Q is placed at a distance b from the center of an insulated uncharged sphere of radius a. Show that the total charge on the part of sphere in front of Q and defined by the polar plane at the point occupied by Q is* $-\frac{1}{2} Q \left[1 + (a^2/b^2) - (1 - a^2/b^2)^{1/2}\right]$.

As explained in problem 9.13 the charge Q at A (Fig. 9.28) has an image charge $-aQ/b$ at the inverse point A' in addition to a charge aQ/b at the center O. From the properties of the inverse relation between A and A', the point A' lies on the polar plane of A. The solid angle at A, A', and O subtended by the spherical cap BCD are,

$$2\pi\,(1-\cos\alpha) = 2\pi\,[1-(b^2-a^2)^{1/2}/b]$$

2π, and

$$2\pi\,(1-\cos\theta) = 2\pi\,(1-a/b)$$

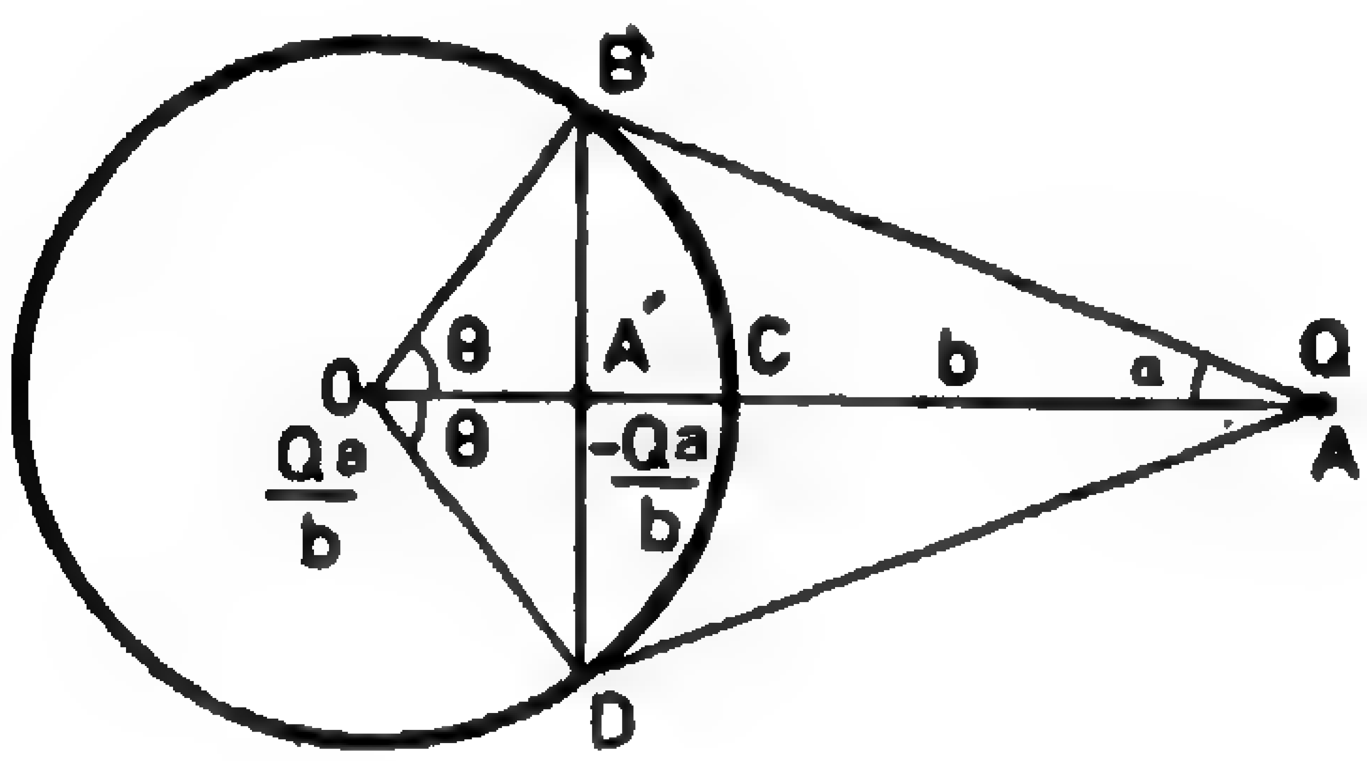


Fig 9.28.

Using the result of the previous problem, the charge induced on the cap is,

$$(1/4\pi)\,\{-2\pi\,[1-(b^2-a^2)^{1/2}/b]\,Q - 2\pi\,aQ/b + 2\pi\,aQ\,(1-a/b)\,/\,b\,\}$$

$$= -\tfrac{1}{2}\,Q\,[1+a^2/b^2-(1-a^2/b^2)^{1/2}]$$

The first solid angle is taken negative since Ω_1, the solid angle subtended by A is negative.

21. *If in the previous problem the sphere is earthed, show that the ratio of charges on the smaller and larger parts of the sphere cut off by the polar plane is* $(b+a)^{1/2}/(b-a)^{1/2}$.

Since the sphere is earthed, the image of Q is $-aQ/b$ at A' (Fig. 9.28). The charge on the part BCD is,

$$q_1 = [Q\Omega_1 + (-aQ/b)\ \Omega_2]/4\pi$$

Ω_1 and Ω_2 are as in the previous problem,

$$q_1 = -\tfrac{1}{2}\ Q\ [1+a/b-(1-a^2/b^2)^{1/2}]$$

Now since the sphere encloses a charge $-aQ/b$, the larger part of the sphere carries a charge

$$q_2 = -aQ/b - q_1$$

$$= -\tfrac{1}{2}Q\ [a/b-1+(1-a^2/b^2)^{1/2}]$$

The required ratio is,

$$r = q_1/q_2 = (b+a)^{1/2}/(b-a)^{1/2}$$

22. *A point charge Q is placed at distances x, and y from two earthed conducting planes at right angles. Find the total induced charges on each of the two planes.*

The charge Q at (x, y), (Fig. 9.29), has the image charges,

$-Q$ at $(-x, y)$

Q at $(-x, -y)$

$-Q$ at $(x, -y)$

The charges on each of the conducting planes can be obtained using the method of Problem 9.19. The solid angle at A due to the plane $y = 0$, $x>0$ is,

$$\Omega_1 = 2\ [\pi - \arctan\ (y/x)] \qquad (1)$$

The same solid angle is subtended at D. The solid angle subtended by the same plane at B and C is,

$$\Omega_2 = 2 \arctan (y/x) \tag{2}$$

The total charge on the plane $y = 0$, $x \geq 0$ is thus,

$$-2 \, (Q/2\pi) \, [\pi - \arctan (y/x)] + 2 \, (Q/2\pi) \arctan (y/x)$$

$$= -(2 \, Q/\pi) \, [\tfrac{1}{2}\pi - \arctan (y/x)]$$

$$= -(2 \, Q/\pi) \arctan (x/y) \tag{3}$$

Similarly the charge induced on the plane $x = 0, y \geq 0$ is $-(2Q/\pi) \arctan (y/x)$.

Note that, the sum of the induced charges on the two planes is $-Q$ since all the lines of force starting from the charge Q end at the two planes.

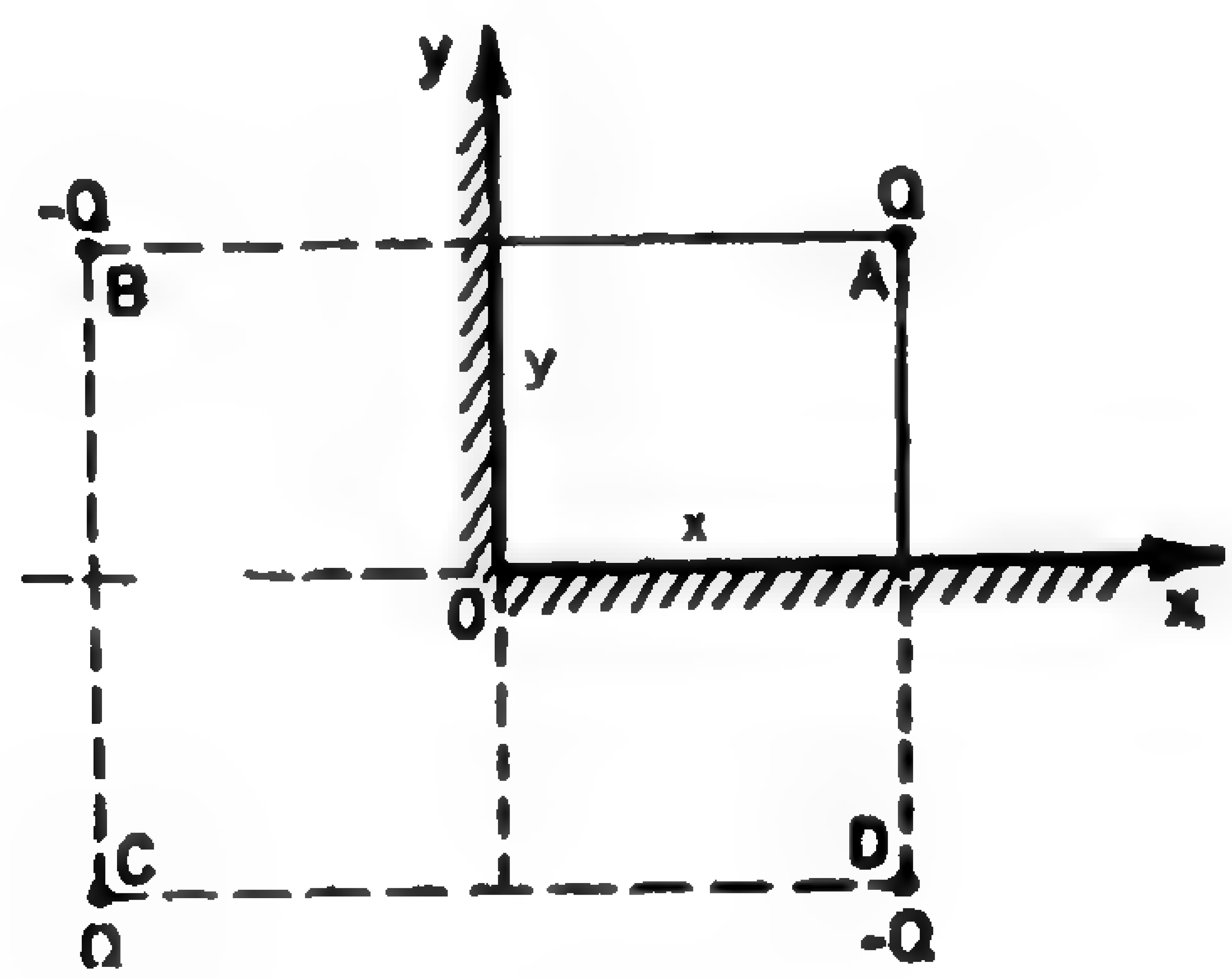


Fig. 9.29.

23. *Two grounded semi-infinite planes are at an angle 60°. If a charge Q is placed at a point between the two planes, find the charges induced on each of the two planes.*

Using the method of images, we have an image system, (Fig. 9.30),

$$Q \text{ at } (r,\theta) , (r,\theta + 2\pi/3) , \text{ and } (r, \theta + 4\pi/3)$$

$$-Q \text{ at } (r, -\theta) , [r, -(\theta + 2\pi/3)] \text{ and } [r, -(\theta + 4\pi/3)$$

with the first plane these charges have the solid angles,

$$\Omega_1 = 2 (\pi-\theta) , \qquad \Omega_2 = 2 (\pi/3+\theta)$$

$$\Omega_3 = 2 (\pi/3-\theta) \qquad \Omega_4 = \Omega_3 , \Omega_5 = \Omega_2$$

$$\Omega_6 = \Omega_1 .$$

Thus the total charge on the first plane is,

$$q_1 = (-Q/4\pi) (\Omega_1 + \Omega_3 + \Omega_4 + \Omega_6 - \Omega_2 - \Omega_5)$$

$$= Q (3\theta/\pi - 1)$$

Since the total induced charges on the two planes is $-Q$ we conclude that the charge induced on the second plane is,

$$q_2 = -Q - q_1 = -Q + Q - (3\theta/\pi) Q$$

$$= -(3\theta/\pi) Q$$

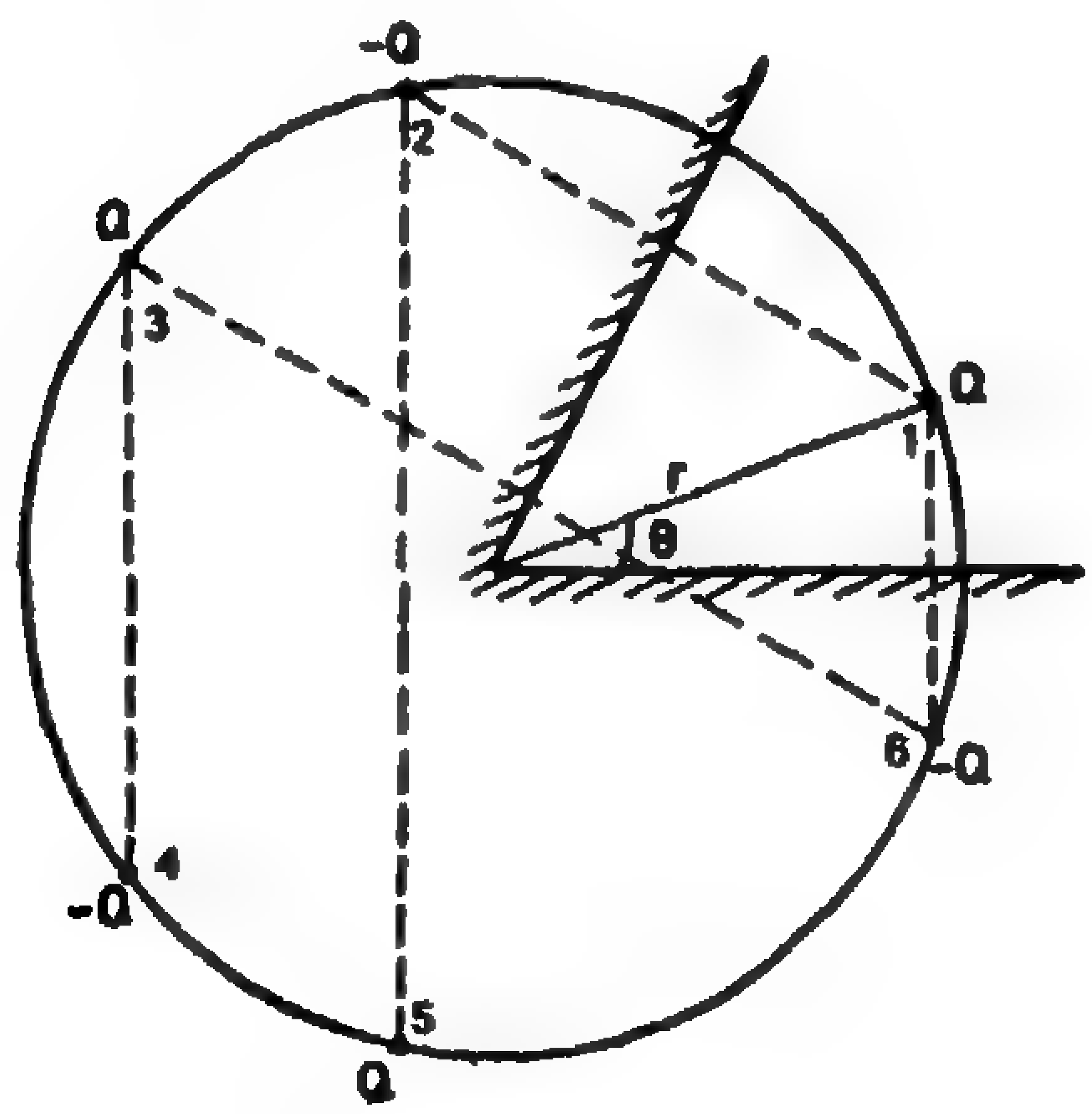


Fig. 9.30.

24. *A two-conductor cable consists of two parallel wires each of radius a at a distance 2d apart and enclosed symmetrically in an earthed metal sheath of radius R; the remaining space is filled with a dielectric of permittivity ϵ. If the core diameters are sufficiently small, show that the capacitance per unit length between the wires is given by $\pi\epsilon / \log\,[2d\,(R^2-d^2)\,/\,a\,(R^2+d^2)]$.*

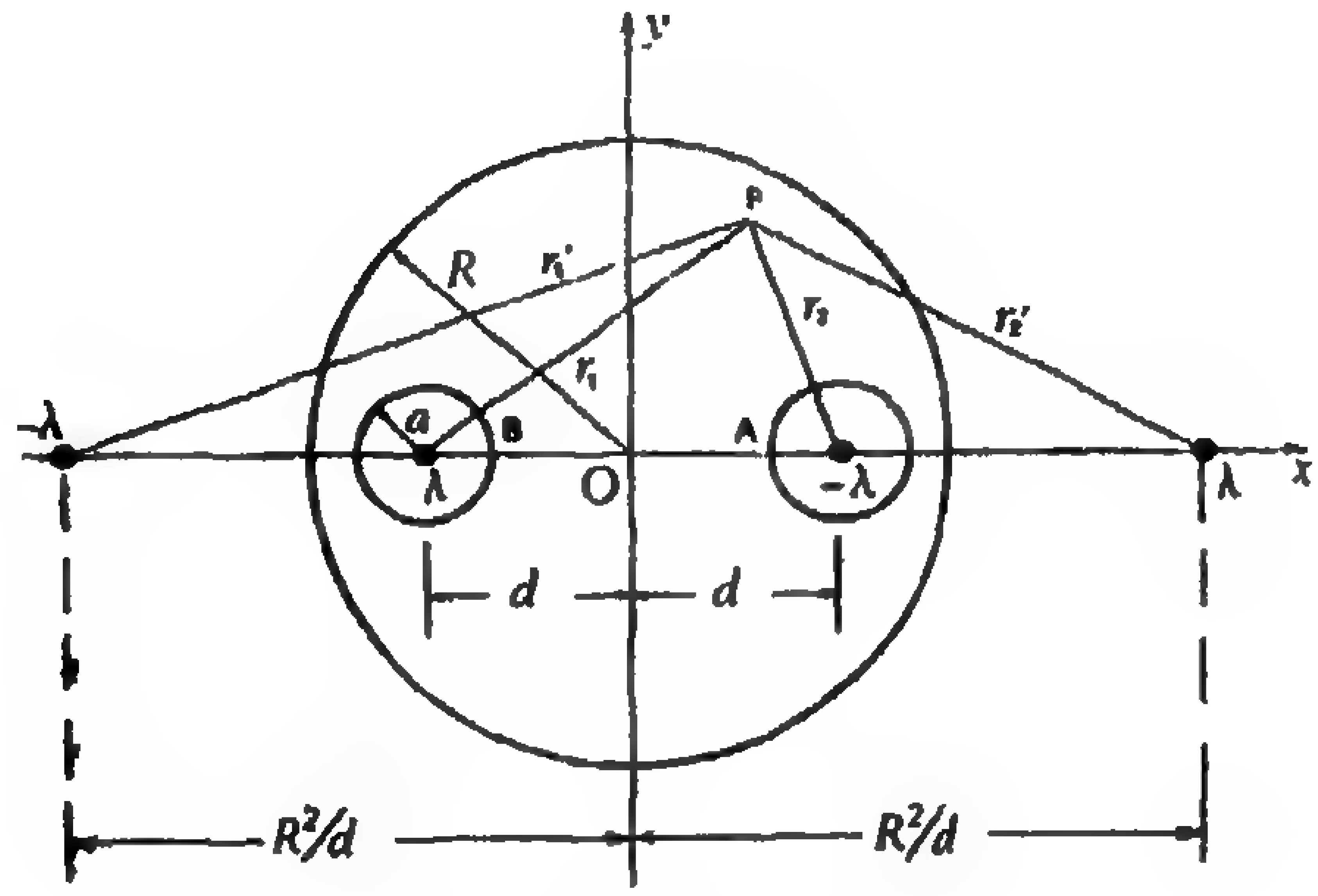


Fig. 9.31.

Since the radii of the inner conductors are small compared to the radius of the outer sheath, the inner conductors will be represented by two line charges $\pm\lambda$ which have images as shown in Fig. 9.31. The electric potential at point P inside the sheath is the same as that due to the four parallel line charges.

$$V_p = (\lambda/2\pi\epsilon_0)\log\,(r_1' r_2 / r_1 r_2')$$

At point A, $V_p = V_1$,

$$r_1 = 2d - a,\; r_2 = a,\; r_1' = R^2/d + d - a$$

$$r_2' = R^2/d - d + a$$

$$V_2 = (\lambda/2\pi\,\epsilon_0)\,\log\left[\frac{a\,(R^2/d + d - a)}{(2\,d - a)\,(R^2/d - d + a)}\right]$$

Similary at B, $V_p = V_2$, $r_1 = a$, $r_2 = 2d—a$, $r_1' = R^2/d—d—a$ and $r_2' = R^2/d+d—a$.

$$V_2 = (\lambda/2\pi\epsilon_0)\ log\left[\frac{(2d—a)\ (R^2/d—d+a)}{a\ (R^2/d+d—a)}\right]$$

$$C = \lambda/(V_2—V_1) = \pi\epsilon_0 / log\left[\frac{(2d—a)\ (R^2—d^2+ad)}{a\ (R^2+d^2—ad)}\right]$$

$$\simeq \pi\epsilon_0 / log\ [2d\ (R^2—d^2) / a\ (R^2+d^2)]$$

25. *An infinite grounded conducting plane has a hemi-spherical boss of radius a on its surface. A point charge* Q *is placed distant h from the center of the hemi-sphere along a line perpendicular to the plane. Find the total charge induced on the boss and the force on the charge* Q.

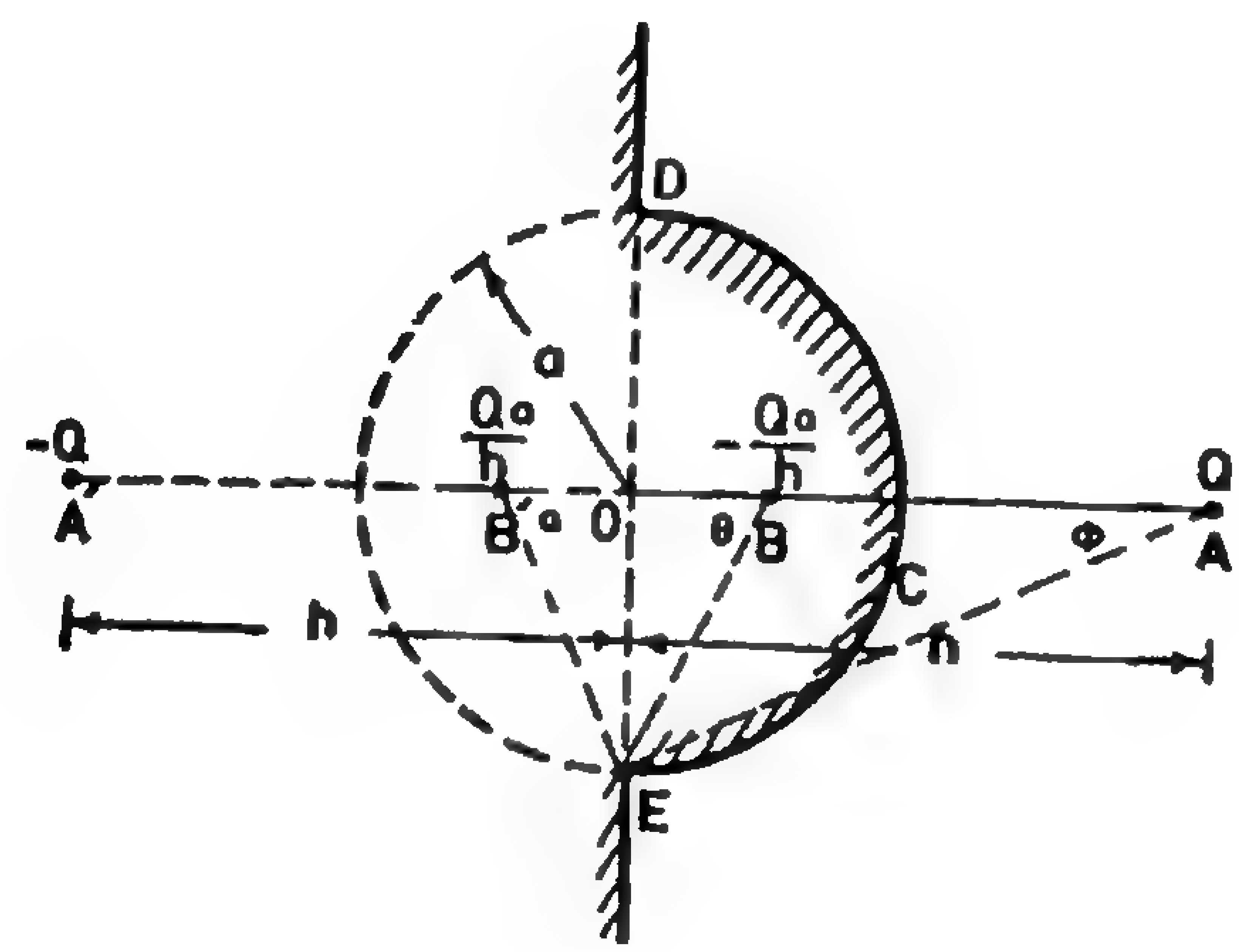


Fig. 9.32.

As indicated in Fig. 9.32 the image system consists of four point charges

(i) the charge Q at A.

(ii) $-a\,Q/h$ at B the image of A on the hemispherical boss.

(iii) $a\,Q/h$ at B' the image of B on the conducting plane.

(iv) $-Q$ at A' the image of A on the conducting plane.

The charge on the hemi-spherical boss is (see problem 9.18),

$$q = -(Q/4\pi)\Omega_1 - (Qa/4\pi h)\Omega_2 + (Qa/4\pi h)\Omega_3 - (Q/4\pi)\Omega_4 \quad (1)$$

where Ω_1, Ω_2, Ω_3, and Ω_4 are the solid angles subtended by the boss at A, B, B', and A' respectively.

$$\Omega_1 = 2\pi\,(1-\cos\phi) = 2\pi\,[1-h/(a^2+h^2)^{1/2}] \quad (2)$$

$$\Omega_2 = 2\pi\,(1-\cos\theta) = 2\pi\,[1+a/(a^2+h^2)^{1/2}] \quad (3)$$

$$\Omega_3 = 2\pi\,(1-\cos\alpha) = 2\pi\,[1-a/(a^2+h^2)^{1/2}] \quad (4)$$

$$\Omega_4 = \Omega_1$$

Substituting for Ω_1, Ω_2, Ω_3, and Ω_4 in (1) we get,

$$q = -Q\,[1-(h^2-a^2)\,/\,h\,(h^2+a^2)^{1/2}]$$

Since the total induced charges on both the plane and the boss must be $-Q$, and since all lines of force out of Q must terminate on the plane, the induced charge on the plane is,

$$-Q - q = -Q\,(h^2-a^2)\,/\,[h\,(h^2+a^2)^{1/2}]$$

The force on the charge Q is due to charges at A', B', and B

$$F = Q^2/16\pi\,\epsilon_0\,h^2 + Q^2a/4\pi\,\epsilon_0\,(h-a^2/h)^2 - Q^2a/4\pi\,\epsilon_0\,h\,(h+a^2/h)^2$$

$$= (Q^2/4\pi\,\epsilon_0\,h^2)\,[1/4 + 4a^3h\,/\,(h^4-a^4)]$$

26. *A dome of conducting material is built in the form of a hemisphere on the ground. A small charged conductor is placed midway between the dome and the ground on the vertical through the center of the dome. Prove that the mechanical force on the small conductor is* $47\,Q^2/900\,\pi\epsilon_0 a^2$ *upwards, where* Q *is the charge on the conductor and a the radius of the dome.*

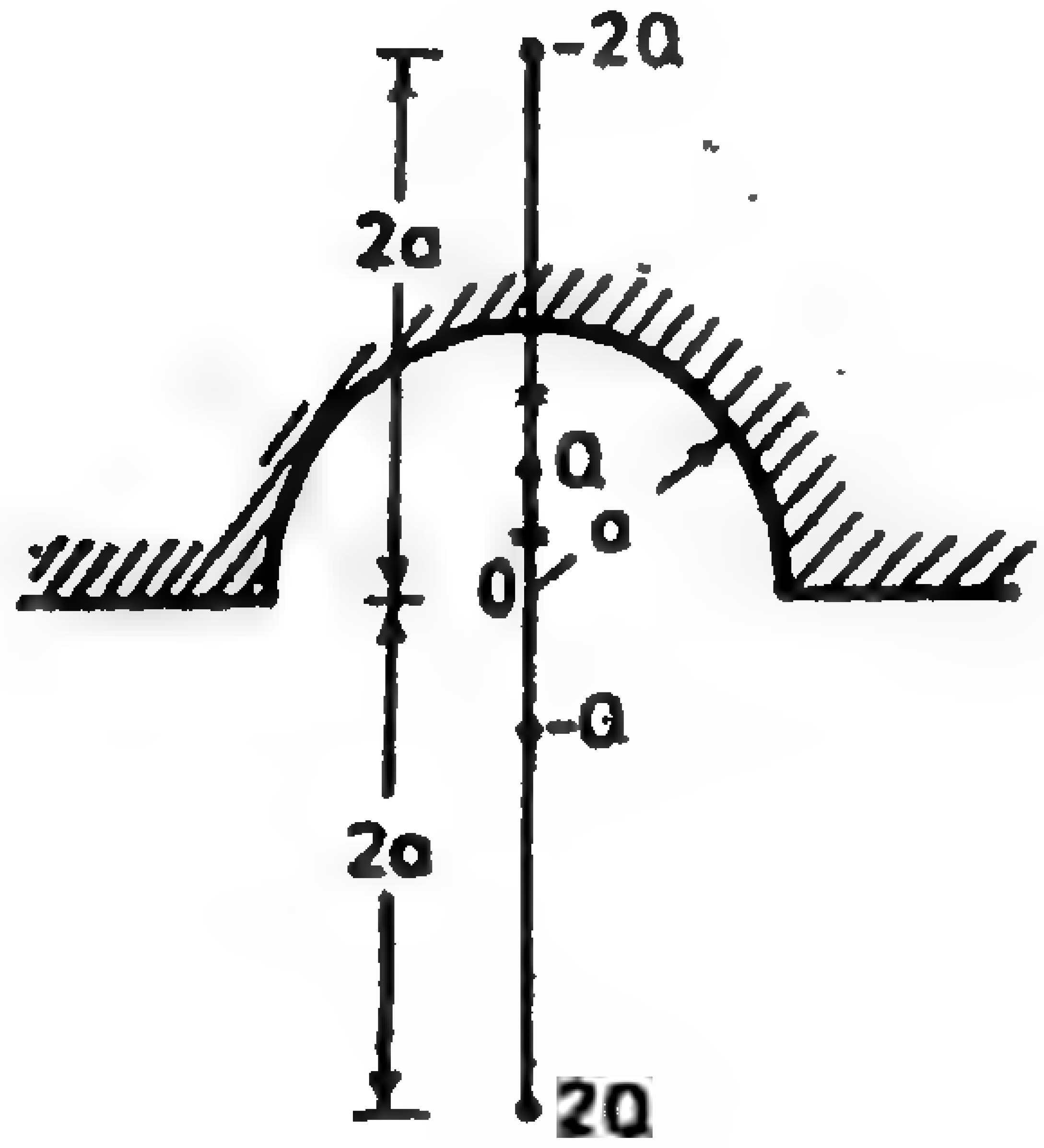


Fig. 9.33.

The image system is shown in Fig. 9.33. Q will have an image on the dome of $-2Q$ at distance $2a$, and both Q and $-2Q$ have images $-Q$ and $2Q$ on the plane as shown. The force on the charged conductor is,

$$F = 2Q^2/4\pi\epsilon_0(2a+\tfrac{1}{2}a)^2 + 2Q^2/4\pi\epsilon_0(2a-\tfrac{1}{2}a)^2 - Q^2/4\pi\epsilon_0 a^2$$

in the upward direction. After simple manipulation we get the required result.

27. *A point charge Q is placed at distance h from the center of a hemispherical boss of radius a on an infinite conducting plane of zero potential. Find expressions for the surface charge densities at points on the boss and the plane.*

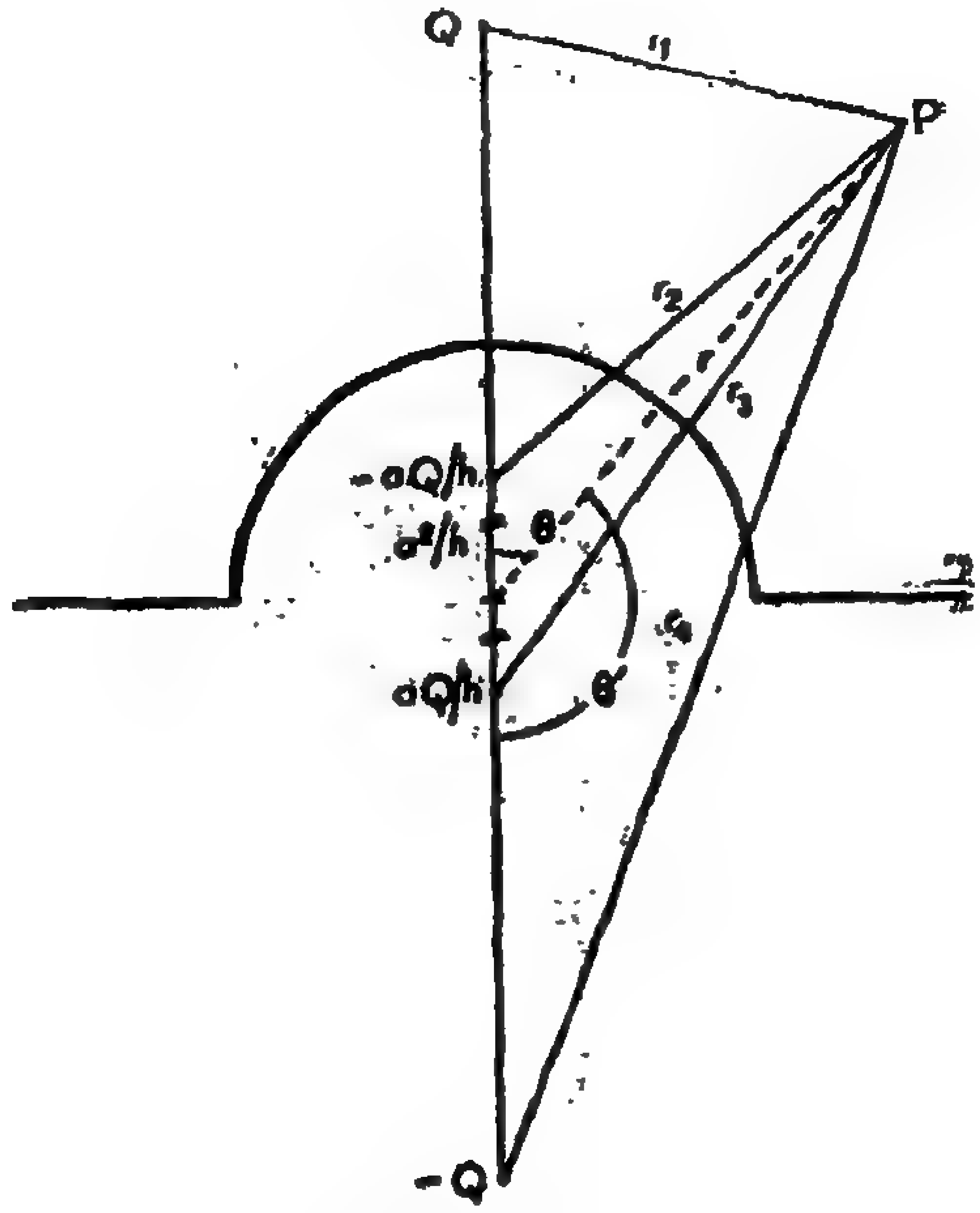


Fig. 9.34.

As explained in problem 9.25 the image system is shown in Fig. 9.34. The potential at any general point P is,

$$V_P = (1/4\pi\epsilon_0)\,(Q/r_1 - aQ/hr_2 + aQ/hr_3 - Q/r_4) \tag{1}$$

where,

$$r_1^2 = r^2 + h^2 - 2\,rh\cos\theta$$
$$r_2^2 = r^2 + \delta^2 - 2r\delta\cos\theta$$
$$r_3^2 = r^2 + \delta^2 - 2r\delta\cos\theta'$$
$$r_4^2 = r^2 + h^2 - 2r\,h\cos\theta'$$
$$\delta = a^2/h,\ \theta' = \pi - \theta$$

The charge distribution over the boss is obtained from,

$$\sigma_1 = \epsilon_0 E_n = -\epsilon_0 \partial V / \partial r \text{ at } r = a.$$

$$= [(h^2-a^2) Q/4\pi a] \, [1/(a^2+h^2+2a h \cos \theta)^{3/2} - 1/(a^2+h^2-2a h \cos \theta)^{3/2}]$$

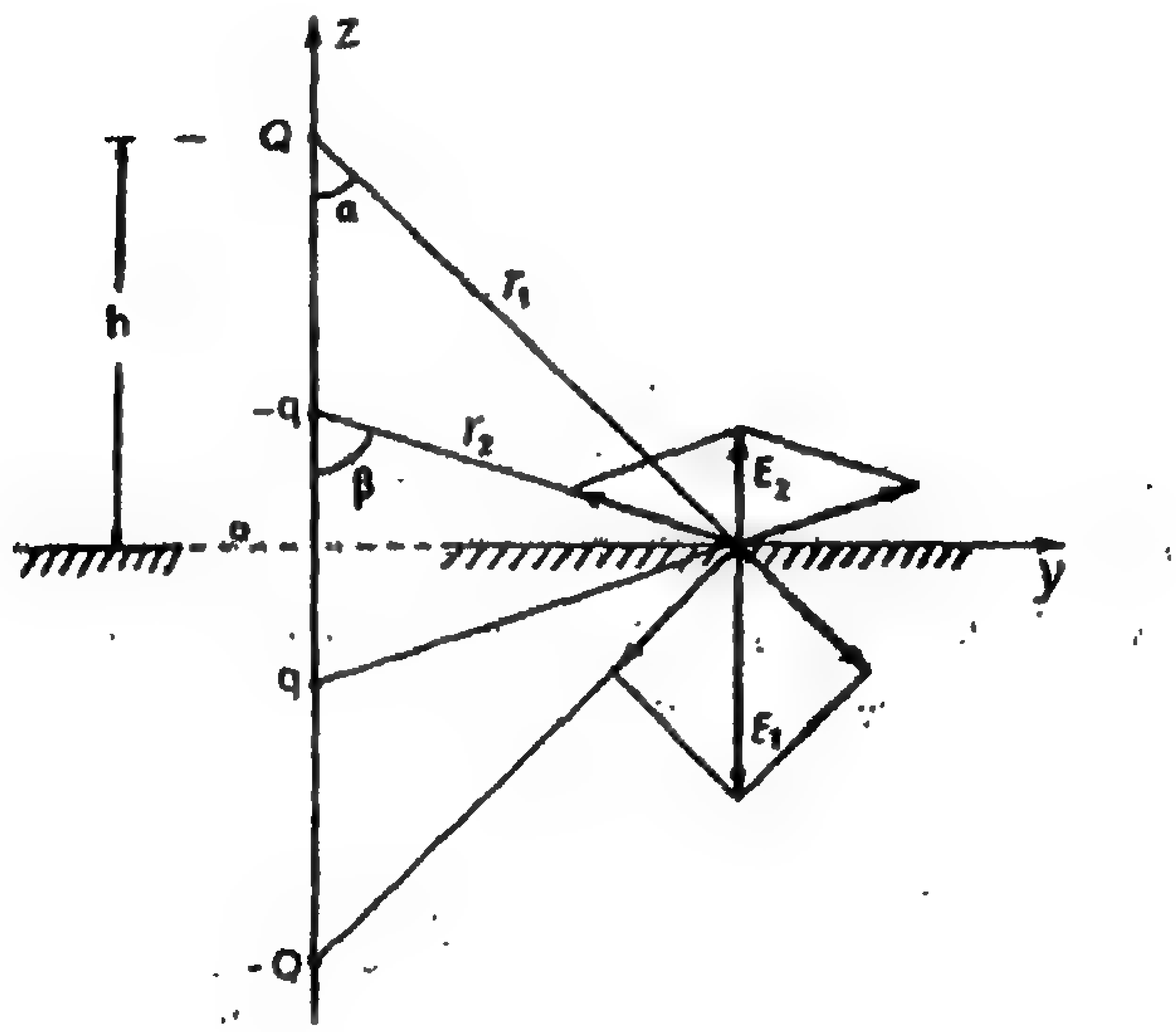


Fig. 9.35

The charge distribution over the conducting plane can be obtained with the aid of Fig. 9.35. The electric field due to Q and —Q at a point on the conducting plane is,

$$E_1 = -(2Q \cos \alpha / 4\pi \epsilon_0 r_1^2) \, \mathbf{a}_n$$

$$= -Qh / 2\pi \epsilon_0 (h^2+y^2)^{3/2} \, \mathbf{a}_n$$

The charges — Qa/h, + Qa/h have,

$$E_2 = [2 (Qa/h) \cos \beta / 4\pi \epsilon_0 r_2^2] \, \mathbf{a}_n$$

$$= [Qa^3 / 2\pi \epsilon_0 h^2 (a^4/h^2 + y^2)^{3/2}] \, \mathbf{a}_n$$

Hence

$$\sigma_2 = \epsilon_o E_n$$
$$= (Qh/2\pi) \, [(a^3/h^3) \, / \, (a^4/h^2+y^2)^{3/2} - 1/ \, (h^2+y^2)^{3/2})]$$
$$y \geq a \qquad (3)$$

On the boundary between plane and boss the surface densities are,

$$\sigma_1 \, (\theta = \pi/2) = 0$$

$$\sigma_2 \, (y = a) = 0$$

28. *A point charge Q is placed at a distance h from the center of a grounded conducting sphere of radius a. Use Coulomb's law to find an expression for the surface charge density on the sphere. If instead of the sphere we have an infinite conducting plane with hemispherical boss, deduce the expression of the surface density on the boss and find the total charge on it.*

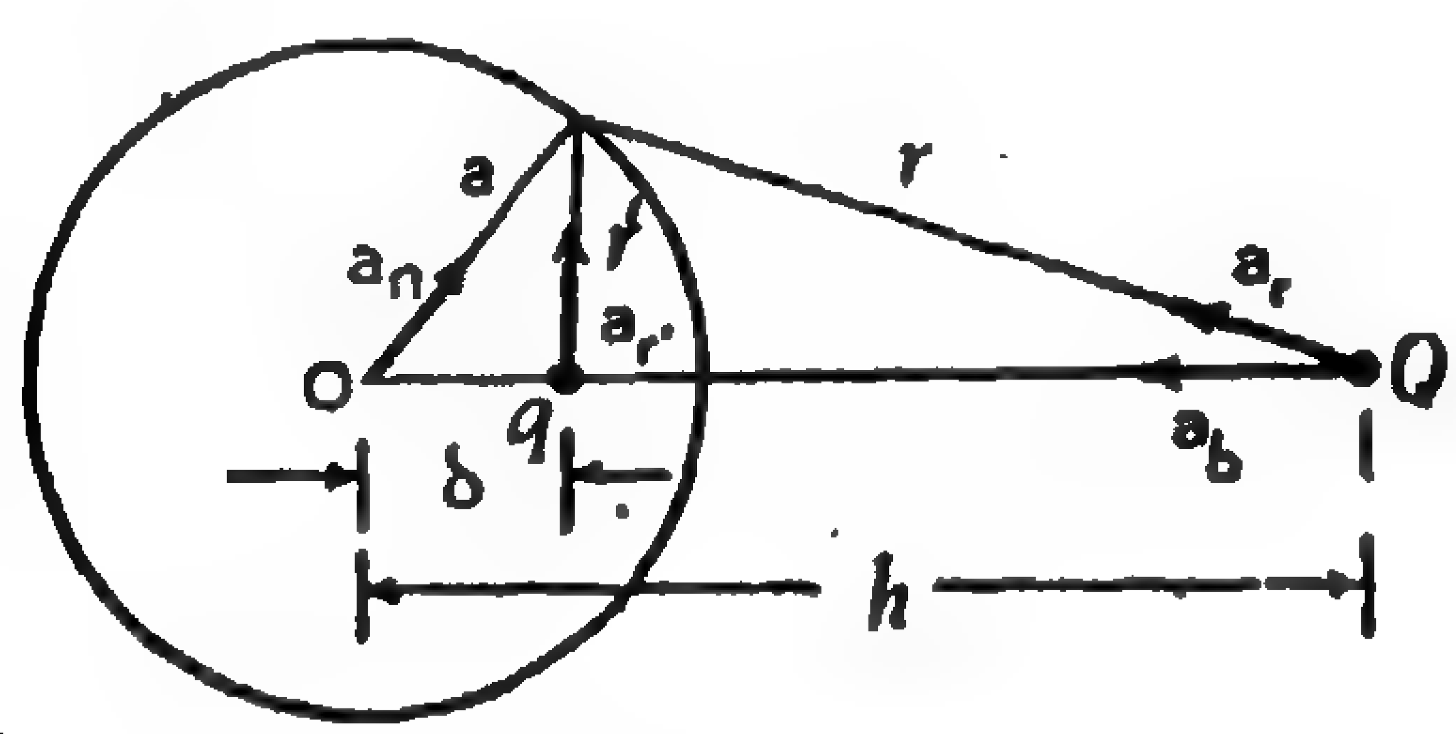


Fig. 9 36.

As shown in Fig. 9.36 the displacement density at a point on the sphere is,

$$\mathbf{D} = \epsilon \mathbf{E} = -(Q/4\pi \, r^2) \, \mathbf{a}_r + (Qa/4\pi \, h \, r'^2) \, \mathbf{a}_r' \qquad (1)$$

From Fig. 9.36 we have that,

$$r \, \mathbf{a}_r = h \, \mathbf{a}_b + a \, \mathbf{a}_n$$

$$r' \, \mathbf{a}_r' = \delta \, \mathbf{a}_b + a \, \mathbf{a}_n$$

with, $h/r = a/r'$, and $\delta = a^2/h$ we get that,

$$\mathbf{D} = (Q/4\pi)\,[-(a/r^3 - a^2/br'^3)\,\mathbf{a}_n + (h/r^3 - \delta a/hr'^3)\,\mathbf{a}_z]$$

$$\delta\, a/hr'^3 = (a^2/h)\, a/h\,(ar/h^2)^3 = h/r^3, \text{ so that we have}$$

$$\mathbf{D} = -[Q\,(h^2 - a^2)\,/\,4\pi a r^3]\,\mathbf{a}_n$$

$$\sigma = D_n = -Q\,(h^2 - a^2)\,/\,4\pi a r^3$$

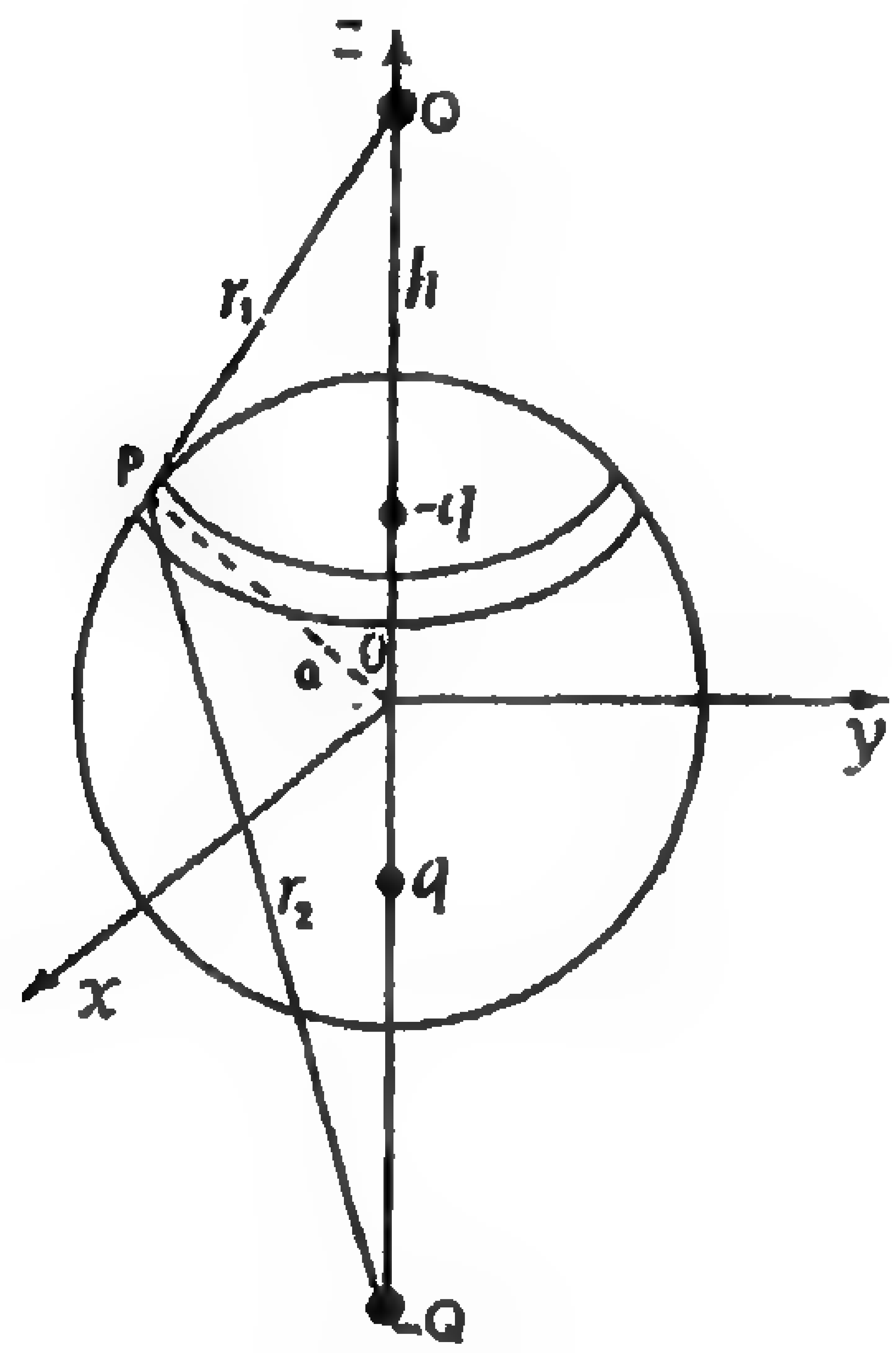


Fig. 9.37.

If instead of the sphere we have an infinite plane with hemispherical boss, then as shown in Fig. 9.37 the surface distribution due to Q and $-q = -Qa/h$ is,

$$\sigma_1 = -Q\,(h^2 - a^2)\,/\,4\pi a r_1^3$$

while that due to $-Q$ and $+q$ at the same point P is

$$\sigma_2 = Q\,(h^2 - a^2)\,/\,4\pi a r_2^3$$

where,

$$r_1^2 = (a^2 + h^2 - 2\, a\, h \cos\theta)$$

$$r_2^2 = (a^2 + h^2 + 2\, a\, h \cos\theta)$$

so that,

$$\sigma = \sigma_1 + \sigma_2$$

$$[(h^2-a^2)\, Q/4\pi a]\, [1/(a^2+h^2+2\, a\, h \cos\theta)^{3/2} - 1/(a^2+h^2-2\, a\, h \cos\theta)^{3/2}] \tag{2}$$

The total charge on the boss is,

$$q = \int\int \sigma\, dS$$

$$= 2\pi a^2 \int_0^{\pi/2} \sigma \sin\theta\, d\theta$$

$$= 2\pi a^2 \int_0^1 \sigma\, d\mu \;, \quad \mu = \cos\theta$$

$$q = \tfrac{1}{2} a\, (h^2-a^2)\, Q\, [\int_0^1 d\mu/(A+B\mu)^{3/2} - \int_0^1 d\mu/(A-B\mu)^{3/2}]$$

where, $A = a^2+h^2$ and $B = 2\, a\, h$. Integrating we get,

$$q = -Q\, [1 - (h^2-a^2)\, /\, h\, (a^2+h^2)^{1/2}] \tag{3}$$

If $h >> a$, this expression gives,

$$q = -Q\, [1 - h\, /\, (a^2+h^2)^{1/2}]$$

which is the expression for the total charge on a circular region of radius a for a system consisting of a charge Q at distance h in front of a grounded infinite plane.

29. ***An infinite plane conductor at zero potential with hemispherical boss of radius a coincides with the x y-plane. Half the space in front of the plane is filled with dielectric of relative permittivity ϵ_r, while the other half is air. If a point charge Q is placed at a distance b (b>a) from the center of the boss in the air region, find the image system so formed.***

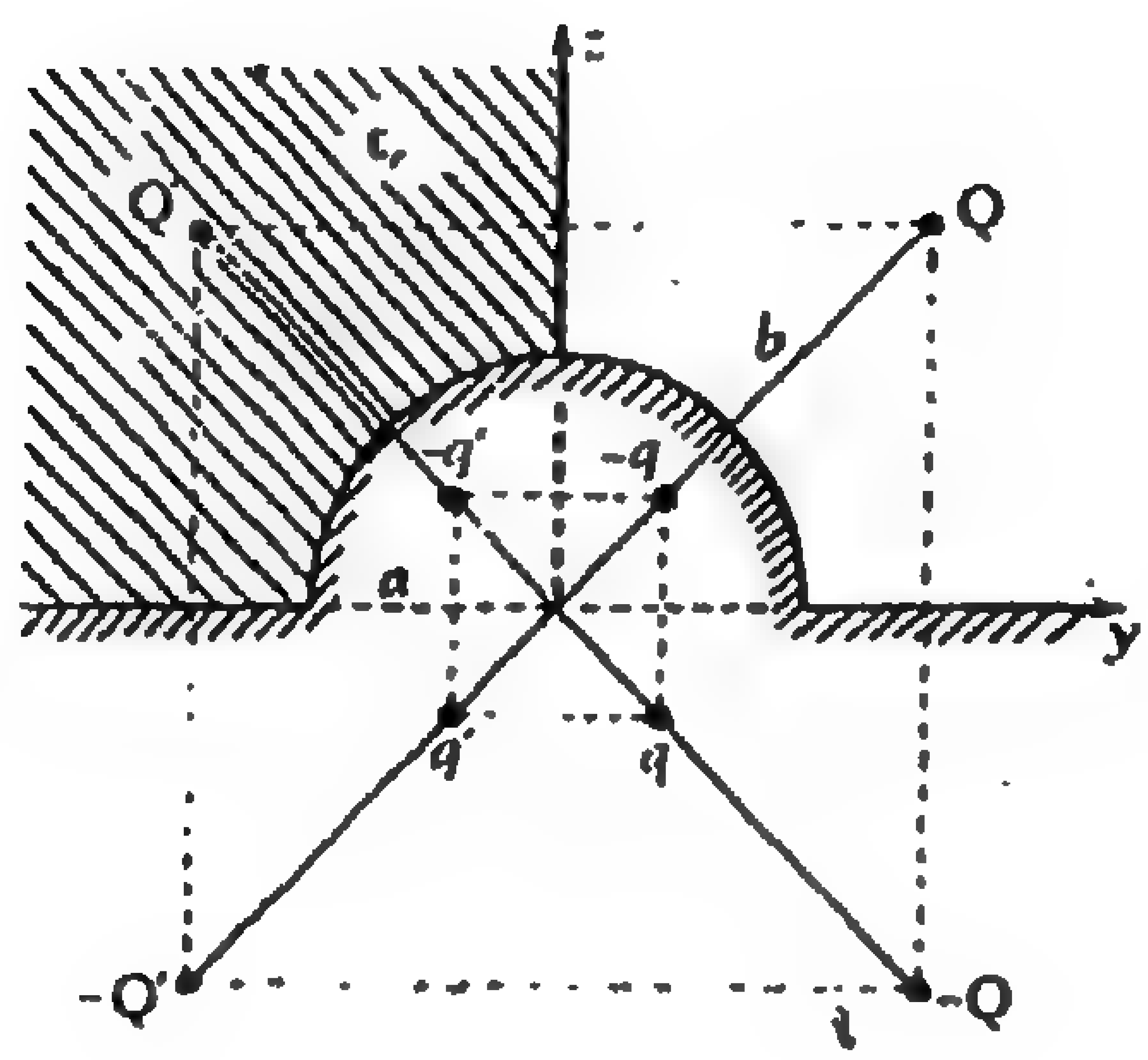


Fig. 9.38.

Aided with Fig. 9.38 the image system for field calculation in the region, $r>a$, $0<\phi<\pi$, $0<\theta<\pi/2$ is,

$$Q \text{ at } (b,\theta,\phi)\ , \qquad Q' \text{ at } (b,\theta,\phi+\pi)$$

$$-Q \text{ at } (b,\pi-\theta,\phi)\ ,\ -Q' \text{ at } (b,\pi-\theta,\phi+\pi)$$

$$-aQ/b \text{ at } (a^2/b,\theta,\phi),\quad aQ/b \text{ at } (a^2/b,\pi-\theta,\phi)$$

$$-aQ'/b \text{ at } (b,\theta,\phi+\pi),\quad aQ'/b \text{ at } (a^2/b,\pi-\theta,\phi+\pi)$$

where, $Q' = (1-\epsilon_r)\,Q/(1+\epsilon_r)$. For field calculations in the dielectric region, the images behind the xy—plane vanish while Q in the above system is changed to $Q'' = 2Q/(1+\epsilon_r)$ and all the half medium is considered as air.

39. *A conducting surface consists of two infinite planes which meet at right angles and a quarter of a sphere of radius a fitted into the right angle. If the conductor is at zero potential and a point charge Q is symmetrically placed with regard to the planes and the spherical surface at a great distance b from the center, show that the charge induced on the spherical portion is approximately* $-5Q\,a^3/\pi b^3$.

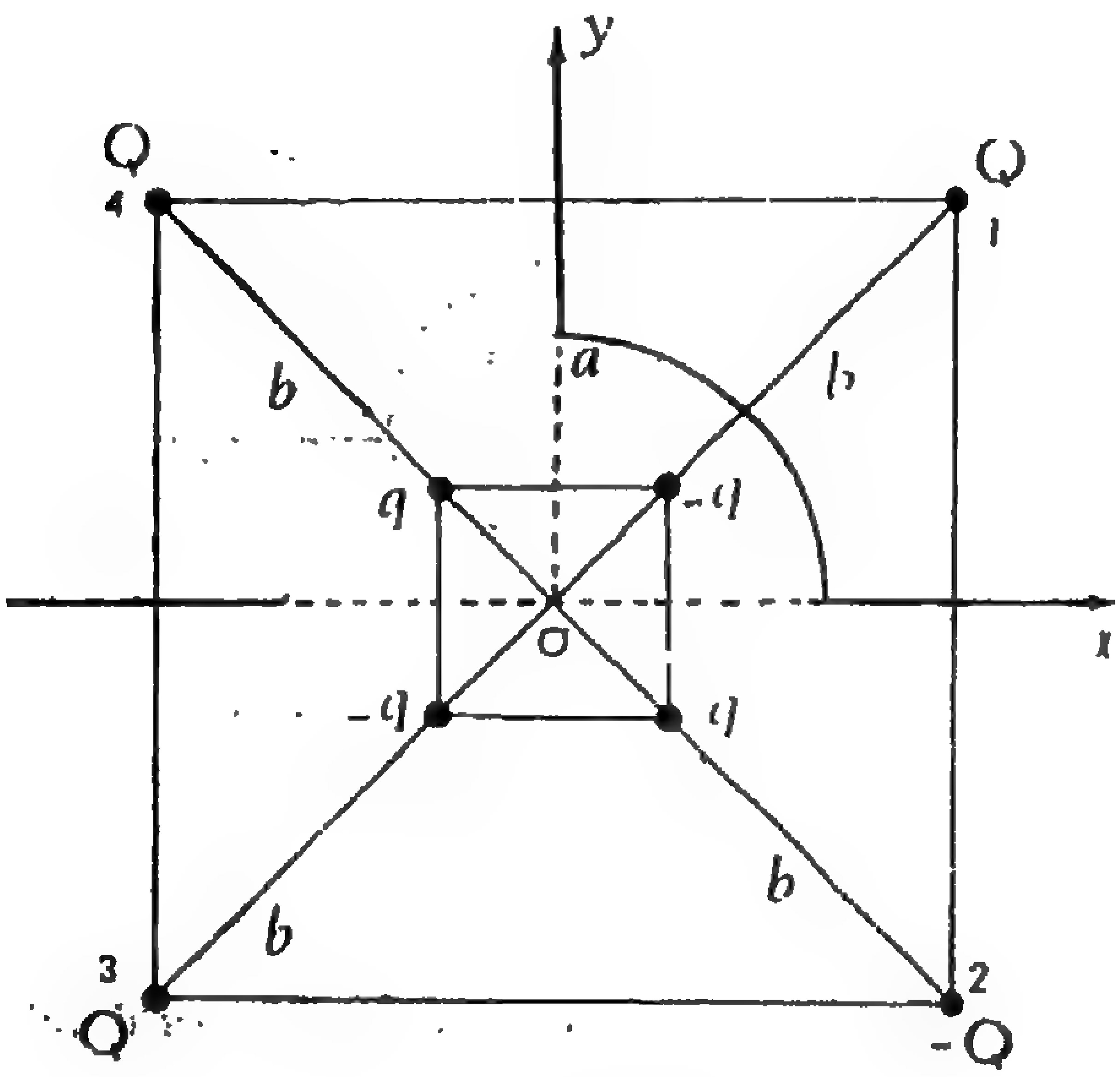


Fig. 9 39.a

As shown in Fig. 9.39a,b the charge Q is at point P ($r = b$, $\theta=\pi/2$, $\phi=\pi/4$). This charge has the image system (see problem 9.29).

$$Q \text{ at } (b,\ \pi/2,\ \pi/4),\quad -q \text{ at } (a^2/b,\ \pi/2,\ \pi/4)$$
$$Q \text{ at } (b,\ \pi/2,\ 5\pi/4),\quad -q \text{ at } (a^2/b,\ \pi/2,\ 5\pi/4)$$
$$-Q \text{ at } (b,\ \pi/2,\ 3\pi/4),\quad +q \text{ at } (a^2/b,\ \pi/2,\ 3\pi/4)$$
$$-Q \text{ at } (b,\ \pi/2,\ 7\pi/4),\quad +q \text{ at } (a^2/b,\ \pi/2,\ 7\pi/4)$$

where $q = aQ/b$.

The surface charge density due to the two charges at 1,3 (problem 9.28) at a general point A on the spherical surface is,

$$\sigma = -Q\,(b^2-a^2)\,/\,4\pi\, a r^3 \tag{1}$$

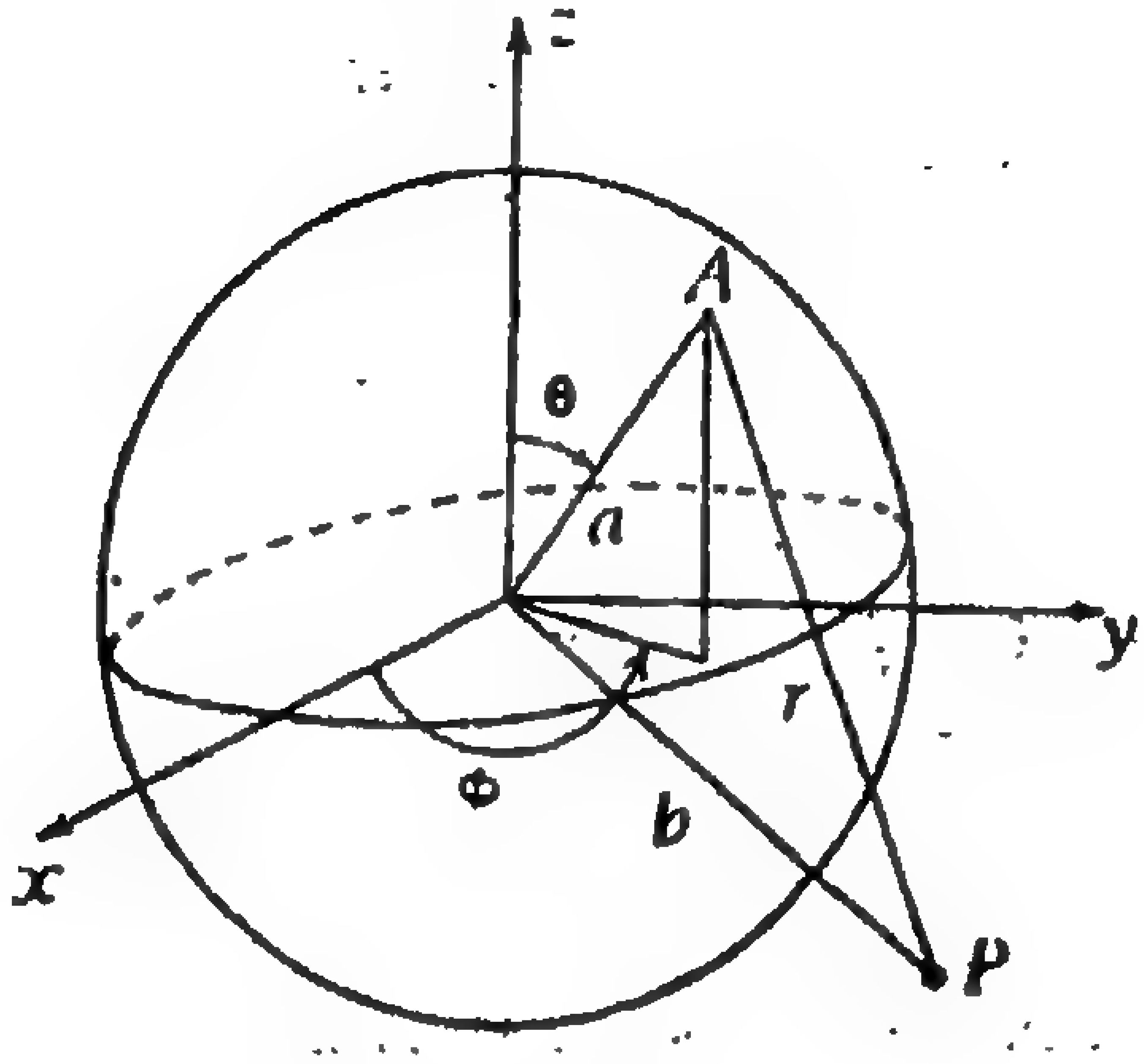


Fig. 9.39 b

The coordinates of A are (Fig. 9.39b),

$$x = a \sin\theta \cos\phi, \quad y = a \sin\theta \sin\phi, \quad z = a \cos\theta$$

The location of the charge at point 1 is,

$$x = b/\sqrt{2}, \quad y = b/\sqrt{2}, \quad z = 0$$

so that we have,

$$r^3 = [(b/\sqrt{2} - a \sin\theta \cos\phi)^2 + (b/\sqrt{2} - a \sin\theta \sin\phi)^2 + (0 - a \cos\theta)^2]^{3/2}$$

$$= [b^2 + a^2 - \sqrt{2}\, a\, b \sin\theta\, (\sin\phi + \cos\phi)]^{3/2}$$

Neglecting a^2 with respect to b^2 we have that,

$$1/r^3 = (1/b^3)\,[1 - (\sqrt{2}\, a/b) \sin\theta\, (\sin\phi + \cos\phi)]^{-3/2}$$

$$\simeq (1/b^3)\,[1 + (3a/\sqrt{2}\, b) \sin\theta\, (\sin\phi + \cos\phi)$$

$$+ (15a^2/b^2) \sin^2\theta\, (\sin\phi + \cos\phi)^2 + \ldots] \tag{2}$$

The total charge on the quarter sphere due to charges at 1, image, 2, image, 3, image, and 4, image is,

$$q = q_1 + q_2 + q_3 + q_4 \quad \text{where} \tag{3}$$

$$q_1 = \int_{\theta=o}^{\pi} \int_{\phi=o}^{\pi/2} \sigma\, a^2 \sin\theta\, d\theta\, d\phi \tag{4}$$

$$q_2 = -\int_{o}^{\pi} \int_{\pi/2}^{\pi} \sigma\, a^2 \sin\theta\, d\theta\, d\phi \tag{5}$$

$$q_3 = \int_{o}^{\pi} \int_{\pi}^{3\pi/2} \sigma\, a^2 \sin\theta\, d\theta\, d\phi \tag{6}$$

$$q_4 \quad q_2 \tag{7}$$

The first two terms of (2) when substituted in (4) to (7) gives $q=0$, so that the third term of (2) must be considered. When substituded in (4) to (7) we get that,

$$\begin{aligned} q &= -[Q(b^2-a^2)/4\pi a]\,[-(5a^4/b^6)(\pi/2+1)+ \\ &\quad 2(5a^4/b^6)(\pi/2-1)+(5a^4/b^6)(\pi/2+1)] \\ &= -[Q(b^2-a^2)/4\pi a]\,(20\,a^4/b^6) \\ &\simeq -(Q\,b^2/4\pi a)\,(20\,a^4/b^6) = -5a^3\,Q/\pi b^3 \end{aligned}$$

which is the required result.

31. *A point charge Q is placed between two earthed concentric spheres of radii a and c at distance b from their common center, c>b>a. Prove that the induced charges on the inner and outer spheres are,*

$$-a\,(c-b)\,Q/b\,(c-a) \quad \textit{and} \quad -c\,(b-a)\,Q/b\,(c-a).$$

With reference to Fig. 9.40, Q induces an image $-aQ/b$ at a^2/b from the center O. This image induces a positive charge $(c\,b/a^2)\,(aQ/b)$

at c^2b/a^2 from center. The second image in turn induces on the smaller sphere a charge $(-aQ/b)\ (a/c)$ at $a^4/c^2 b$. Repeating we conclude that all the negative charges on the inner sphere have the value,

$$q_1 = -(aQ/b)\ [1+a/c+(a/c)^2+\ldots] = -(a/b)\,Q[c/(c-a)]$$

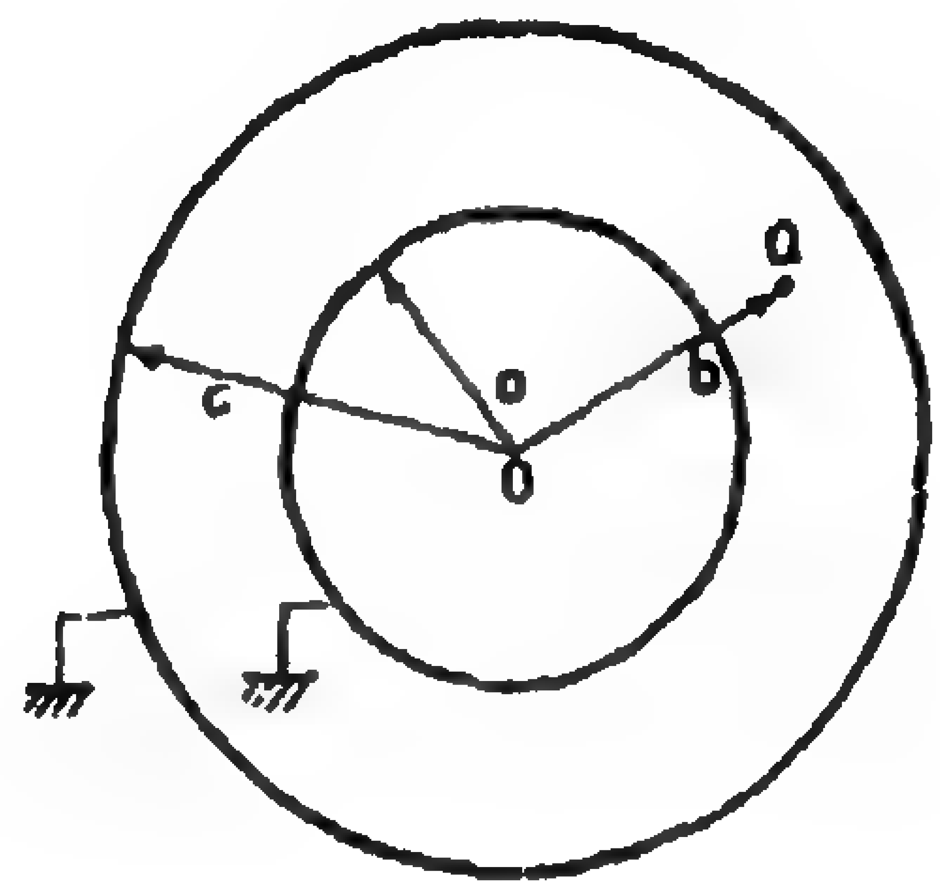


Fig. 9.40.

The charge Q induces a charge $-cQ/b$ on the outer sphere at c^2/b and this induces a positive charge aQ/c on the inner sphere. This induces an image charge on the outer sphere which again induces a charge $(a/c)^2Q$ and so on. The positive charges on the inner sphere is thus,

$$q_2 = (aQ/c)\ [1+a/c+(a/c)^2+\ldots] = (a/c)\ Q[c/(c-a)]$$

Thus the net negative charge on the inner sphere is,

$$q_{in} = q_1+q_2 = -a\ (c-b)\ Q/b\ (c-a).$$

Since all the lines of force out of Q must terminate on the two spheres we conclude that the total induced charges on the two spheres is $-Q$ and hence the other sphere carries a charge $q_{out} = -[Q+q_{in}] = -c\ (b-a)\ Q/b\ (c-a)$.

32. *A dipole of moment p is placed at a point distance r from the center of an earthed conducting sphere. If the angle between the dipole and the vector r is θ, find the image system due to the dipole.*

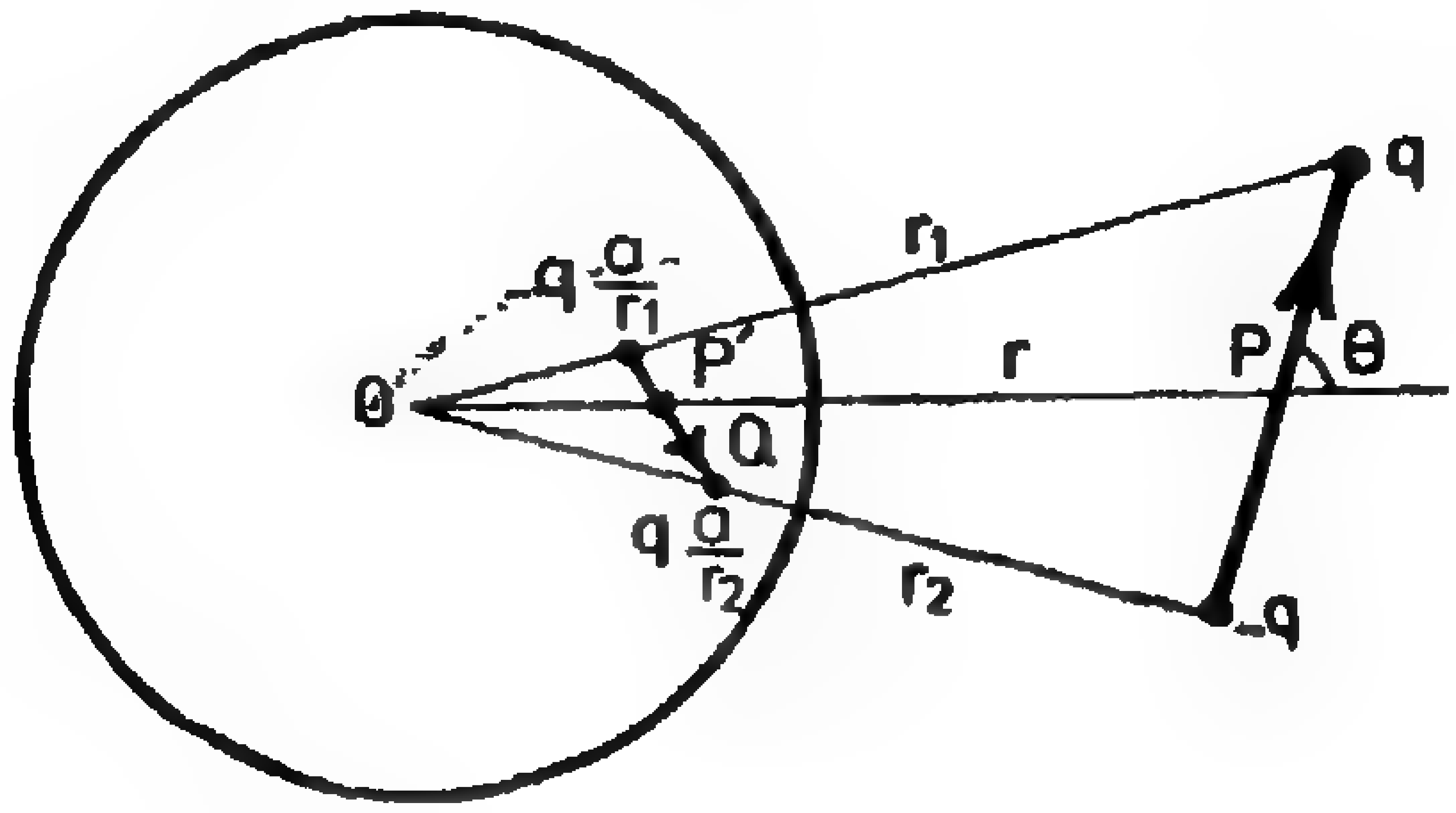


Fig. 9.41.

The dipole can be considered to consist of two charges $\pm q$ at an incremental distance l as shown in Fig. 9.41. The charge q at r_1 from O has an image charge $-q\ a/r_1$ at distance a^2/r_1 from O, also $-q$ has an image charge $+qa/r_2$ at distance a^2/r_2 from O.

Thus we can conclude that the dipole $p = q\ l$ has an image dipole of moment p' in addition to a charge Q given by,

$$\begin{aligned} Q &= q\ a/r_2 - q\ a/r_1 = q\ a\ (1/r_2 - 1/r_1) \\ &= q\ a\ [(r - \tfrac{1}{2}\ l\ \cos\theta)^{-1} - (r + \tfrac{1}{2}\ l\ \cos\theta)^{-1}] \\ &= (qa/r)\ [1 + (l/2r)\cos\theta - 1 + (l/2r)\cos\theta] \\ &= (qa/r)\ (l/r)\cos\theta = p\ (a/r^2)\cos\theta \end{aligned} \tag{1}$$

The dipole moment p' is given by

$$\begin{aligned} p' &= l'\ (q\ a/r_1) \simeq q\ (a/r)\ l' \\ &= q\ l\ (a/r)\ (l'/l) = p\ (a/r)\ (l'/l) \end{aligned} \tag{2}$$

Where l' is the length of the image dipole. From the geometry of the problem it is clear that $l' = l\ (a/r)^2$ so that the image dipole has moment,

$$p' = p\ (a/r)^3$$

33. *An electric dipole of moment* $\mathbf{p}$ *is placed in front of an infinite conducting plane. Determine the force and torque on the dipole and the potential energy due to interaction with the induced charge on the conducting plane.*

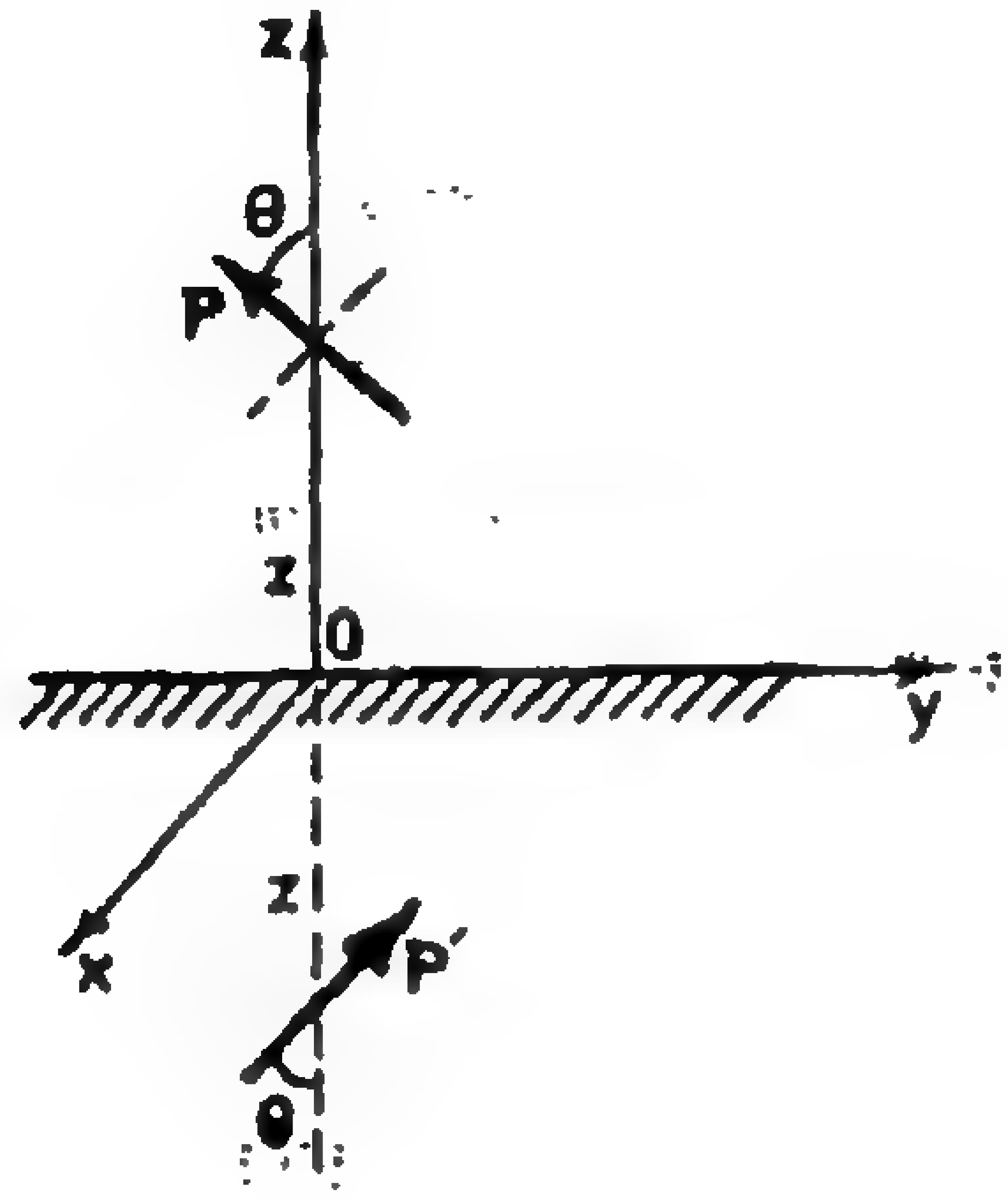


Fig. 9.42.

Let the dipole be located in the xz-plane at a distance z from the conducting plane as shown in Fig. 9.42. Thus we can write,

$$\mathbf{p} = p \sin\theta\, \mathbf{a}_x + p \cos\theta\, \mathbf{a}_z \tag{1}$$

Hence its image is

$$\mathbf{p}' = -p \sin\theta\, \mathbf{a}_x + p \cos\theta\, \mathbf{a}_z \tag{2}$$

To find the torque and force on the dipole we first find the electric field due to $\mathbf{p}'$ at $\mathbf{p}$.

$$\mathbf{E} = (1/4\pi\epsilon_0)\,[3(\mathbf{r}\cdot\mathbf{p}')\mathbf{r}/r^5 - \mathbf{p}'/r^3] \tag{3}$$

with $\mathbf{r} = 2z\, \mathbf{a}_z$

$$\mathbf{r}\cdot\mathbf{p}' = 2z\, \mathbf{a}_z \cdot (-p \sin\theta\, \mathbf{a}_x + p\cos\theta\, \mathbf{a}_z) = 2pz\cos\theta$$

Substituting in (3) we get,

$$\mathbf{E} \ = \ (p \ /32\pi \ \epsilon_o z^3) \ (\sin\theta \, \mathbf{a}_x + 2 \cos\theta \, \mathbf{a}_z) \tag{4}$$

Thus the torque on the dipole is (Problem 2.44 Vol. I),

$$\mathbf{T} \ = \ \mathbf{p} \times \mathbf{E} \tag{5}$$

Substituting from (1) and (4) in (5) we get,

$$\mathbf{T} \ = \ (-p^2 \ \sin 2\theta \,/\, 64\pi \ \epsilon_o \ z^3) \ \mathbf{a}_y \tag{6}$$

The potential energy of the dipole is (Problem 2.44 Vol. I),

$$U \ = \ -\mathbf{p}.\mathbf{E}$$

Substituting from (1) and (4) we get,

$$U \ = \ -p^2 \ (1 + \cos^2\theta) \ / \ 32\pi \ \epsilon_o \ z^3 \tag{7}$$

Note that since $\mathbf{E}$ given by (4) is proportional to p and when $\mathbf{p}$ is increased by $d\,\mathbf{p}$ at the same orientation the interaction energy increases by

$$dU = \ -\mathbf{E} \ d\mathbf{p}$$

hence $$U \ = \ \int_0^p dU = -\tfrac{1}{2} \ \mathbf{E}.\mathbf{p}$$

so that the actual value of the energy is half that given by (7). The force on the dipole can also be obtained,

$$\begin{aligned} \mathbf{F} \ &= \ -(\partial U/\partial z) \ \mathbf{a}_z \\ &= \ -3 \ [p^2 \ (1 + \cos^2\theta) \ / \ 64\pi \ \epsilon_o z^4] \ \mathbf{a}_z \end{aligned} \tag{8}$$

From the equation of the torque it is clear that the torque tends to align the dipole with the z-axis either in the positive or the negative directions *i.e.* $\theta = 0$ or π. If $\theta = \pi/2$, $T = 0$ but this is an unstable equilibrium position of the dipole.

34. *An electric dipole of moment* $\mathbf{p}$ *is placed at a distance r from the center of an earthed conducting sphere of radius a. Find the force, the couple, and the interaction energy between the dipole and the sphere. Also consider the case when a tends to infinity,* $(r > a)$.

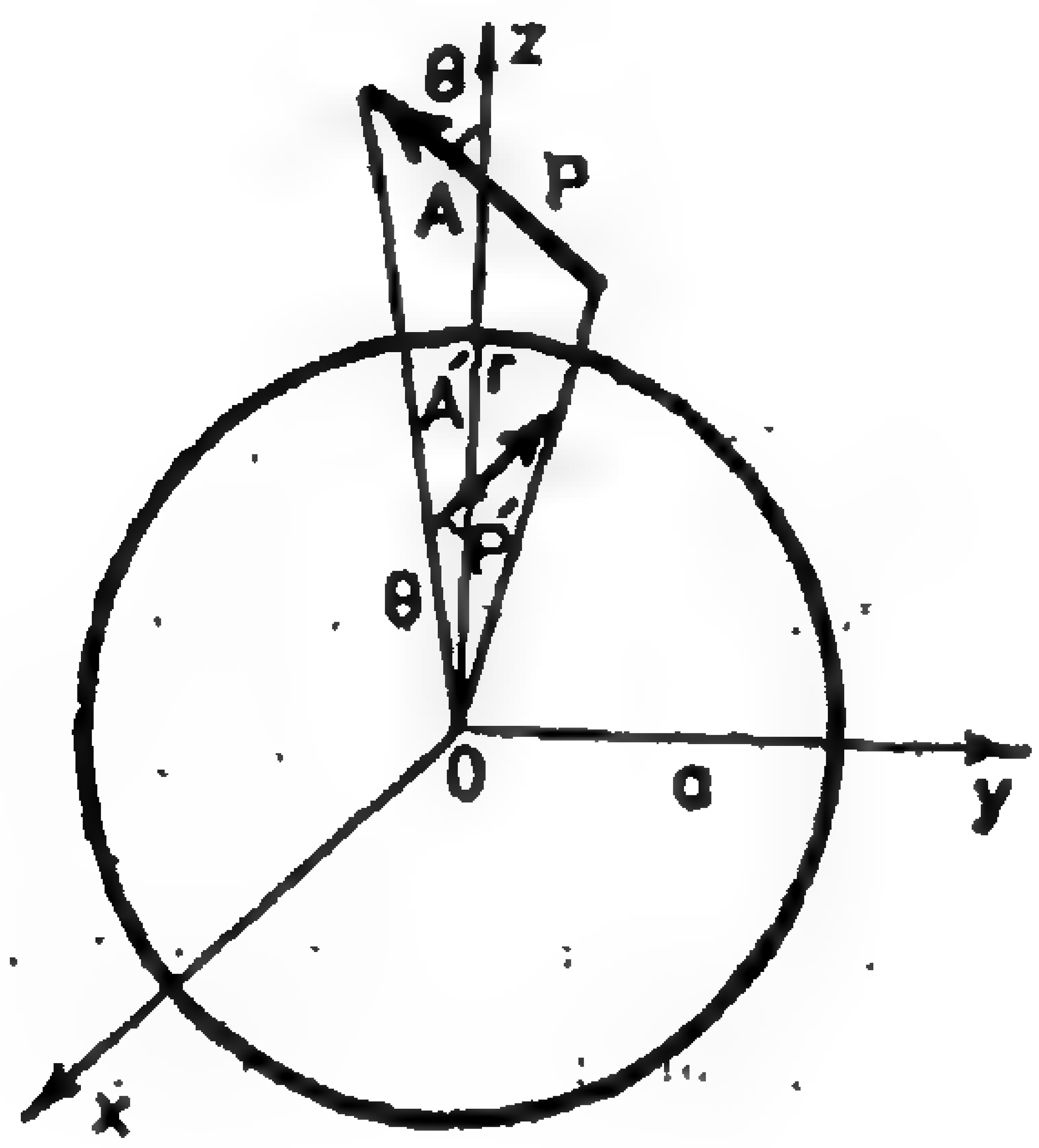


Fig. 9.43.

Let the dipole be in the xz-plane as shown in Fig. 9.43.

$$\mathbf{p} = (p \sin\theta\, \mathbf{a}_x + p \cos\theta\, \mathbf{a}_z) \tag{1}$$

As in problem 9.32, its image is a point charge

$$Q = p\,(a/r^2) \cos\theta \tag{2}$$

and a dipole of moment

$$\mathbf{p}' = p\,(a/r)^3\,(-\sin\theta\, \mathbf{a}_x + \cos\theta\, \mathbf{a}_z) \tag{3}$$

both are at the inverse point A'. The electric field at point A is due to Q and the dipole $\mathbf{p}'$ (see problem 2.43. Volume I).

$$\mathbf{E} = (1/4\pi\epsilon_0)\,[(3\mathbf{R}.\mathbf{p}'/R^5)\,\mathbf{R} - \mathbf{p}'/R^3] + (p\cos\theta/4\pi\,\epsilon_0\, r^2 R^2)\,\mathbf{a}_r$$

where $\mathbf{R} = \mathbf{r} - \mathbf{r}'$, and $r' = OA'$.

Substituting for $\mathbf{p}'$ and $\mathbf{R}, \mathbf{p}'$ we get,

$$\mathbf{E} = (pa^3/4\pi\epsilon_0 r^3 R^3)\ [\sin\theta\ \mathbf{a}_x + (2+rR/a^2)\cos\theta\ \mathbf{a}_z] \quad (4)$$

The potential energy of the dipole is,

$$U \quad -\tfrac{1}{2}\,\mathbf{p}\cdot\mathbf{E} = -\,p^2 a\,(a^2+r^2\cos^2\theta)\ /\ [8\pi\epsilon_0 (r^2-a^2)^3] \quad (5)$$

The force on the dipole is,

$$\mathbf{F} = -\,\partial U/\partial z\ \mathbf{a}_z = -\,\partial U/\partial r\ \mathbf{a}_z$$

$$= +\,[p^2 ar/4\pi\epsilon_0 (r^2-a^2)^4]\ [3a^2 + (a^2+2r^2)\cos^2\theta]\ \mathbf{a}_z \quad (6)$$

The torque on the dipole can also be obtained,

$$\mathbf{T} = \mathbf{p}\times\mathbf{E} = -\{\ p^2 ar^2 \sin 2\theta\ /\ [8\pi\epsilon_0 (r^2-a^2)^3]\ \}\ \mathbf{a}_y \quad (7)$$

In order to consider the case when a tends to infinity, let $r = a+z$ and let a tend to infinity in (4)—(7) we get the same results as the previous problem.

35. *A point charge Q is placed near a plane dielectric interface. Show that outside the dielectric the field is the same as if we have in addition to the given charge Q, a charge Q′ at the image point of Q, and in the dielectric the field is the same as if we have a charge Q″ instead of Q where* $Q' = -Q(\epsilon_r-1)/(\epsilon_r+1)$ *and* $Q'' = 2Q/(\epsilon_r+1)$, *with* ϵ_r *the relative permittivity of the dielectric. Also find the pulling force on the dielectric.*

What will be the case if the point charges Q, Q′, and Q″ are replaced by parallel infinite line charges λ, λ' *and* λ'' *respectively.*

The electric field in air due to the assumed distribution is (see Fig. 9.44),

$$E_p = (Q/4\pi\epsilon_0 r^2)\,\mathbf{a}_r + (Q'/4\pi\epsilon_0 r'^2)\,\mathbf{a}_{r'} \quad (1)$$

If the point P lies on the interface $r =$ the normal and tangential components at the interface are,

$$E_{pn} = (Q'-Q) \cos \theta/4\pi \epsilon_o r^2 \tag{2}$$

$$E_{pt} = (Q'+Q) \sin \theta/4\pi \epsilon_o r^2 \tag{3}$$

In the dielectric the tangential and normal components at the interface are,

$$E_{Dt} = Q'' \sin \theta/4\pi \epsilon_o r^2 \tag{4}$$

$$E_{Dn} = -Q'' \cos \theta/4\pi \epsilon_o r^2 \tag{5}$$

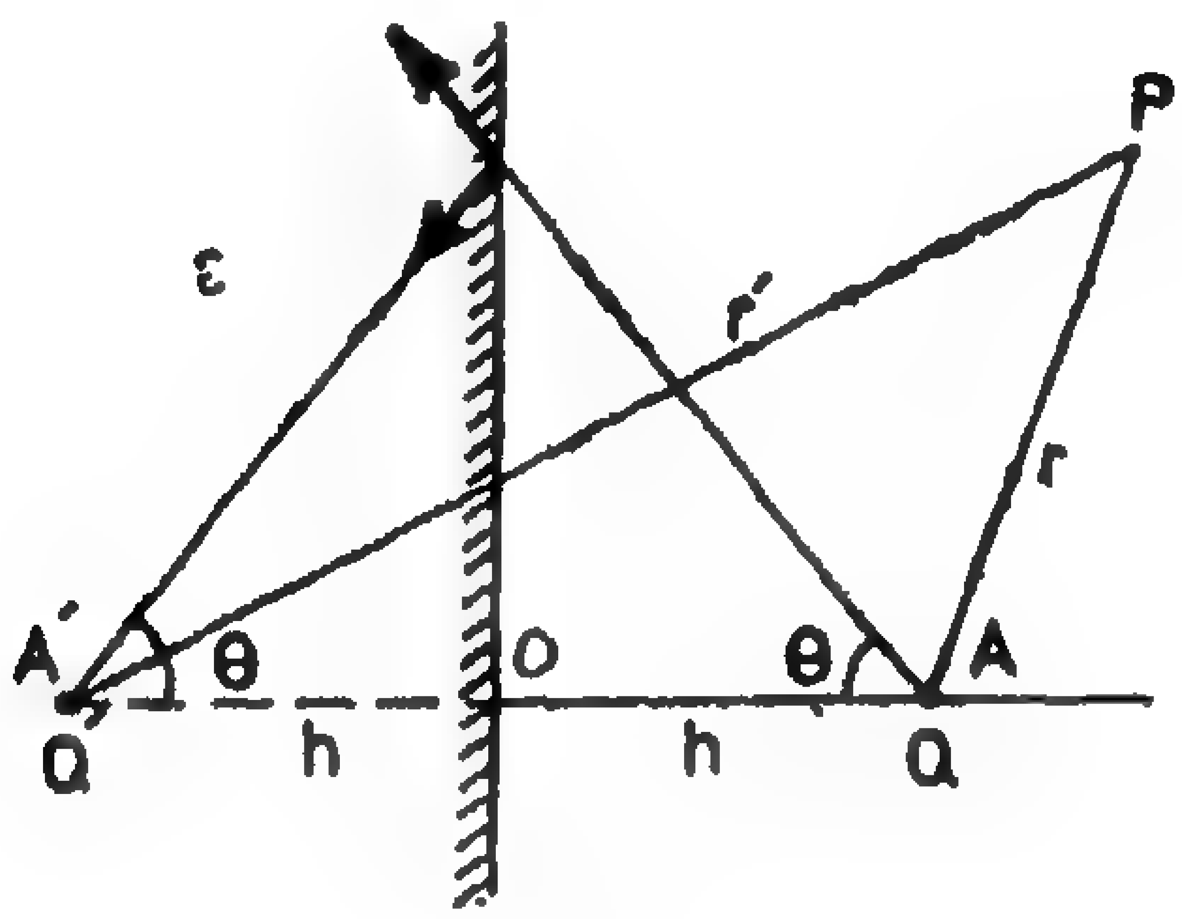


Fig. 9.44.

The required relations are obtained from the boundary conditions at the interface namely,

$$E_{pt} = E_{Dt} \ , \ \epsilon_r E_{Dn} \quad E_{pn} \tag{6}$$

Substituting from (2)—(5) in (6) we get,

$$Q+Q' = Q''$$
$$Q'-Q = -\epsilon_r Q''$$

Solving for Q' and Q'' we get,

$$Q' = -(\epsilon_r-1) Q/(\epsilon_r+1) \tag{7.a}$$
$$Q'' = 2Q/(\epsilon_r+1) \tag{7.b}$$

The force between Q and Q$'$ is,

$$F = QQ' / 4\pi\epsilon_0 (2h)^2 = (\epsilon_r - 1)\, Q^2/16\pi\, \epsilon_0\, (\epsilon_r + 1)\, h^2 \qquad (8)$$

If the point charges Q, Q$'$, and Q$''$ are replaced by infinite line charges λ, λ', λ'' respectively we get equations similar to (1) — (7) only with r^2 in the demominator of (1) — (5) replaced by r, hence $\lambda' = (\epsilon_r - 1)\, \lambda/(\epsilon_r + 1)$, and $\lambda'' = 2\,\lambda/(\epsilon_r + 1)$. The force between the dielectric and the line charge λ per unit length is,

$$F = \lambda\lambda'/2\pi\, \epsilon_0\, (2h) = (\epsilon_0 - 1)\, \lambda^2/4\pi\, \epsilon_0\, (\epsilon_r + 1)\, h$$

36. *A point charge* Q *is placed at a distance a in front of an infinite dielectric slab bounded by a plane interface. Let ϕ be the angle between a line of force in the dielectric and the normal to the plane interface, and ψ the angle between the same two lines near the charge* Q. *Show that ϕ and ψ are related by;*

$$\sin(\psi/2) = [2\, \epsilon/\, (\epsilon_0 + \epsilon)]^{1/2} \sin(\phi/2).$$

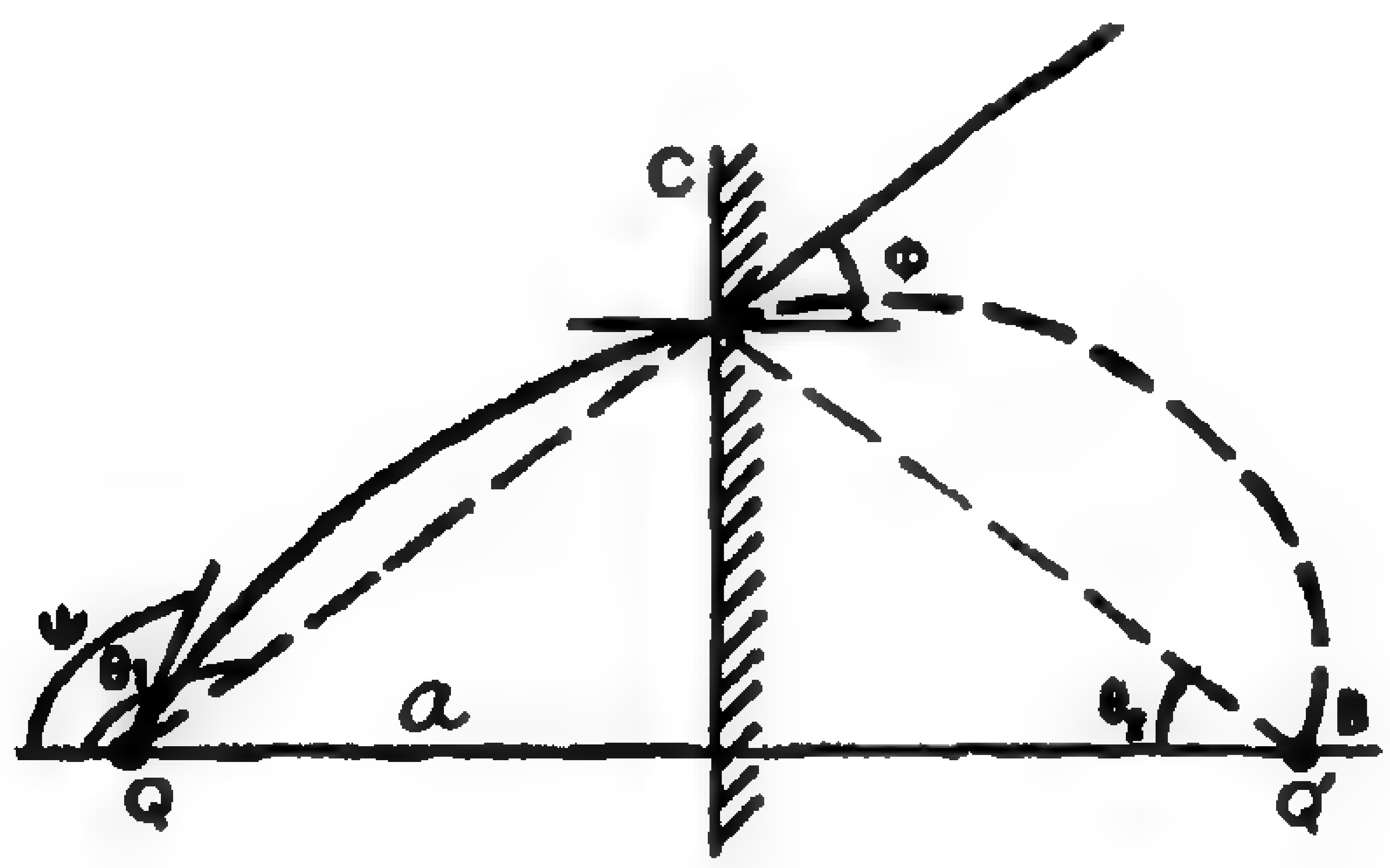


Fig. 9.45.

As shown in problem 9.?5, the field in air is the same as if we have a charge $Q' = -(\epsilon - \epsilon_0)\, Q/(\epsilon + \epsilon_0)$ at B the image point of A in addition to Q at A (Fig. 9.45). The equation of a line of force in

the air-region only, passing by Q is, (see problem 2.15, Vol. I),

$$Q \cos \theta_1 + Q' \cos \theta_2 = k$$

At point A, $\theta_1 = \psi$, $\theta_2 = 0$

$$Q \cos \theta_1 + Q' \cos \theta_2 = Q \cos \psi + Q' \qquad k \tag{1}$$

At point C, $\theta_2 = \pi - \theta_1$ so that,

$$Q \cos \theta_1 - Q' \cos \theta_1 \doteq k \tag{2}$$

Now in the dielectric region the field is that same as if all the region is free space with a charge Q'' at A. Thus the line of force in the dielectric is in the direction AC. Hence $\phi = \pi - \theta_1 = \theta_2$. Substituting in the left hand side of (2) we have that,

$$(Q - Q').\cos(\pi - \phi) = Q \cos \psi + Q'$$

Substituting for the above value of Q' we get after simple manipulation,

$$\sin(\psi/2) = [2\epsilon/(\epsilon + \epsilon_0)]^{1/2} \sin(\phi/2)$$

37. *If the charge Q of problem 9.35 is replaced by an electric dipole of moment* **p**, *determine the image system, also show that the torque on the dipole is,*

$$T = p^2 (\epsilon_r - 1) \sin 2\theta / 64\pi \epsilon_0 (\epsilon_r + 1) h^3$$

where θ is the angle between the dipole and the plane interface.

From the results of the previous problem we conclude that p has an image dipole of moment p'. The value of p' is obtained by considering p to consist of two charges $+Q$ and $-Q$ at distance l apart. The charge Q has an image Q' and $-Q$ has an image $-Q'$ (see equation (7) of problem 9.35), so that p' has two charges Q' and $-Q'$ at distance l (Fig. 9.46),

$$\begin{aligned} p' &= Q' l = Q\, l\, (\epsilon_r - 1) \,/\, (\epsilon_r + 1) \\ &= p\, (\epsilon_r - 1) \,/\, (\epsilon_r + 1) \end{aligned}$$

To determine the torque on dipole p we first find the electric field due to p' at P (see problem 2.43 Volume I),

$$\mathbf{E} = (1/4\pi\,\epsilon_0)\,[3\,(\mathbf{r}.\mathbf{p}'/r^5)\,\mathbf{r} - \mathbf{p}'/r^3]$$

The torque on the dipole p is thus,

$$\mathbf{T} = \mathbf{p}\times\mathbf{E} = (1/4\pi\,\epsilon_0)\,[3\,(\mathbf{r}.\mathbf{p}'/r^5)\,\mathbf{p}\times\mathbf{r} - (1/r^3)\,\mathbf{p}\times\mathbf{p}']$$

Here we have,

$$\begin{aligned} \mathbf{r}.\mathbf{p}' &= 2\,h\,p\,(\epsilon_r - 1)\,\sin\theta \,/\, (\epsilon_r + 1) \\ \mathbf{p}\times\mathbf{r} &= 2\,h\,p\,\cos\theta \\ \mathbf{p}\times\mathbf{p}' &= p^2\,(\epsilon_r - 1)\,\sin 2\theta \,/\, (\epsilon_r + 1) \end{aligned}$$

Note also that $\mathbf{p}\times\mathbf{r}$ and $\mathbf{p}\times\mathbf{p}'$ are in the same direction. Substituting in the torque equation we get,

$$T = p^3\,(\epsilon_r - 1)\,\sin 2\theta \,/\, 64\pi\,\epsilon_0\,(\epsilon_r + 1)\,h^3$$

in the direction normal to the paper, inwards.

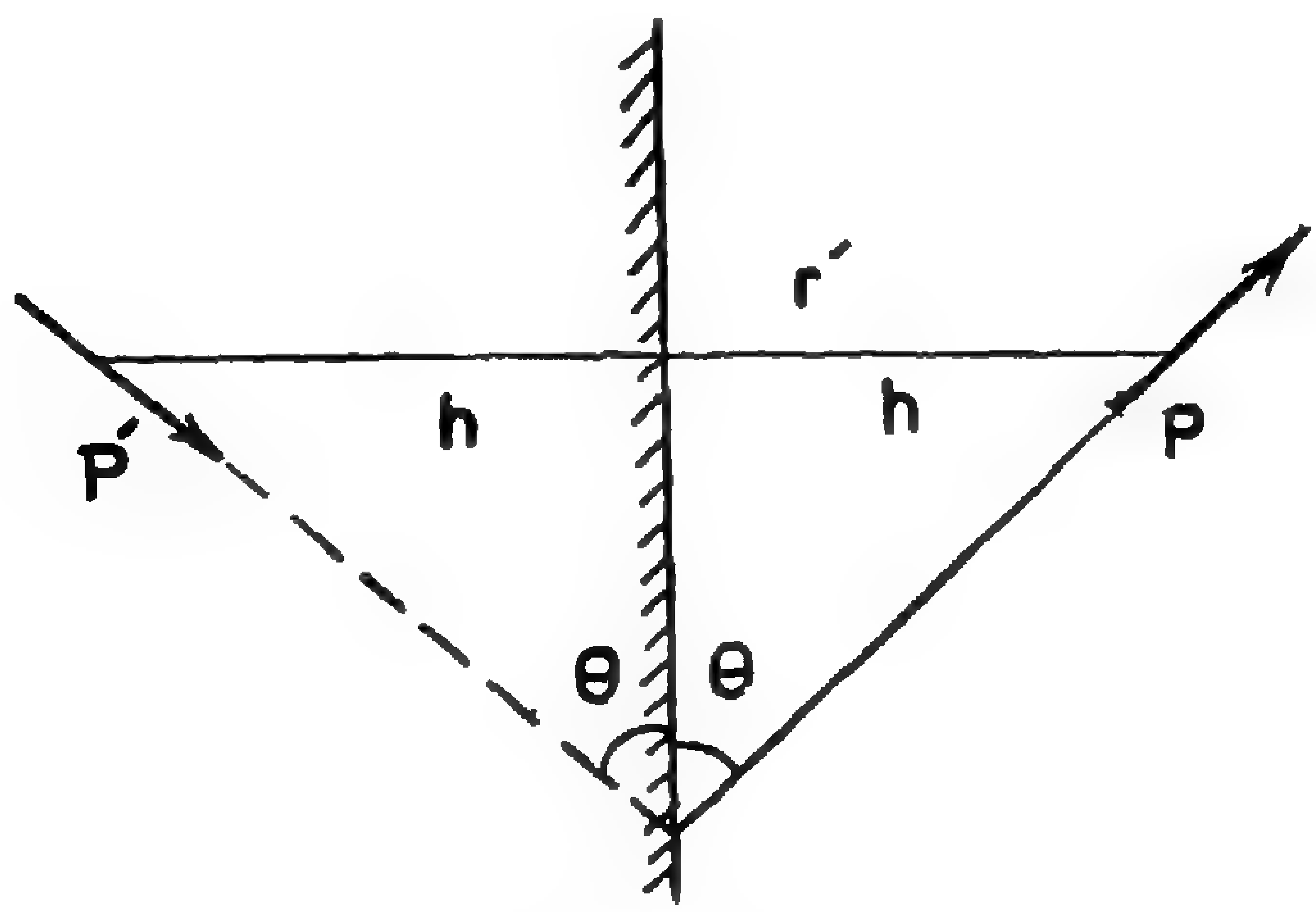


Fig. 9.46.

38. *A magnetic dipole of moment p is placed at distance h from an infinite slab of relative permeability μ_r. If the dipole is parallel to the plane face of the slab, show that the magnitude of force acting on the dipole is,*

$$3\, p^2 (\mu_r - 1)\, /\, 64\pi\, \mu_0\, (\mu_r + 1)\, h^4$$

This problem can be solved as the previous one. The image of dipole p is p' where,

$$\mathbf{p}' = -\,[(\mu_r - 1)/(\mu_r + 1)]\ \mathbf{p}$$

The magnetic field due to $\mathbf{p}'$ at $\mathbf{p}$ is

$$\mathbf{H} = (1/4\pi\mu_0)\ [3\ (\mathbf{r} \cdot \mathbf{p}'/r^5)\ \mathbf{r} - (\mathbf{p}'/r^3)],$$

$$= -\,(1/4\pi\mu_0\ r^3)\ \mathbf{p}'$$

The potential energy due to $\mathbf{p}$ is

$$U = -\tfrac{1}{2}\ \mathbf{p} . \mathbf{H} = -\ p^2\ (\mu_r - 1)\ /\ 8\pi\ \mu_0\ (\mu_r + 1)\ r^3$$

so that the force on $\mathbf{p}$ is,

$$F = -\ \partial U/\partial h = -3\ p^2\ (\mu_r - 1)\ /\ 64\pi\ \mu_0\ (\mu_r + 1)\ h^4$$

which has the required magnitude.

39. *An infinite plane boundary separates two media of permeabilities μ_1 and μ_2. In medium 1 and at distance h from the boundary there is a long wire which carries a current I. Show that the wire experiences a force per unit length given by,*

$$F = \mu_1\ (\mu_2 - \mu_1)\ I^2\ /\ 4\pi\ a(\mu_2 + \mu_1)$$

This problem is similar to problem 9.35. The current I will have an image current,

$$I' = -\ (\mu_2 - \mu_1)\ I/(\mu_2 + \mu_1)$$

This current I' has a field at I given by,

$$\mathbf{H} = (I'/4\pi a)\,\mathbf{a}_t$$

$$= -[(\mu_2-\mu_1)\,I/4\pi a\,(\mu_2+\mu_1)]\,\mathbf{a}_t$$

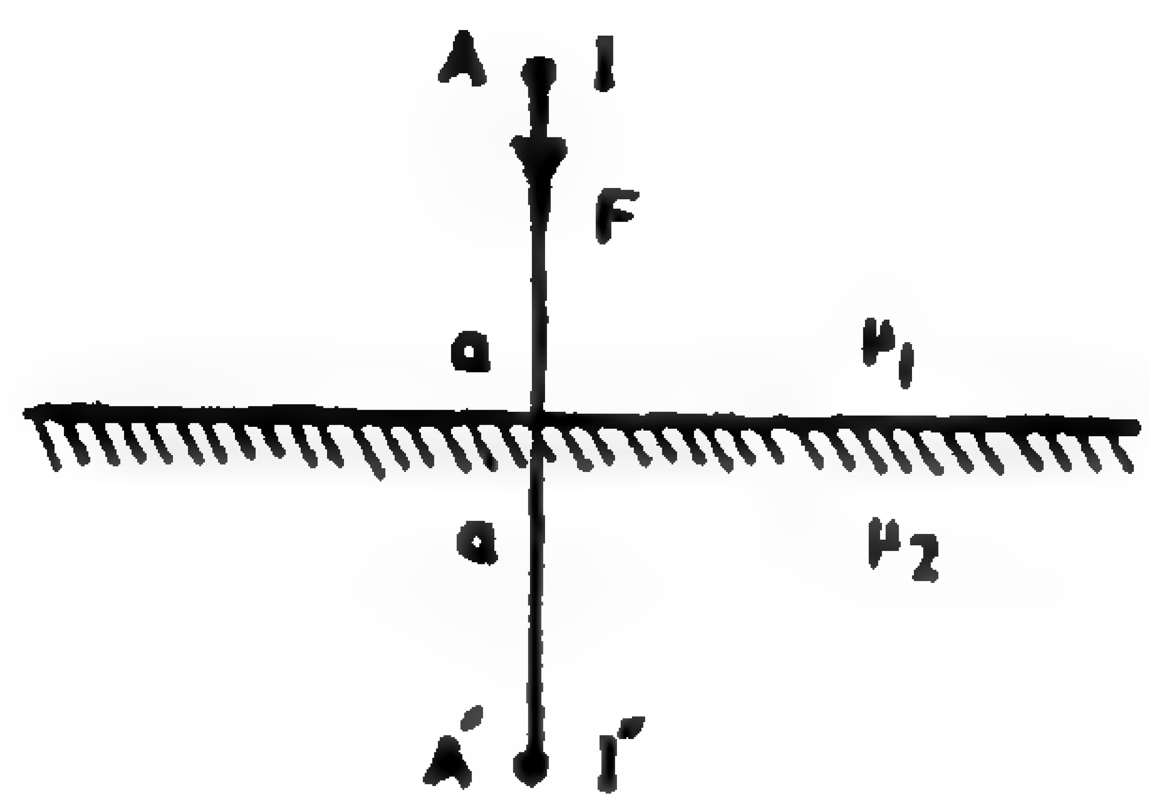


Fig. 9.47.

As shown in Fig. 9.47, the force per unit length on I is,

$$F = BI = \mu_1 HI$$

$$= -\mu_1(\mu_2-\mu_1)\,I^2 / 4\pi a\,(\mu_1+\mu_2)$$

in the direction $A'A$ or $-F$ in the direction AA'.

40. *A region of permeability μ_1 is separated from a region of permeability μ_2 by a boundary of arbitrary shape. Determine the distribution of poles which when lying along the boundary give the same effect on the field in region 1 as does the presence of the interface.*

Refering to Fig. 9.48 the boundary is considered to be a very thin layer of permeability μ_o in which the equivalent pole distribution lies. Let H_n be the normal component due to sources in either or both regions in absence of the magnetized media 1 and 2. The effect of the magnetized media is accounted for by a normal component H'_n

at the boundary in the same direction as H_n. The resulting component in region 1 is $H_n + H_n'$ and that in region 2 is $H_n - H_n'$. Now since the normal component of flux is continuous,

$$\mu_1 (H_n + H'_n) = \mu_2 (H_n - H'_n)$$

This gives

$$H'_n = [(\mu_2 - \mu_1) / (\mu_2 + \mu_1)] H_n$$

The equivalent pole density σ_s which gives rise to H_n' in the μ_o region and hence to the effect of the boundary is given by,

$$\sigma_s = \mu_o [H'_n - (-H'_n)] = 2\mu_o H'_n$$

Note: This method can be used in many of the electrostatic and magnetostatic problems.

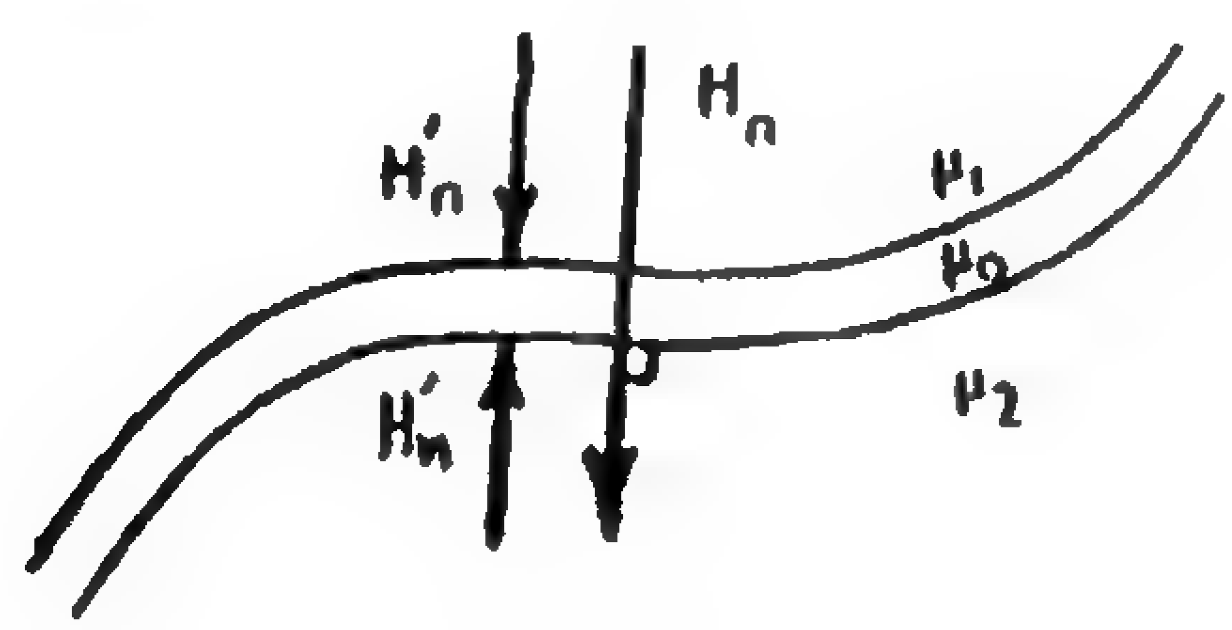


Fig. 9.48.

41. *A charge Q is placed near an infinite plane boundary. The charge Q is located in region 1 of permittivity ϵ_1, region 2 having permittivity ϵ_2. Use the method of the previous problem to detemine the field in the two regions.*

At any point P on the boundary the normal component of the field E_n is (Fig. 9.49),

$$E_n = Q \cos\theta / 4\pi \epsilon_1 (x^2 + a^2)$$
$$\cos\theta = a / (x^2 + a^2)^{1/2}$$

so that

$$E_n = Qa \,/\, 4\pi \epsilon_1 \,(x^2+a^2)^{3/2}$$

This component induces a charge along the boundary. The normal field component due to this surface distribution is,

$$E'_n = [(\epsilon_2-\epsilon_1) \,/\, (\epsilon_2+\epsilon_1)]\; E_n$$

$$= (Q/4\pi\epsilon_1)\; [(\epsilon_2-\epsilon_1) \,/\, (\epsilon_2+\epsilon_1)]\; a \,/\, (x^2+a^2)^{3/2}$$

This field is the same as if we have a charge $-Q\ (\epsilon_2-\epsilon_1) / (\epsilon_2+\epsilon_1)$ at a distance a from the boundary in region 2 or from a charge Q $(\epsilon_2-\epsilon_1) / (\epsilon_2+\epsilon_1)$ at distance a from the boundary in region 1. Only one of them will be used in the field representation. In region 1 the field is determined from the first charge while in region 2 the field is determined from the second charge. Thus the total system is,

(i) A charge $Q - Q(\epsilon_2-\epsilon_1) \,/\, (\epsilon_2+\epsilon_1) = [2\,\epsilon_1/(\epsilon_2+\epsilon_1)]\; Q$

located at point $z=a$ in region 1. It is used in determining the field at any point in the dielectric as if it did not exist *i.e.* all the medium is of permittivity ϵ_1.

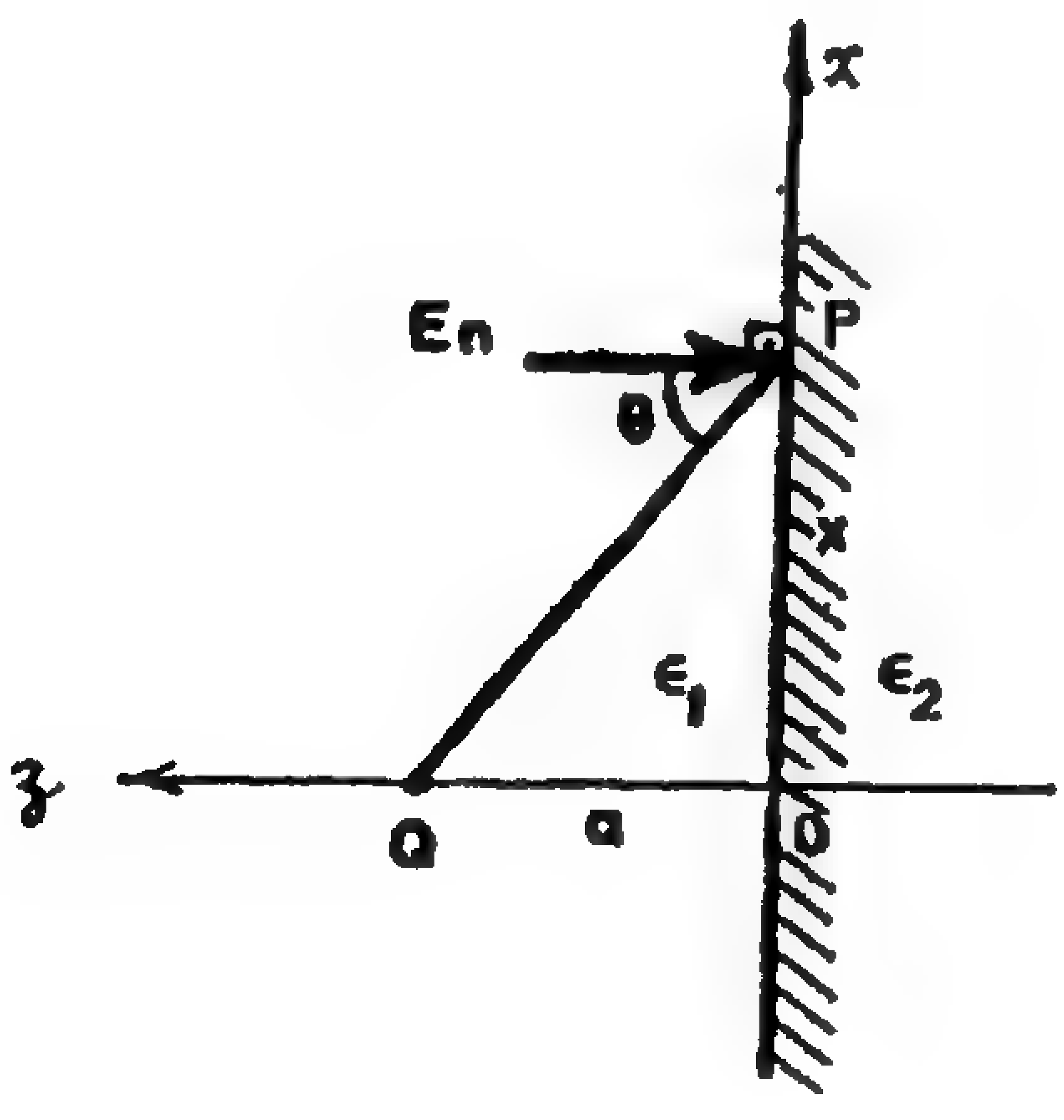


Fig. 9.49.

(ii) A charge Q at a in region 1 and a charge $-Q(\epsilon_2-\epsilon_1)/(\epsilon_2+\epsilon_1)$ in region 2. These are used to determine the field at any point in region 1.

Note that the above discussion is based on the fact that the field in region 1 is $E_n+E'_n$ and in region 2 is $E_n-E'_n$ on the boundary surface.

42. *A line charge of density λ is placed in a medium of permittivity ϵ_1 near a circular infinite cylinder of permittivity ϵ_2. Discuss the image system and consider the case when the line charge is inside the cylinder.*

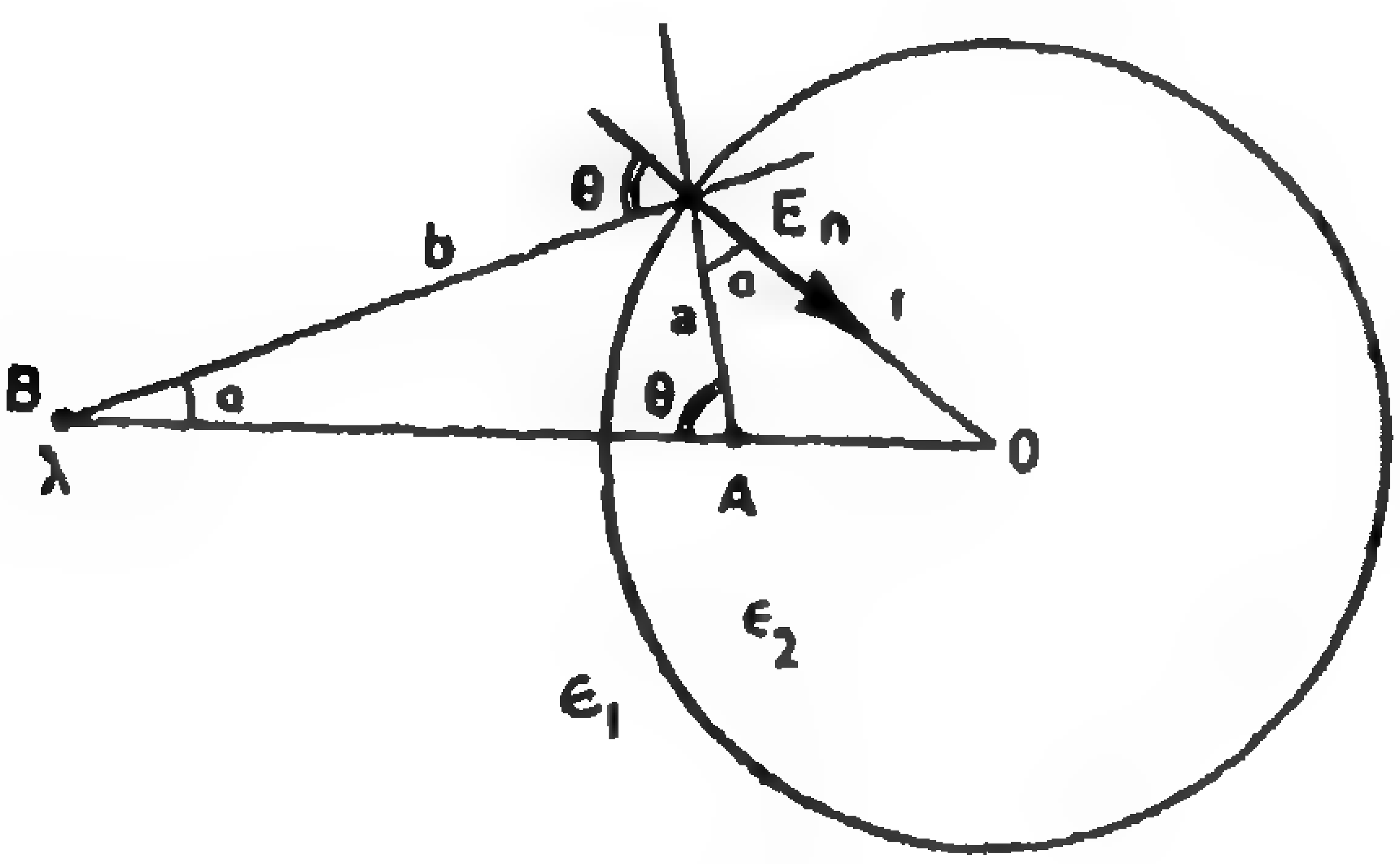


Fig. 9.50.

As shown in Fig. 9.50 the normal field component to the cylindrical surface due to the charge λ is,

$$E_n = \lambda \cos\theta / 2\pi\epsilon_1 b$$

Using the results of the previous problem we conclude that the normal field due to the equivalent surface charge distribution is,

$$E'_n = \lambda \cos\theta\,(\epsilon_2-\epsilon_1)\,/\,2\pi\epsilon_1 b\,(\epsilon_2+\epsilon_1) \qquad (1)$$

From Fig. 9.50 we have, (A is the inverse of B),

$$b \sin \alpha = r \sin (\theta - \alpha)$$

Hence,

$$b/\cos \theta = r \sin (\theta - \alpha) / \sin \alpha \cos \theta$$

This gives

$$\cos \theta / b = (1/r) [-1 + \sin \theta \cos \alpha / \sin (\theta - \alpha)]$$

But we have,

$$r/\sin \theta = a/\sin (\theta - \alpha)$$

Thus

$$\cos \theta / b = \cos \alpha / a - 1/r$$

Using this result in equation (1) we get,

$$E'_n = \lambda (\epsilon_2 - \epsilon_1) \cos \alpha / 2\pi \epsilon_1 (\epsilon_2 + \epsilon_1) a - \lambda (\epsilon_2 - \epsilon_1) / 2\pi \epsilon_1 (\epsilon_2 + \epsilon_1) r$$

This shows that the normal component of field is due to two charges. The first $\lambda (\epsilon_2 - \epsilon_1) / (\epsilon_2 + \epsilon_1)$ located at origin and the second $-\lambda (\epsilon_2 - \epsilon_1) / (\epsilon_2 + \epsilon_1)$ at the inverse point. Now since the two charges are inside the boundary of the cylinder they are the required image charges which with the outside charge give the field outside the cylinder. (all medium ϵ_1)

In the region inside the cylinder the images giving rise to E'_n must be outside the cylinder. From the equation (1) it is clear that the image charge is $-\lambda(\epsilon_2 - \epsilon_1) / (\epsilon_2 + \epsilon_1)$ at the same point as λ.

Note : A similar procedure can be used if the charge is inside the boundary. The field inside the cylinder is due to an image $\lambda (\epsilon_2 - \epsilon_1) / (\epsilon_2 + \epsilon_1)$ at the outer inverse point and the field outside is given by an image of magnitude $-\lambda (\epsilon_2 - \epsilon_1) / (\epsilon_2 + \epsilon_1)$ at the same point of λ together with an image $\lambda (\epsilon_2 - \epsilon_1) / (\epsilon_2 + \epsilon_1)$ at the center.

linear

43. *A dipole of strength p is placed in a medium of permittivity ϵ_1 near a cylinder of radius R and permittivity ϵ_2. Discuss the image system related to this problem.*

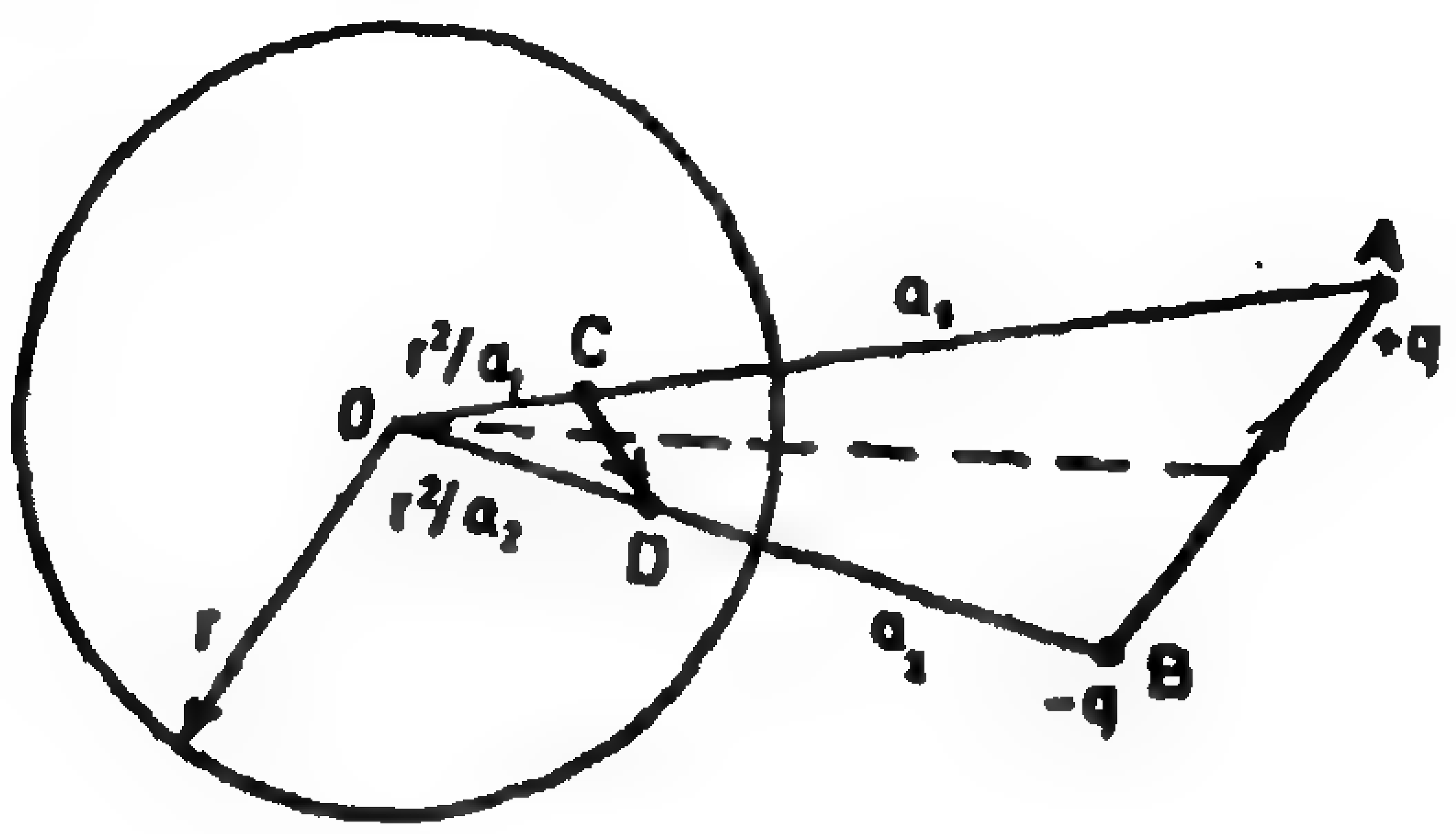


Fig.9.51.

The images of the two charges $\pm q$ constituting the dipole will first be obtained. The positive charge $+q$ at A has an image $-q(\epsilon_2-\epsilon_1)/(\epsilon_2+\epsilon_1)$ at the inverse point of A and an image $q(\epsilon_2-\epsilon_1)/(\epsilon_2+\epsilon_1)$ at the center O. For the negative charge $-q$ we have an image $q(\epsilon_2-\epsilon_1)/(\epsilon_2+\epsilon_1)$ at the image point of B and a charge $-q(\epsilon_2-\epsilon_1)/(\epsilon_2+\epsilon_1)$ at center O. Thus the two charges at center cancel one another, and we have a pair of charges at the inverse point. This pair constitute a dipole of moment (Fig. 9.51),

$$p' = -\lim_{A \to B} (CD)\, q\, (\epsilon_2-\epsilon_1)/(\epsilon_2+\epsilon_1)$$

$$= -\lim_{A \to B} (CD/AB)\, p\, (\epsilon_2-\epsilon_1)/(\epsilon_2+\epsilon_1)$$

But from the geometry of the problem $CD/AB = OC/OB = r^2/a_1a_2$

$$p' = -(r^2/a_1a_2)\, p\, (\epsilon_2-\epsilon_1)/(\epsilon_2+\epsilon_1)$$

44. *An infinite line charge of strength λ is placed at a distance b parallel to the axis of an infinitely long conducting cylinder. Find the potential at any point, and the surface charge density on the conducting cylinder. Consider two cases, (a) the cylinder is earthed, and (b) the cylinder is insulated.*

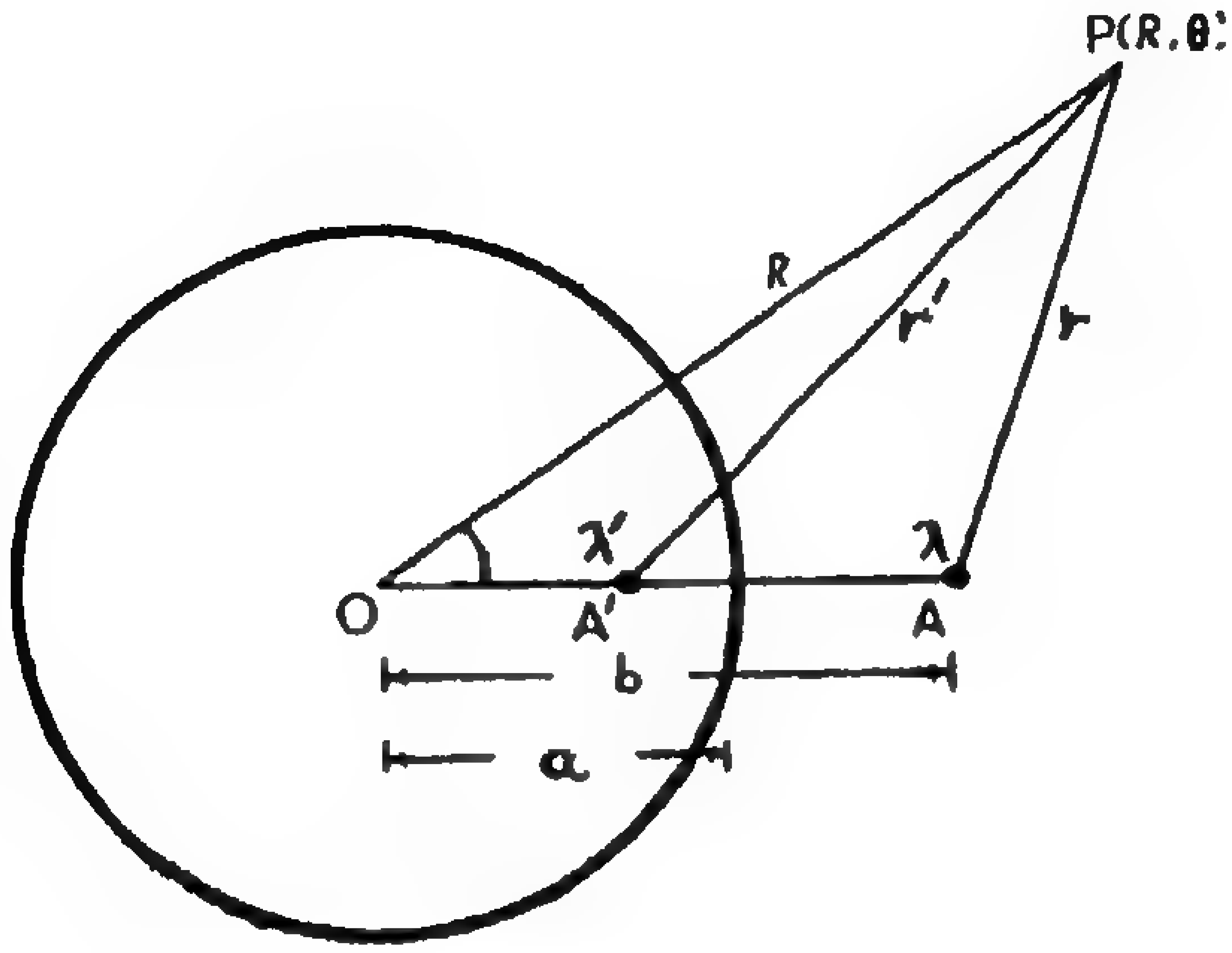


Fig. 9.52.

The conducting cylinder represents an equipotential surface so that the line charge λ will have an image λ' at the inverse point to A. Thus the system reduces to two line charges at a and a^2/b from the center O, Fig. 9.52. The potential at any external point P distant r from A and r' from A' is,

$$V = \int (\lambda/2\pi \epsilon_0 r)\, dr + \int (\lambda'/2\pi \epsilon_0 r')\, dr'$$

$$= k + (1/2\pi \epsilon_0)\, [\lambda \log r + \lambda' \log r'] \qquad (1)$$

where k is the constant of integration. It can be determined as follows : If P lies on the cylinder the potential is constant K. Also from the geometry of the problem we have $r'/r = b/a$ so that (1) gives,

$$K = k + (1/2\pi \epsilon_0)[\lambda \log r + \lambda' \log r + \lambda' \log (b/a)] \qquad (2)$$

Now since P is any arbitrary point on the cylinder, that is, (2) is valid for any r, we conclude that the coefficient of $\log r$ must be zero. Thus $\lambda' = -\lambda$. The relation between K and k is

$$K = k - (\lambda/2\pi\epsilon_0) \log (b/a) \tag{3}$$

(a) *The cylinder is earthed* : $(K = 0)$

In this case, $k = (\lambda/2\pi\epsilon_0) \log (b/a)$ and (1) becomes,

$$V = (\lambda/2\pi\epsilon_0) \log (br/ar') \tag{4}$$

From the geometry of Fig. 9.52 we have that,

$$r^2 = R^2 + b^2 - 2a\ R \cos\theta \tag{5}$$

$$r'^2 = R^2 + a^4/b^2 - 2\ (a^2R/b) \cos\theta \tag{6}$$

so that the potential at any point $P\ (R, \theta)$, $R \geq a$, is

$$V = (\lambda/4\pi\epsilon_0) \log \{ (b^2R^2 + b^4 - 2b^3 R \cos\theta) / [a^2R^2 + a^6/b^2 - 2\,(a^4R/b) \cos\theta] \} \tag{7}$$

The surface charge density at any point on the conducting cylinder is,

$$\sigma = \epsilon_0 E_n = -\epsilon_0\,(\partial V/\partial R) \qquad \text{at } R = a \tag{8}$$

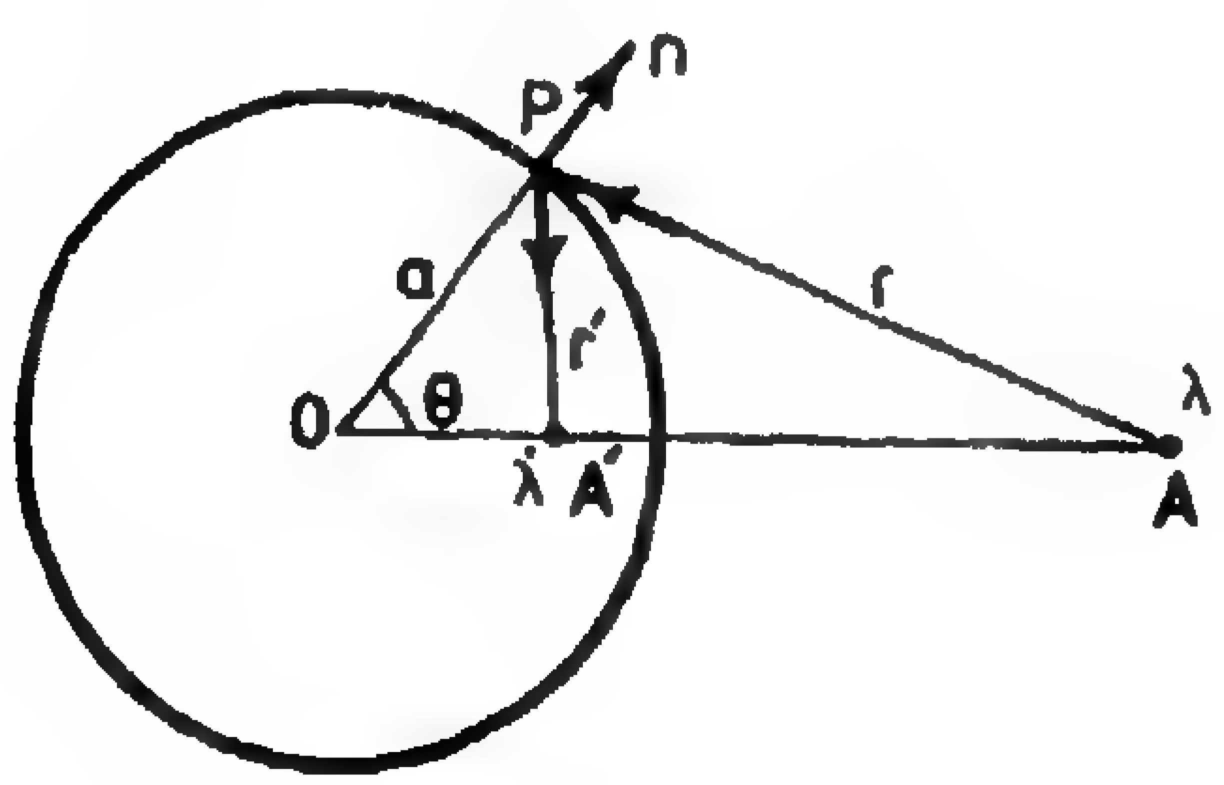


Fig. 9.53.

As the expression $-\partial V/\partial R$ is laborious, the value of E_n at $R = a$ can be obtained as follows, (Fig. 9.53),

$$E_n = -(\lambda/2\pi\,\epsilon_o\,r)\ \mathbf{n}.\mathbf{r}^* - (\lambda/2\pi\,\epsilon_o\,r')\ \mathbf{n}.\mathbf{r}^{*\prime} \tag{9}$$

We have that,

$$\mathbf{n}.\mathbf{r}^* = (b\ cos\ \theta - a)\ /\ r \tag{10}$$

$$\mathbf{n}.\mathbf{r}^{*\prime} = [a - (a^2/b)\ cos\ \theta]\ /\ r' \tag{11}$$

Using (5), (6) at $R = a$, (10), and (11), and equation (9) gives,

$$E_n = -(\lambda/2\pi\,\epsilon_o\ a)\ [(b^2 - a^2)\ /\ (a^2 + b^2 - 2\ a\ b\ cos\ \theta)] \tag{12}$$

Thus (8) becomes,

$$\sigma = -(\lambda/2\pi a)\ (b^2 - a^2)\ /\ (a^2 + b^2 - 2a\ b\ cos\ \theta) \tag{13}$$

(*b*) *The cylinder is insulated* : ($K \neq 0$).

Since the cylinder is insulated the total charge on the cylinder per meter is zero. Thus the previous system is to be modified as follows :

In both cases the potential of the cylinder is constant. Thus we conclude that the shape of the charge distribution must be similar within an additive constant η. This constant is such that,

$$\int_o^{2\pi} (\sigma + \eta)\ .\ (1\ a\ d\ \theta) = zero \tag{14}$$

But by Gauss's law we have that,

$$\int \sigma\ (a\ d\ \theta) = -\lambda$$

Thus (14) gives,

$$\eta \int_o^{2\pi} a\ d\ \theta = \lambda \tag{15}$$

Thus $\eta = \lambda/2\pi\, a$. From this we conclude that in addition to the charge $-\lambda$ at A', we must have another λ at the center O. The surface charge density at any point $P\ (a, \theta)$ is that of (13) plus $\lambda/2\pi\, a$, this gives,

$$\sigma = (\lambda/\pi)\ (a-b \cos \theta)\ /\ (a^2+b^2-2b\ a \cos \theta) \tag{16}$$

The potential at any external point $P\ (R, \theta)$ is that of (1) plus $(\lambda/2\pi\ \epsilon_0)$ $\log R$.

$$V = k + (\lambda/2\pi\ \epsilon)\ [\log r - \log r' + \log R] \tag{17}$$

This is to be zero at certain reference $P\ (R, \theta) = P_o\ (R_o, \theta_o)$

$$0 = k + (\lambda/2\pi\ \epsilon) \log\ (r\ R_o/r')$$

This gives $k = (\lambda/2\pi\ \epsilon) \log\ (r'/rR_o)$ at $P = P_o$. Thus the potential at any external point is,

$$V = (\lambda/4\pi\ \epsilon_o) \log\ (G\ H_o/G_o\ H) \tag{18}$$

where G_o, H_o are G and H at $R = R_o$, $\theta = \theta_o$ and,

$$G = R^2\ (R^2+b^2-2b\ R \cos \theta)$$

$$H = R^2 + (a^4/b^2) - (2a^2\ R/b) \cos \theta$$

45. *In both cases of the previous problem, investigate the existence of the line of no electrification on the surface of the conducting cylinder.*

The line of no electrification is that line which lies on the surface and divides the surface between the positively and negatively charged regions.

When the cylinder is earthed, equation (13) of previous problem gives,

$$\sigma = -(\lambda/2\pi\ a)\ (b^2-a^2)\ /\ (a^2+b^2-2a\ b \cos \theta) \tag{1}$$

It is clear that σ is always negative since the numerator is always negative and the denominator is always positive since a^2+b^2 is always greater than $2a\ b\ cos\ \theta$ for any real θ since

$$a^2+b^2-2a\ b\ cos\ \theta \geq a^2+b^2-2a\ b\ = (a-b)^2$$

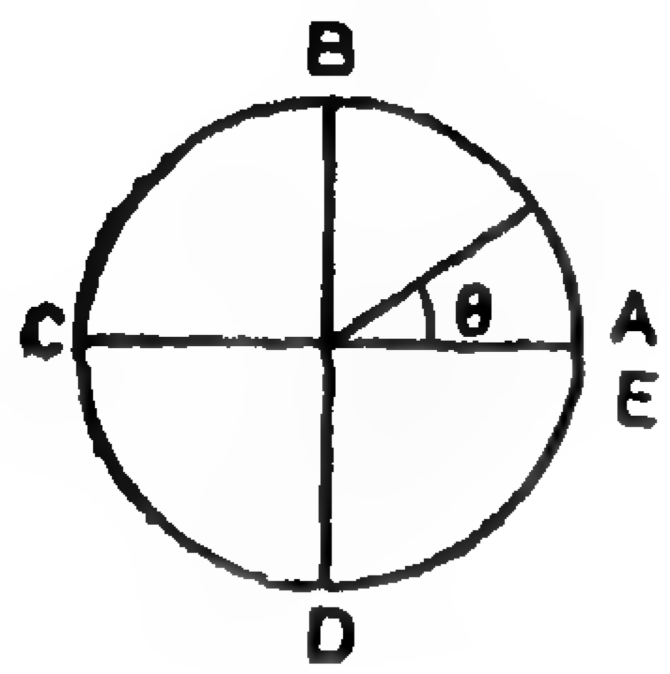


Fig. 9.54. a

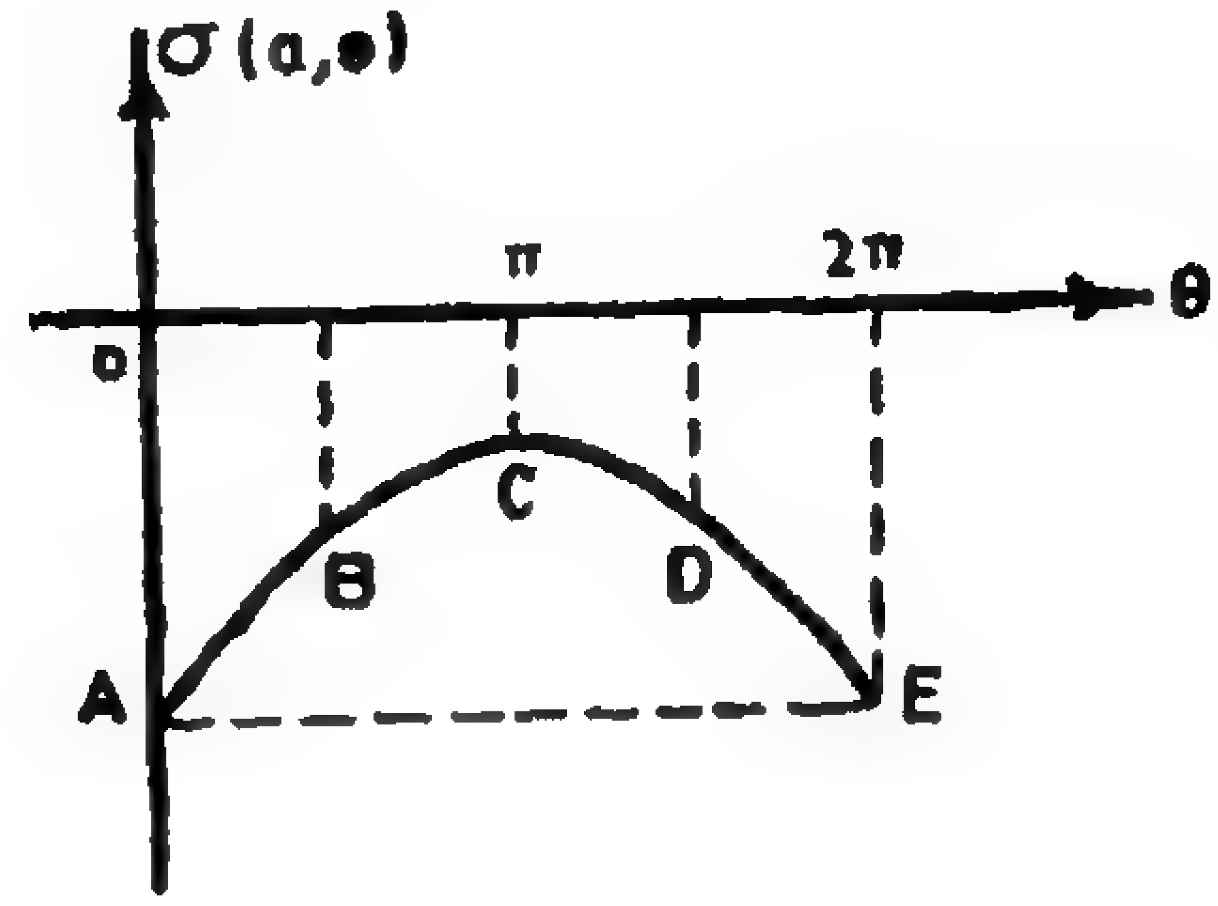


Fig. 9.54. b

Thus in the case of earthed cylinder there is no line of no electrification. Fig. 9.54 a,b shows the variation of the induced surface charge with θ

$$\sigma_A = \sigma E = -\lambda\ (b+a)\ /\ [2\pi . a\ (b-a)]$$

$$\sigma_B = \sigma D = -\lambda\ (b^2-a^2)\ /\ [2\pi\ a\ (b^2+a^2)]$$

$$\sigma C = -\lambda/2\pi\ a$$

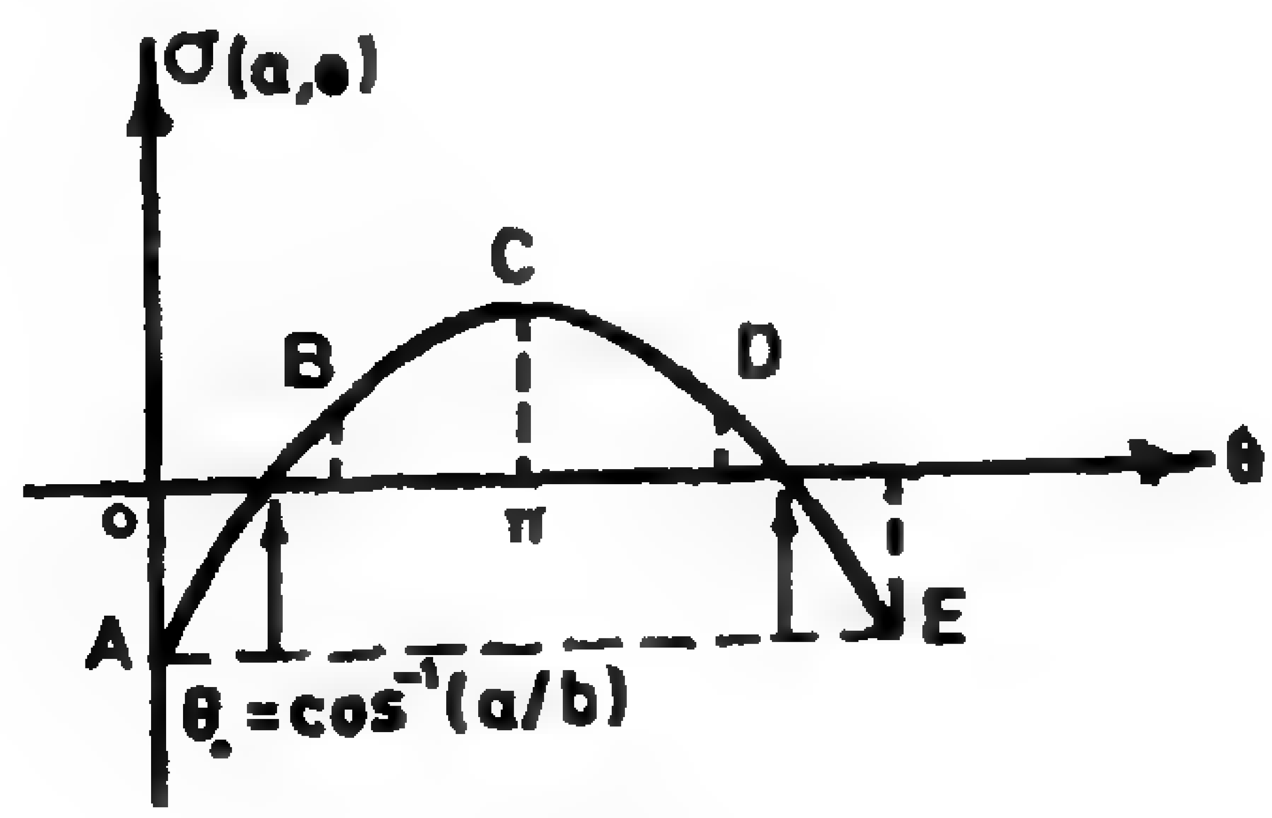


Fig. 9.55.

In the case of insulated cylinder the surface density is (equation (16) problem 9.44),

$$\sigma = \lambda (a-b \cos \theta) / \pi (a^2+b^2-2a b \cos \theta) \tag{2}$$

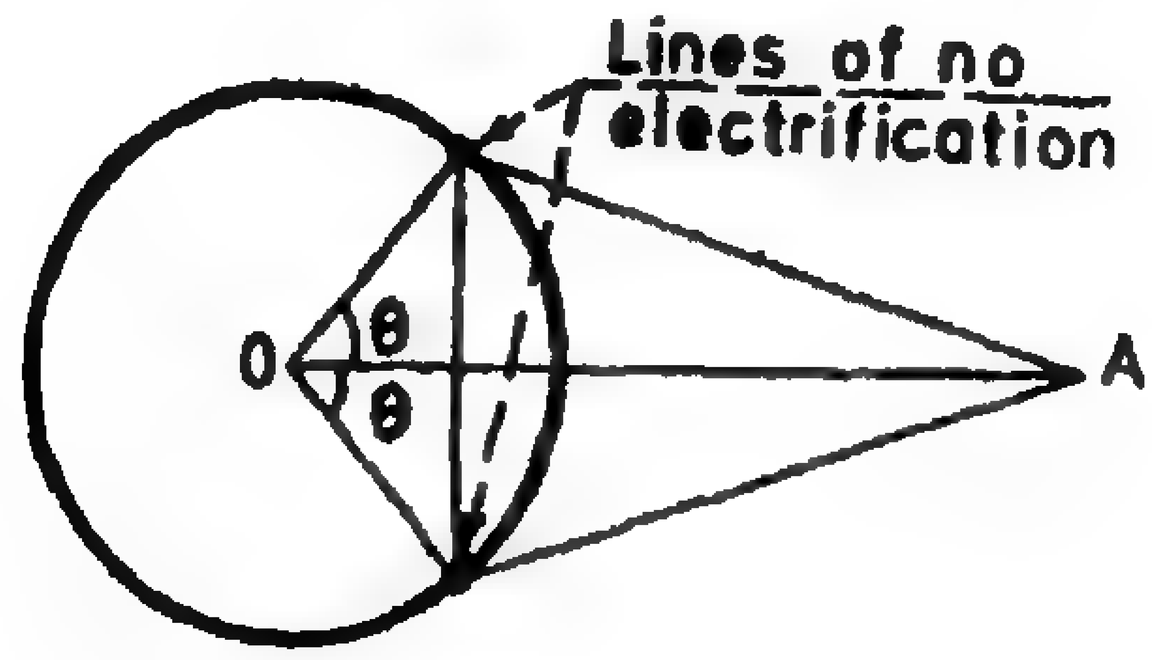


Fig. 9.56.

This is zero when $a = b \cos \theta_0$, *i.e.* when $\theta = \theta_0 = \pm \arccos (a/b)$. This gives the equation of the line of no electrification, (Fig. 9.56). The relation between σ and θ is sketched in Fig. 9.55.

From equation (2),

$$\sigma_A = \sigma = -\lambda/\pi (b-a)$$

$$\sigma_B = \sigma_D = \lambda a/(b^2+a^2)$$

$$\sigma_C = \lambda/\pi (b+a)$$

46. *Two long thin parallel wires with charge densities $\pm\lambda$ are at distance x apart. They are symmetrically enclosed by a thin hollow earthed conducting cylinder of radius R. Show that the electric force between the wires is zero if $x = 0.98\ R$.*

The shield can be removed and replaced by two image charges $+\lambda$, $-\lambda$ at distances $2R^2/x$ on both sides of the cylinder as shown in Fig. 9.57. The force on the line charge at A per unit length is,

$$F = \lambda^2/2\pi\epsilon_0 x - \lambda^2/2\pi\epsilon_0 [2R^2/x + x/2] -$$

$$\lambda^2/2\pi\epsilon_0 [2R^2/x - x/2]$$

$$(\lambda^2/2\pi\epsilon_0) [(16R^4 - 16R^2x^2 - x^4) / x (16R^4 - x^4)]$$

in the direction OA. There is an equal but opposite force on the wire **B.** An optimum value of x can be obtained for minimum force between the two wires. Equating F to zero we get,

$$16R^4 - 16R^2x^2 - x^4 = 0$$

solving for x^2 we get

$$x^2 = (-8+4\sqrt{5})\, R^2, \qquad (-8-4\sqrt{5})\, R^2$$

since x is real, only the positive sign will be considered,

$$x \quad (4\sqrt{5}-8)^{1/2} R = 0.98\, R.$$

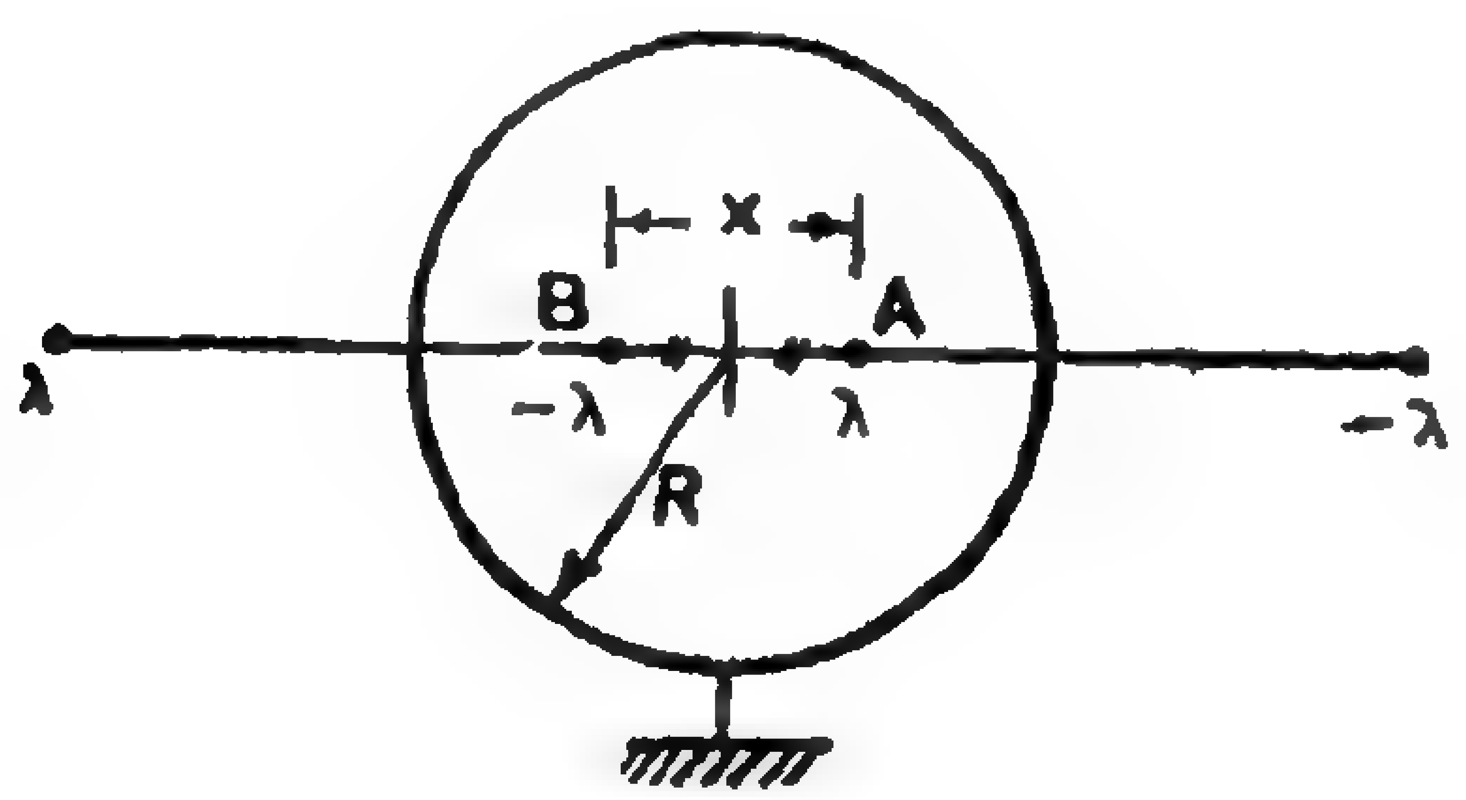


Fig. 9.57.

47. *The radii of two infinitely long conducting cylinders of circular cross sections, are a and b and the distance between the centers is c. The potential difference between the cylinders is V_0 volts. Obtain an expression for the charge density and for the mechanical force exerted on one cylinder by the other per unit length. Solve using the method of images.*

We know that the equipotential surfaces due to two infinite line charges of opposite densities are cylinders. Our problem is to define the linear charge densities which provide the potential difference $V_0 = V_b - V_a$, and the positions of these charge densities.

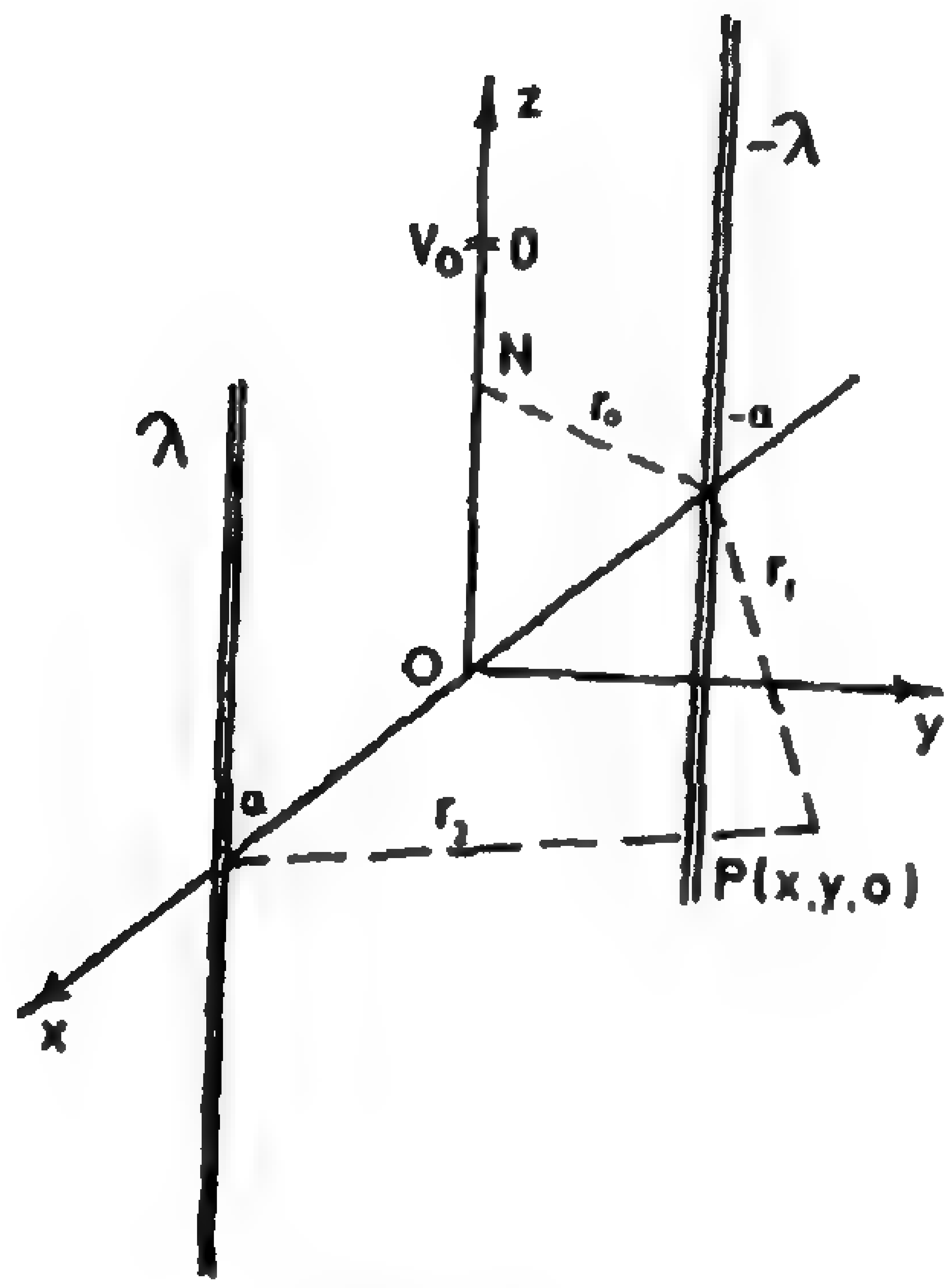


Fig. 9.58.

First consider two line charges with opposite charge densities shown in Fig. 9.58. Let the reference point whose distance from the line is r_0 be taken equidistant from both lines. The potential due to the positively charged line is,

$$V_+ = (\lambda/2\pi\epsilon) \log (r_0/r_2)$$

while that for the negatively charged line is

$$V_- = (\lambda/2\pi\epsilon) \log (r_0/r_1)$$

The total potential is,

$$V_0 = V_+ - V_- = (\lambda/2\pi\epsilon) \log (r_1/r_2) \tag{1}$$

The equation of the equipotential surfaces can be obtained from,

$$r_1/r_2 = exp\,(2\pi\, \epsilon V_o/\lambda) = k$$

substituting for r_1 and r_2,

$$r_1 = [(x+a)^2 + y^2]^{1/2}$$

$$r_2 = [(x-a)^2 + y^2]^{1/2}$$

after simple manipulation we get,

$$(x-x_k)^2 + y^2 = R^2_k \tag{2}$$

where

$$x_k = (k^2+1)\, a/(k^2-1) \tag{3}$$

$$R_k = 2k\, a/\,|\, k^2-1\,| \tag{4}$$

which is the equation of a family of circles in the parameter k. It is important to note that,

$$x^2_k - R^2_k = [(k^2+1)^2 - 4k^2]\, a^2/(k^2-1)^2 = a^2$$

Then

$$x_k^{\,2} = R^{\,2} + a^2 \tag{5}$$

Now referring to Fig. 9.59, we can write,

$$x_a^2 = a^2 + a^2 \tag{6}$$

$$x_b^2 = b^2 + a^2 \tag{7}$$

Hence

$$x_a^2 - x_b^2 = (c - x_b)^2 - x_b^2 = a^2 - b^2$$

with

$$x_a + x_b = c$$

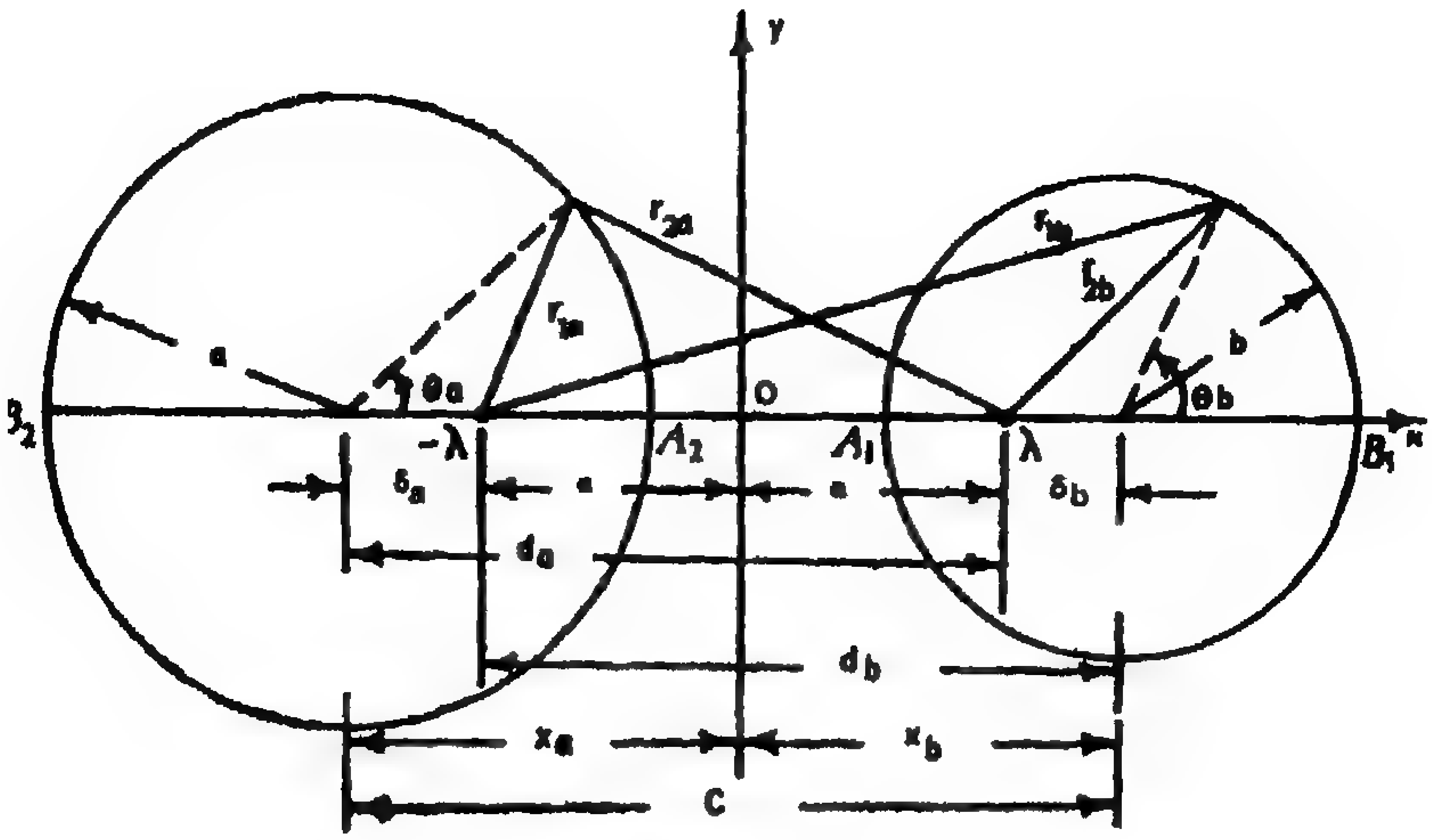


Fig. 9.59.

(13)

Solving for x_a and x, we get,

(14) $$x_a = (c^2+a^2-b^2) / 2c \qquad (8)$$

$$x_b = (c^2+b^2-a^2) / 2c \qquad (9)$$

These two equations determine the position [of the coordinate axes, which specify the equations of the equipotentials. Now we have to calculate the distance a of the line charges to the coordinate axes. Using equations (6) and (8) we have,

$$a^2 = [(c^2+a^2-b^2)^2 / 4c^2] - a^2$$

Then,

(15) $$a = (1/2c)\ [(a+b+c)\ (a+c-b)(a-c+b)\ (a+c-b)]^{1/2} \qquad (10)$$

and as,

(16) $$\delta_a = x_a - a$$

$$\delta_b = x_b - a$$

we have,

$$\delta_a = (1/2c)\,(c^2+a^2-b^2) - [(c^2+a^2-b^2)^2/4c^2-a^2]^{1/2} \tag{11}$$

$$\delta_b = (1/2c)\,(c^2+a^2-b^2) - [(c^2+a^2-b^2)^2/4c^2-b^2]^{1/2} \tag{12}$$

These two equations define the positions of the two line charges.

If V_a and V_b are, respectively, the potentials of the cylinders of radii a, and b we have,

$$V_a = (\lambda/2\pi\epsilon)\ log\ (r_{1a}/r_{2a}) = (\lambda/2\pi\epsilon)\ log\ k_a$$

$$V_b = (\lambda/2\pi\epsilon)\ log\ (r_{1b}/r_{2b}) = (\lambda/2\pi\epsilon)\ log\ k_b$$

Then,

$$V_o = V_b - V_a = (\lambda/2\pi\epsilon)\ log\ (k_b/k_a)$$

and

$$k_b/k_a = exp\ (2\pi\epsilon V_o/\lambda)$$

From (3) and (4)

$$x_a/a = (k_a^2+1)\ /\ (k_a^2-1)$$

$$a/2k_a a = 1/\ |\ k_a^2-1\ |$$

Hence

$$x_a/a = (a/2k_a\ a)\ (k_a^2+1)$$

$$a\ k_a^2-2\ x_a\ k_a + a = 0$$

This gives,

$$k_a = |\ x_a\ |/a - [(x_a/a)^2 - 1]^{1/2}$$

Similarly,

$$k_b = |\ x_b\ |/b + [(x_b/b)^2 - 1]^{1/2}$$

Then the value of λ can be obtained

$$\lambda = 2\pi \epsilon V_0 / \log (k_b / k_a) \tag{17}$$

where $k_a < 1$ and $k_b > 1$ in accordance with Fig. 9.59.

Determination of the charge density.

The electric field at the surface of cylinder b can be obtained as in problem 9.44 Here a charge $-\lambda$ is at distance d_b from the center of cylinder b so that,

$$E_b = (\lambda/2\pi \epsilon b) \left[(d_b{}^2 - b^2) / (b^2 + d_b{}^2 - 2\, b\, d_b \cos \theta_b)\right] \tag{18}$$

in the normal direction to its surface, where $d_b = b^2/\delta_b$. The charge on the surface of the cylinder b is,

$$\sigma_b = \epsilon E_b = (\lambda/2\pi b)[(d_b{}^2 - b^2) / (b^2 + d_b{}^2 - 2bd_b \cos \theta_b)] \tag{19}$$

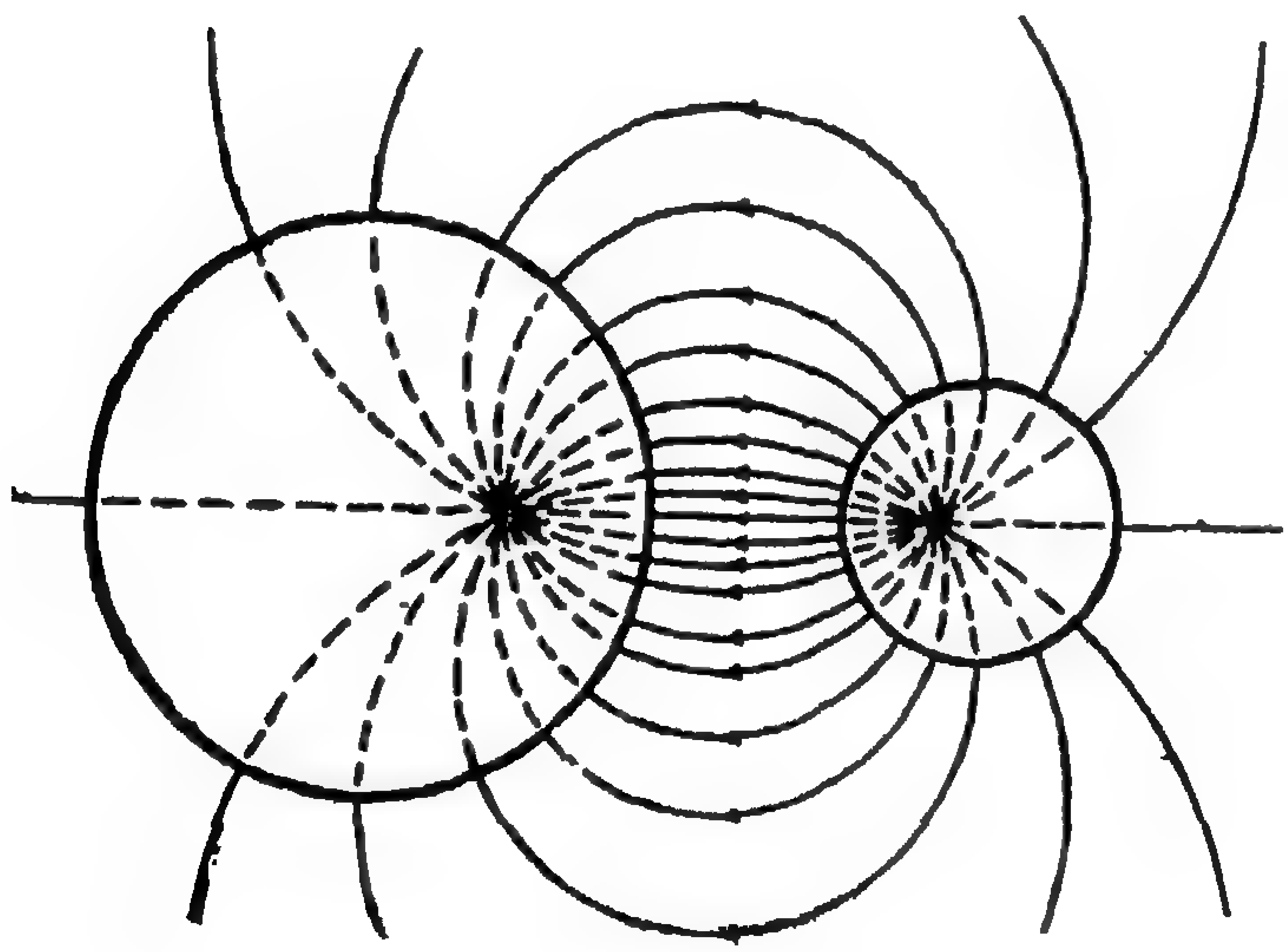


Fig. 9.60.

It is evident that the total charge per unit length on cylinder b is $\lambda^{+} = \lambda$. This can be checked by integrating σ_{-} over its surface per unit length. Similarly for cylinder a we have,

$$E_a = -(\lambda/2\pi\epsilon a)\ [(d_a^2 - a^2) / (d_a^2 + a^2 - 2ad_a \cos\theta_a)] \qquad (20)$$

$$\sigma_a = -(\lambda/2\pi a)\ (d_a^2 - a^2) / (d_a^2 + a^2 - 2ad_a \cos\theta_a) \qquad (21)$$

The total charge per unit length on this cylinder is $-\lambda$ as can be checked by integrating σ_a over a unit length. The charge distributions on the two cylinders a and b are shown in Fig. 9.60. The associated lines of forces are also drawn.

Determination of the mechanical force.

Since the region of application of the method of images is outside the two cylinders, the two cylinders can be substituted by two infinite line charges $\pm\lambda$. The force per unit positive charge located on the line $+\lambda$ is $(\lambda/2\pi\epsilon d)$ in the negative x-direction (Fig. 9.61) so that the force on an element $\lambda\ dz$ is,

$$dF = -(\lambda^2 dz/2\pi\epsilon d)\ \mathbf{a}_x$$

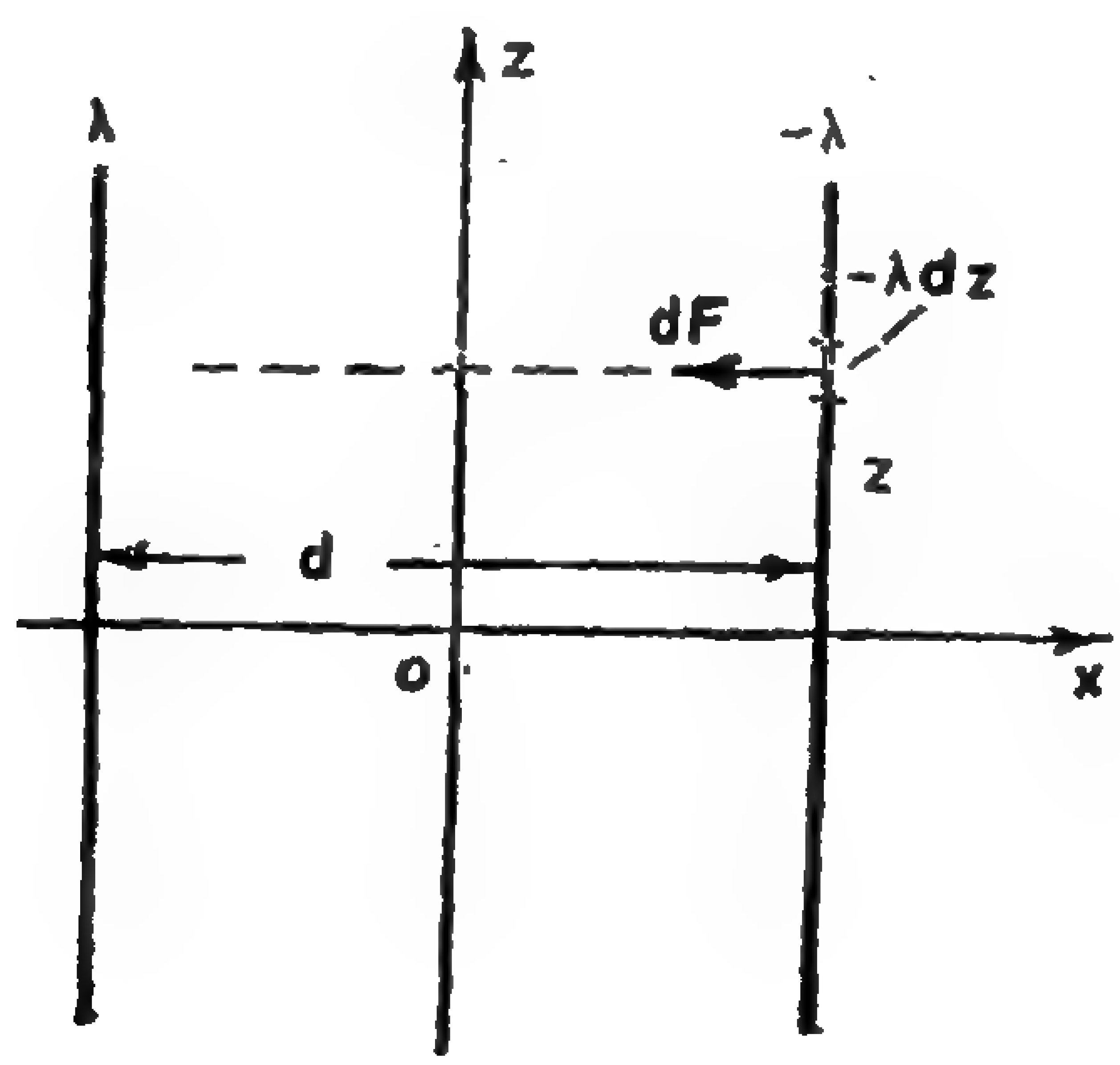


Fig. 9.61.

Thus the force per unit length between the two cylindrical conductors is,

$$\mathbf{F} = -(\lambda^2/2\pi\epsilon d)\,\mathbf{a}_x$$

of negative sign since the force is attractive.

48. *For the two cylindrical conductors given in the previous problem, find an expression for the capacitance per unit length. Also consider the case when the two conductors are eccentric and then proceed to the limit to find the capacitance of a conductor in front of an infinite plane conductor.*

Equation (13) of the previous problem gives the capacitance per unit length of the two conductors

$$C = \lambda/(V_b - V_a) = 2\pi\epsilon_o / \log\,(r_{1b}\, r_{2a} / r_{2b}\, r_{1a}) \tag{1}$$

To determine the capacitance C interms of the geometry of the system, consider points A_1 and B_1 in Fig. 9.59,

$$\begin{aligned} r_{1b}/r_{2b} &= (c-b-\delta_a) / (b-\delta_b) \\ &= (c+b-\delta_a) / (b+\delta_b) \end{aligned}$$

From this relation we conclude that,

$$r_{1b}/r_{2b} = (c-\delta_a) / b = b/\delta_b \tag{2}$$

Also at A_2 and B_2 we have,

$$\begin{aligned} r_{1a}/r_{2a} &= (a-\delta_a) / (c-a-\delta_b) \\ &= (a+\delta_a) / (c+a-\delta_b) \end{aligned}$$

Again we conclude that,

$$r_{1a}/r_{2a} = a / (c-\delta_b) = \delta_a/a \tag{3}$$

Thus (1) becomes,

$$C = 2\pi\epsilon_o / \log\,(ab/\delta_a\delta_b \tag{4}$$

From (2) and (3) we have,

$$b^2+\delta_a\delta_b = \delta_b c$$

$$a^2+\delta_a\delta_b = \delta_a c$$

Multiplying the two equations we get,

$$u^2-2\beta u+1 = 0 \qquad (5)$$

where, $u \quad ab/\delta_a\delta_b$, and $\beta = (c^2-a^2-b^2) \quad 2ab$

Solving for u we get,

$$u_1, u_2 = \beta \pm (\beta^2-1)^{1/2}$$

Now $\delta_a < a$ and $\delta_b < b$ we conclude that u must be greater than unity. But from (5) we see that $u_1 u_2 = 1$; thus only the positive sign is considered. The capacitance between the two conductors is,

$$C \quad 2\pi\epsilon_0 / \log [\beta+(\beta^2-1)^{1/2}] \qquad (6)$$

$$2\pi\epsilon_0 / \operatorname{arccosh} \beta \qquad (7)$$

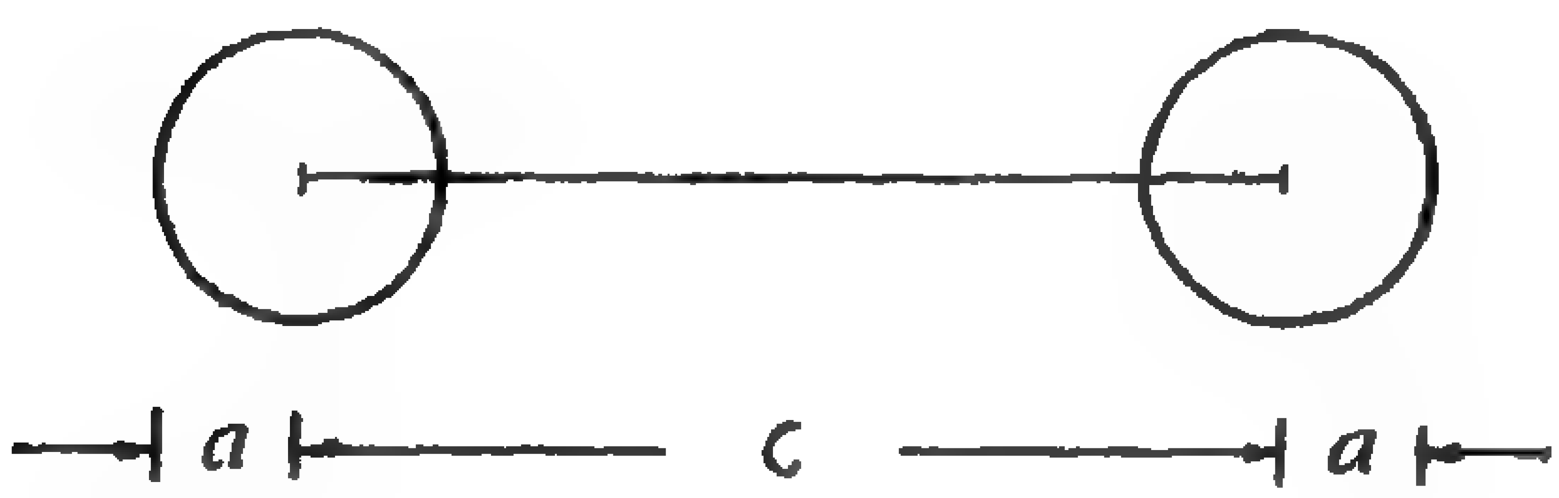


Fig. 9.62

If the two cylinders have equal radii, (Fig. 9.62),

$$\beta = (c^2/2a^2)-1$$

so that

$$C = \pi\epsilon_0 / \log \{ c/2a + [(c/2a)^2-1]^{1/2} \} \qquad (8)$$

$$= \pi\epsilon_0 / \operatorname{arccosh}(c/2a)$$

when $a << c$

$$C = \pi \epsilon_0 / \log (c/a) \tag{9}$$

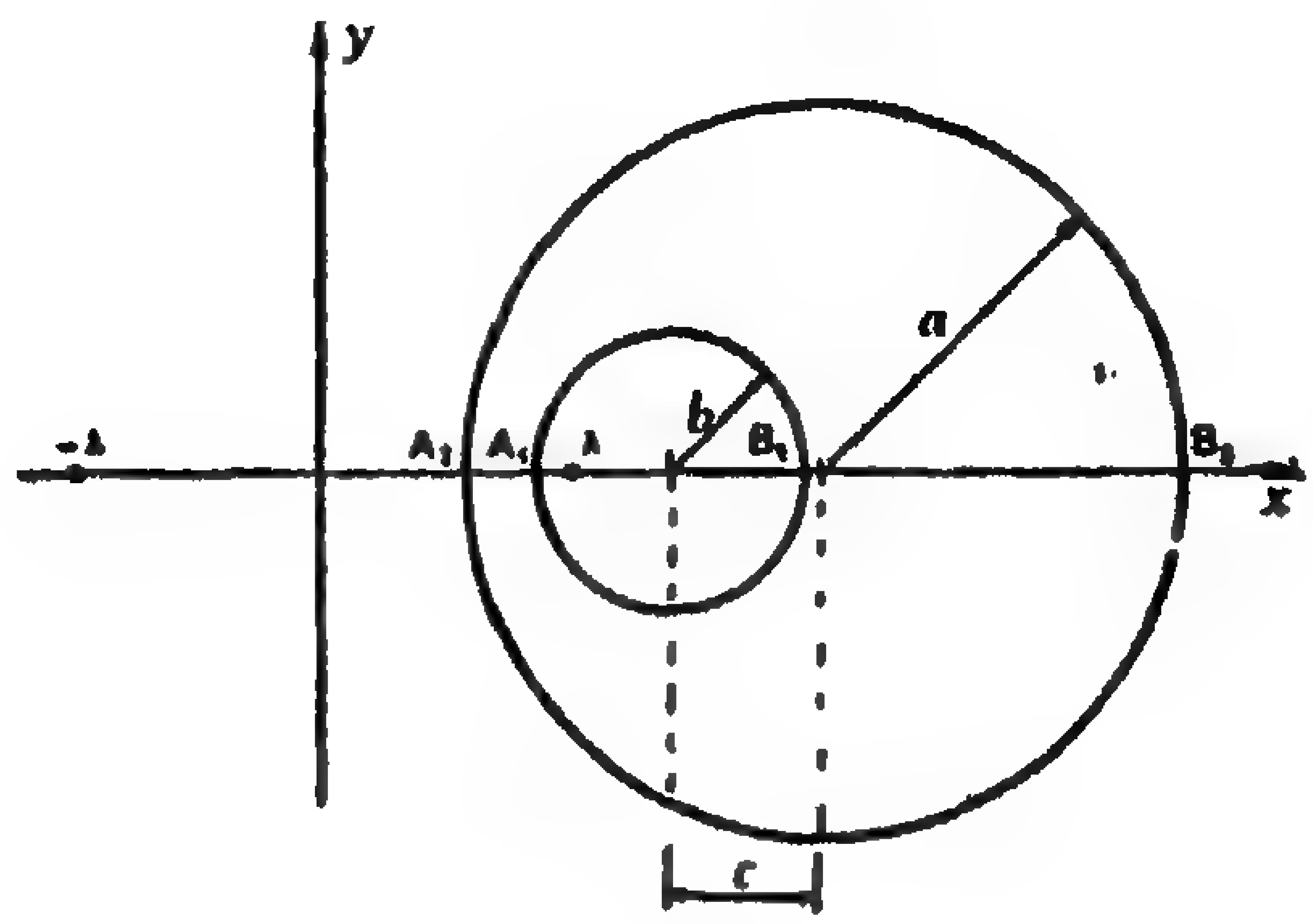


Fig. 9.63.

If the two cylinders are eccentric with their centers at distance c, (Fig. 9.63), we have the same equations as above only with $-\beta$ instead of β,

$$-\beta = (a^2+b^2-c^2) / 2ab = \beta'$$

as can be shown by deriving (2)—(5) again. Hence,

$$C = 2\pi \epsilon_0 / \log [\beta' + \sqrt{\beta'^2-1}]$$

$$= 2\pi \epsilon_0 / \operatorname{arccosh} [(a^2+b^2-c^2) / 2ab]$$

If the two cylinders are concentric, $c = 0$ so that,

$$C = 2\pi \epsilon_0 / \log (a/b)$$

which is a well known result.

If the radius b tends to ∞ the problem is reduced to that of a cylindrical conductor at distance h from an infinite plane, (Fig. 9.64). Here we have,

$$\beta \simeq (c^2-b^2) \;/\; 2ab = (c-b)\,(c+b) \;/\; 2ab$$

As b tends to ∞ we have,

$$c-b = h$$

$$c+b = (b+h)+b = 2b+h$$

Thus,

$$\beta \quad h\,(2b+h) \;/\; 2ab \simeq h/a$$

and

$$C = 2\pi\epsilon_0 \;/\; \log \left\{ h/a + [(h/a)^2-1]^{1/2} \right\} \qquad F/m$$

Again if $h >> a$,

$$C = 2\pi\epsilon_0 \;/\; \log\,(2h/a) \qquad F/m$$

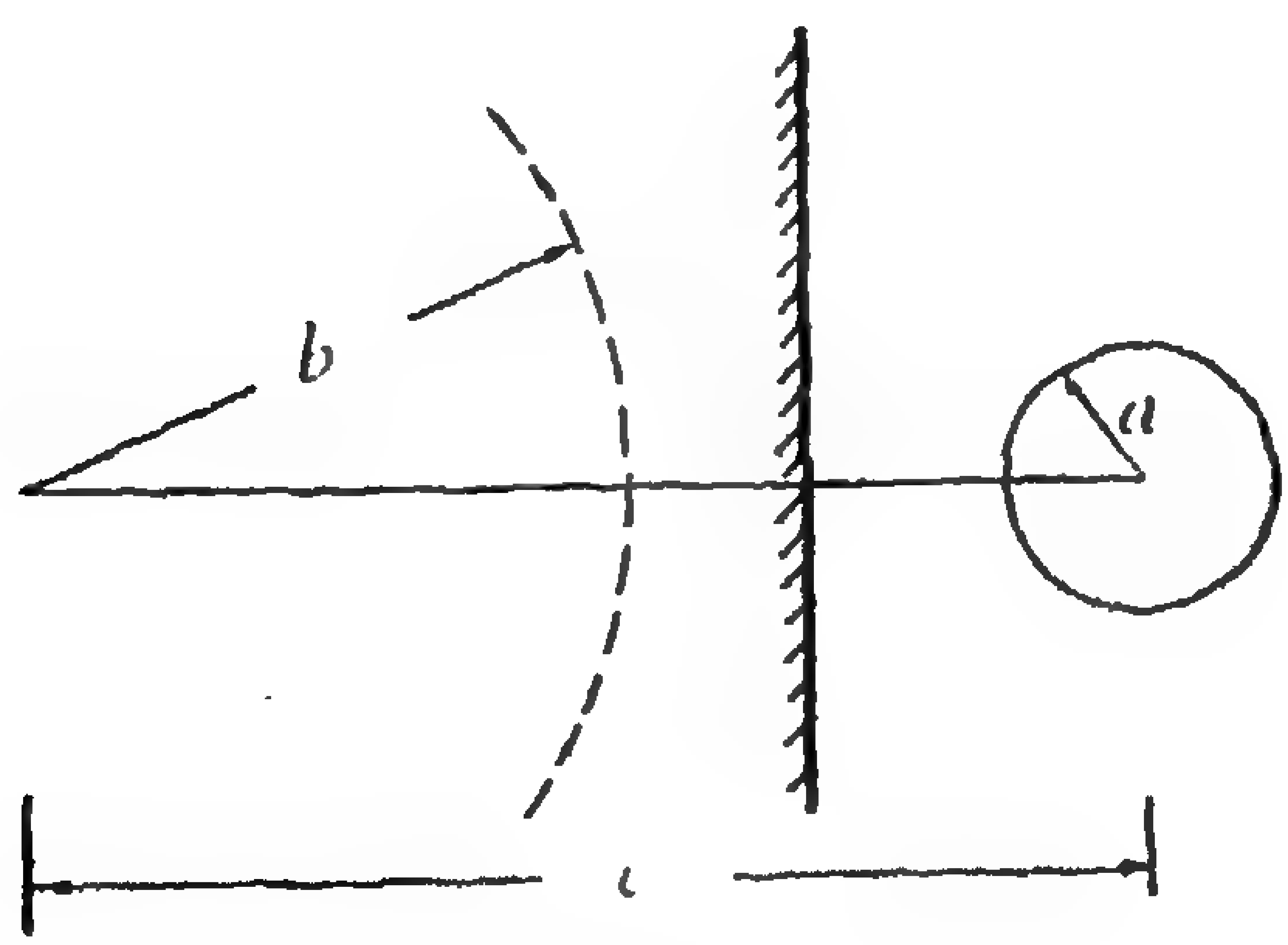


Fig. 9.64.

49. *A conducting sphere of radius a is placed at a distance d from an infinitely earthed conducting plane. Find an exact expression for the capacity of the sphere, then proceed to the limit when $d >> a$.*

Let the conducting sphere be kept at potential V_0. In the absence of the conducting plane this potential is equivalent to a charge $Q = 4\pi \epsilon_0 a\ V_0$. In the presence of the conducting plane the equivalent charge Q does not satisfy the constant potential condition on the plane, and we need search for a system of image charges to represent the problem correctly.

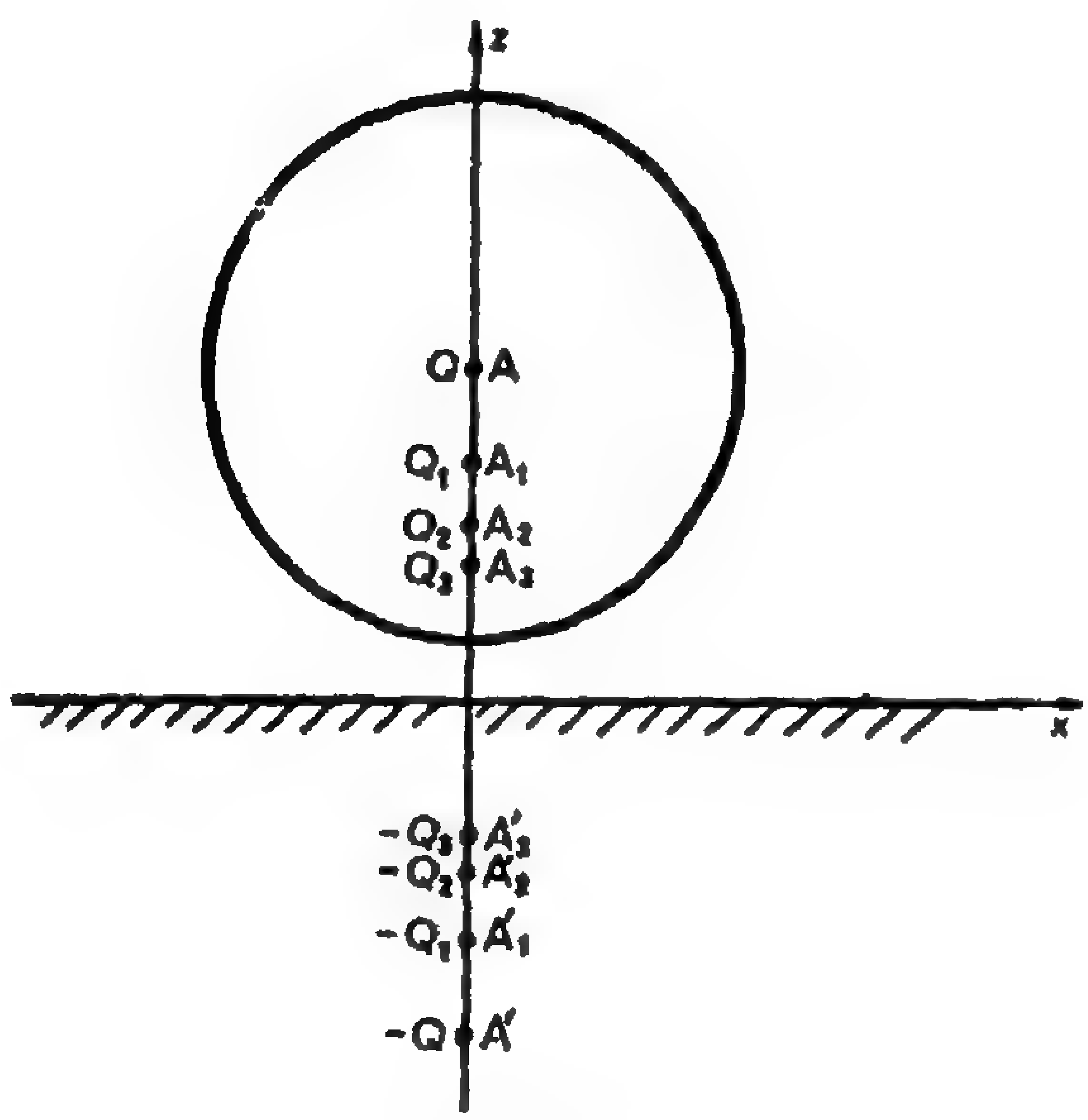


Fig. 9.65.

The charge Q at A, (Fig. 9.65), has an image $-Q$ in the plane at distance d below the plane. This charge has an image on the sphere $Q_1 = Q\ a/2d$ at point A_1 where $AA_1 = a^2/2d$. Again Q_1 has an image in the plane $-Q_1$ at distance $d - AA_1$ below it. The image

of $-Q_1$ on the sphere is $Q_2 = +Q_1\, a \,/\, (d+OA_1)$ at $AA_2 = a^2 \,/\, (d+OA_1)$. Repeating this process n times we get,

$$Q_{n+1} = Q_n\, a \,/\, (d+OA_n) \tag{1}$$

$$AA_{n+1} = d - OA_{n+1} = a^2 \,/\, (d+OA_n) \tag{2}$$

Using equation (1)

$$Q_n/Q_{n-1} = a \,/\, (d+OA_{n-1}) \tag{3}$$

$$Q_n/Q_{n+1} = (d+OA_n) \,/\, a \tag{4}$$

From (2) we have that,

$$a \,/\, (d+OA_{n-1}) = (d-OA_n) \,/\, a \tag{5}$$

Using (3)—(5) we get a difference equation satisfying the charge Q_n,

$$Q_n/Q_{n-1} + Q_n/Q_{n+1} = 2d/a$$

$$1/Q_{n+1} - (2d/a) \,/\, Q_n + 1/Q_{n-1} = 0$$

or,

$$S_{n+1} - (2d/a)\ S_n + S_{n-1} = 0 \tag{6}$$

with

$$S_n = 1/Q_n$$

This represents a linear difference equation of constant coefficients. A trial solution to this equation is,

$$S_n = (m)^n$$

Substituting in (6) we get,

$$m^2 - (2d/a)\ m + 1 = 0 \tag{7}$$

The roots of this auxiliary equation are,

$$m_1 = d/a + [(d/a)^2 - 1]^{1/2} \tag{8}$$

$$m_2 = d/a - [(d/a)^2 - 1]^{1/2} \tag{9}$$

It is clear from (7)—(9) that $m_1 = 1/m_2$. Thus the general solution of (6) is,

$$S_n = A\, m_1^{\,n} + B\, m_1^{\,-n} = 1/Q_n$$

$$= A \exp(n \log m_1) + B \exp(-n \log m_1)$$

$$1/Q_n = C \cosh[n \operatorname{arccosh}(d/a)] + E \sinh[n \operatorname{arccosh}(d/a)] \quad (10)$$

The constants C and E can be obtained from the fact that,

for $n = 0$, $1/Q_n = 1/Q$

$n = 1$, $1/Q_1 = 2d/aQ$

These give, $C = 1/Q$ and,

$$2d/aQ = (1/Q)(d/a) + E \sinh[\operatorname{arccosh}(d/a)]$$

$$E = (d/aQ) / \sinh[\operatorname{arccosh}(d/a)]$$

$$= (1/Q) \coth[\operatorname{arccosh}(d/a)] \quad (11)$$

Substituting for C and E in (10) we get,

$$Q_n = Q \sinh[\operatorname{arccosh}(d/a)] / \sinh[(n+1) \operatorname{arccosh}(d/a)]$$

The total charge on the conducting sphere is,

$$q_t = \sum_0^{\infty} Q_n$$

The capacitance of the sphere is,

$$C = q_t/V_o = 4\pi \epsilon_o\, a \sinh[\operatorname{arccosh}(d/a)]$$

$$\sum_0^{\infty} 1/\sinh[(n+1) \operatorname{arccosh}(d/a)] \quad (12)$$

when d/a is large,

$$\operatorname{arccosh}(d/a) \simeq \log(2d/a)$$

$$\sinh[\operatorname{arccosh}(d/a)] \simeq d/a$$

Thus,

$$C = 4\pi \epsilon_0 a \,(d/a) \sum_{0}^{\infty} (a/2d)^{n+1}$$

$$= 4\pi \epsilon_0 a \,(2d/a) \left\{ (a/2d) \,/\, [1-a/2d] \right\}$$

$$= 4\pi \epsilon_0 a \,[2d \,/\, (2d-a)] \tag{13}$$

which is the capacitance between two pheres, each of radius a, at a large distance $2d$ apart.

As d tends to infinity, we have the case of a single sphere in space,

$$C = 4\pi \epsilon_0 a \tag{14}$$

50. ***A small metal ball of mass m and charge Q is suspended from a string of length L. If the point of fixation O, Fig. 9.66, is at a distance H (>> L) from a horizontal conducting plane at zero potential, calculate the period of oscillation.***

At any general position the energy of the pendulum is,

$$W = \tfrac{1}{2} mv^2 + mg\,L\,(1-\cos\theta) + U \tag{1}$$

where U is the electric energy due to the existence of the charge Q in the field of its image $-$Q.

$$U = QV = Q\,[-Q \,/\, 8\pi \epsilon_0 \,(H - L\cos\theta)]$$

$$\simeq -(Q^2/8\pi \epsilon_0 H)\,[1+(L/H)\cos\theta]$$

with $v = L\dot{\theta}$ equation (1) becomes,

$$W = \tfrac{1}{2} mL^2 \dot{\theta}^2 + mg\,L\,(1-\cos\theta) - (Q^2/8\pi \epsilon_0 H)$$

$$[1 + (L/H)\cos\theta]$$

Solving for $\dot{\theta}$ we get,

$$\dot{\theta} = (A + B\cos\theta)^{1/2} \tag{2}$$

where,

$$A = 2W/mL^2 - 2g / L + Q^2/8\pi\epsilon_0\, mL^2 H$$

$$B = 2g/L + Q^2/4\pi\epsilon_0\, L H^2$$

The maximum amplitude of oscillation occurs at $\theta = \alpha$ where $\theta^{\cdot} = 0$. Thus α = arccos $(-A/B)$. The period of oscillation T is four times the time required to swing from $\theta = 0$ to α

$$d\theta/dt = (A + B \cos\theta)^{1/2}$$

$$T = 4 \int_0^{\alpha} d\theta \,/\, (A+B\cos\theta)^{1/2}$$

$$= (4/B^{1/2}) \int_0^{\alpha} d\theta \,/\, (\cos\theta - \cos\alpha)^{1/2} \qquad (3)$$

This integral can be transformed to a standard form by introducing a new variable ϕ such that,

$$\sin\phi = \sin(\theta/2) \,/\, \sin(\alpha/2)$$

$$\cos\psi\, d\phi = \tfrac{1}{2} \cos(\theta/2)\, d\theta \,/\, \sin(\alpha/2)$$

Thus (3) reduces to,

$$T = 4/(\tfrac{1}{2}B)^{1/2} \int_0^{\pi/2} d\phi \,/\, [1-k^2 \sin^2\phi]^{1/2} = 2\pi/\omega$$

which is a complete elliptic integral of the first kind

where $k = \sin^2(\alpha/2)$.

$$T = (2\sqrt{2}\,\pi \,/\, \sqrt{B})\, [1+(1/4)\sin^2(\alpha/2)+(9/64)\sin^4(\alpha/2)+\ldots$$

$$= (2\pi \,/\, \sqrt{\tfrac{1}{2} B})\, [1 + (1/16)\, \alpha^2]$$

If the angle α is very small,

$$T = 2\pi / \sqrt{\tfrac{1}{2} B} \quad \textit{seconds}$$

$$= 4\pi / [4g/L + Q^2/2\pi \epsilon_0 LH^3]^{1/2}$$

This reduces to $2\pi \sqrt{l/g}$ if the conducting plane is absent or if $Q=0$.

Note that the above solution is valid only if W is considered constant.

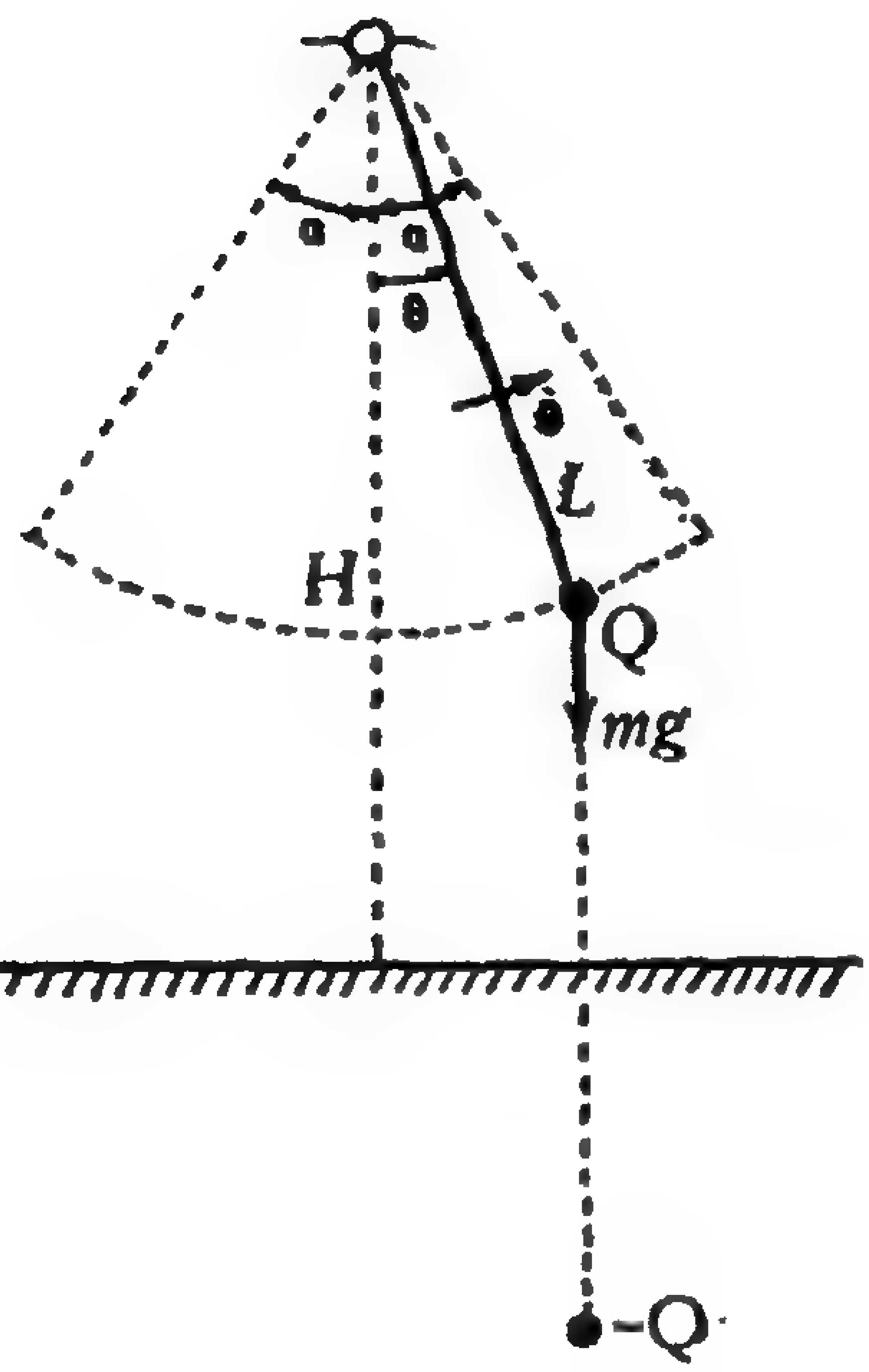


Fig. 9.66.

Application of the Method of Inversion to the Solution of Electrostatic Problems.

51. *Summarize carefully the fundamental geometrical theorems concerning the application of the method of inversion for the solution of static problems.*

These theorems can be summarized in the following steps;

(i) A circle is inverted into another circle unless it passes through the center of inversion. In this case it is inverted into a straight line.

(ii) Inversion preserve the angle between lines or surfaces.

(iii) A sphere is inverted into another sphere if it does not pass through the center of inversion. If it passes through this center it is inverted into a plane.

(iv) If a and b are the radii of two inverted spheres, h and g are the distances of their centers from the center of inversion O (Fig. 9.67) we have the relations,

$$a/b = h/g = (h^2-a^2) / R^2 = R^2 / (g^2-b^2)$$

where h^2-a^2, g^2-b^2 are the powers of the two spheres with respect to O.

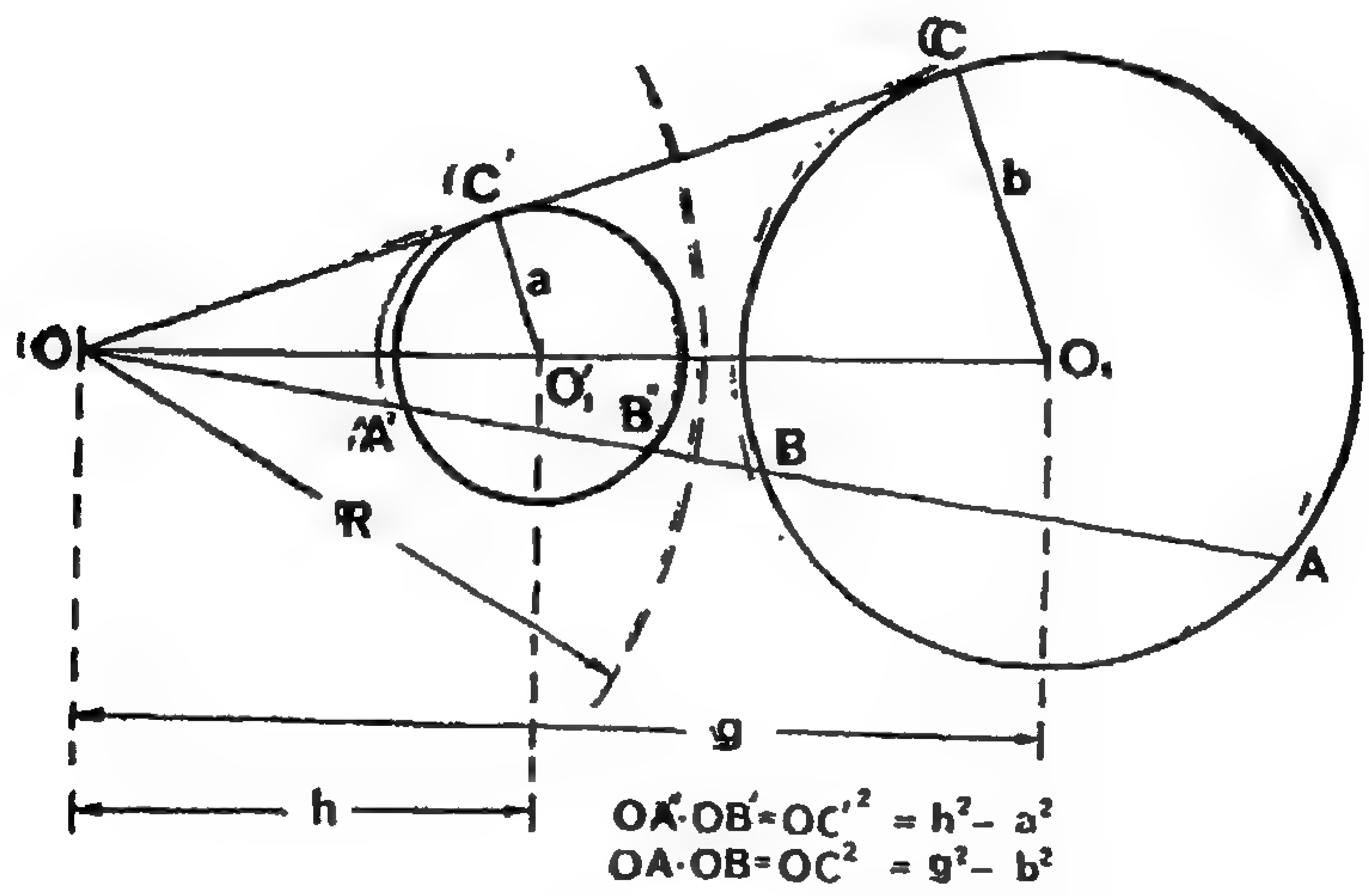


Fig. 9.67.

(v) The image of the center of inversion with respect to one of the spheres is the inverse point of the center of the other sphere. As shown in Fig. 9.68 O' is the image of O in the sphere O_1 and the inverse of O_2 with respect to the sphere of radius R.

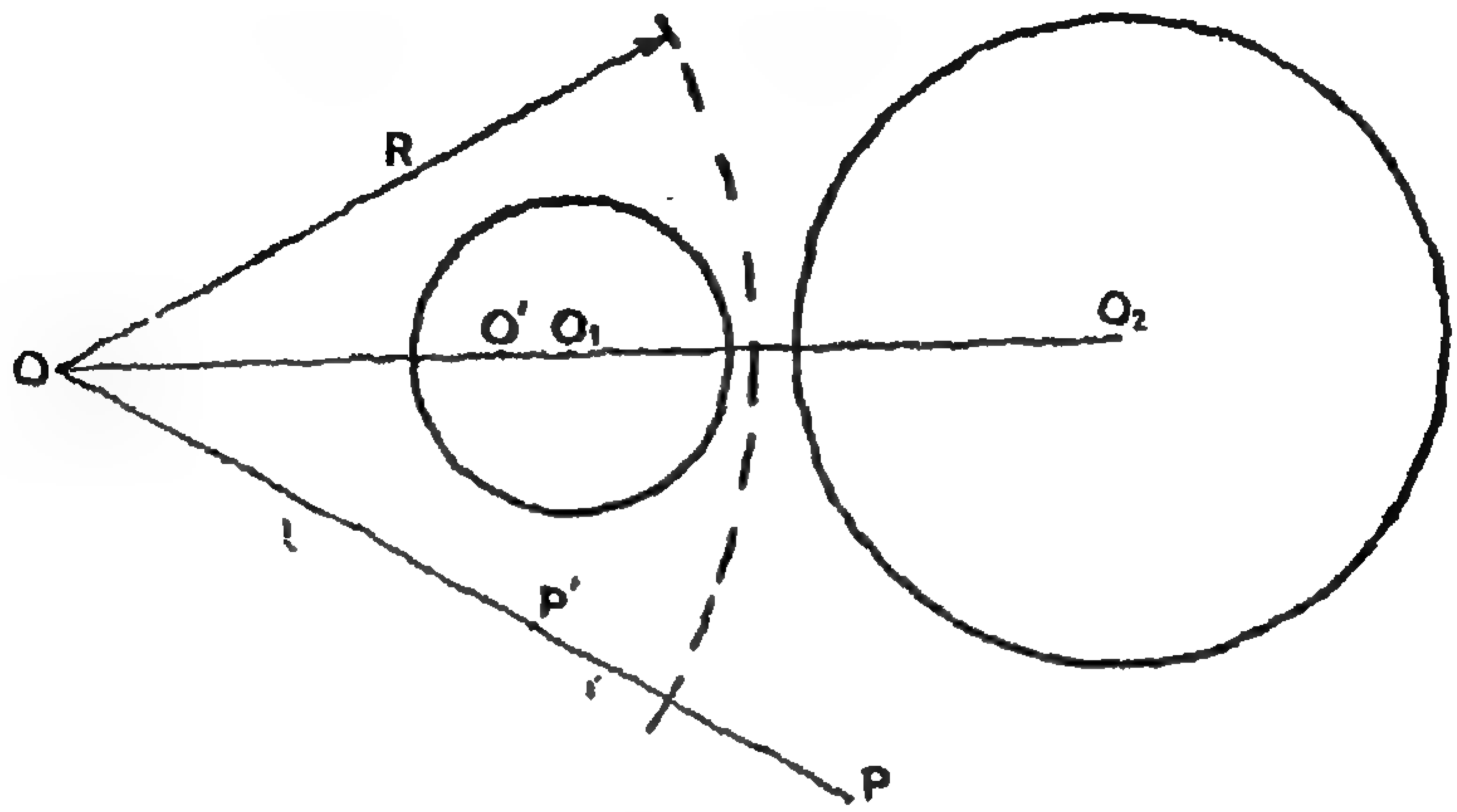


Fig. 9.68.

(vi) Every circle which passes through a point and the image of that point with respect to a sphere cuts the sphere at right angles and vice versa.

(vii) If P and P' are inverse points, L and L', A and A', v and v' are length, area, volume and their corresponding inverses, we have that,

$$r'/r = L'/L = R^2/r^2 = r'^2/R^2$$

$$A'/A = R^4/r^4 = r'^4/R^4$$

$$v'/v = R^6/r^6 = r'^6/R^6$$

52. *Two planes intersect at an angle θ_0. Find the inverse of these planes with respect to a point between the two planes.*

As shown in Fig. 9.69 the inverse system is two spheres intersecting at angle θ_0. A circle C, center A, is transformed into a straight line since the circle passes through O. This straight line is the perpendicular bisector for OA'.

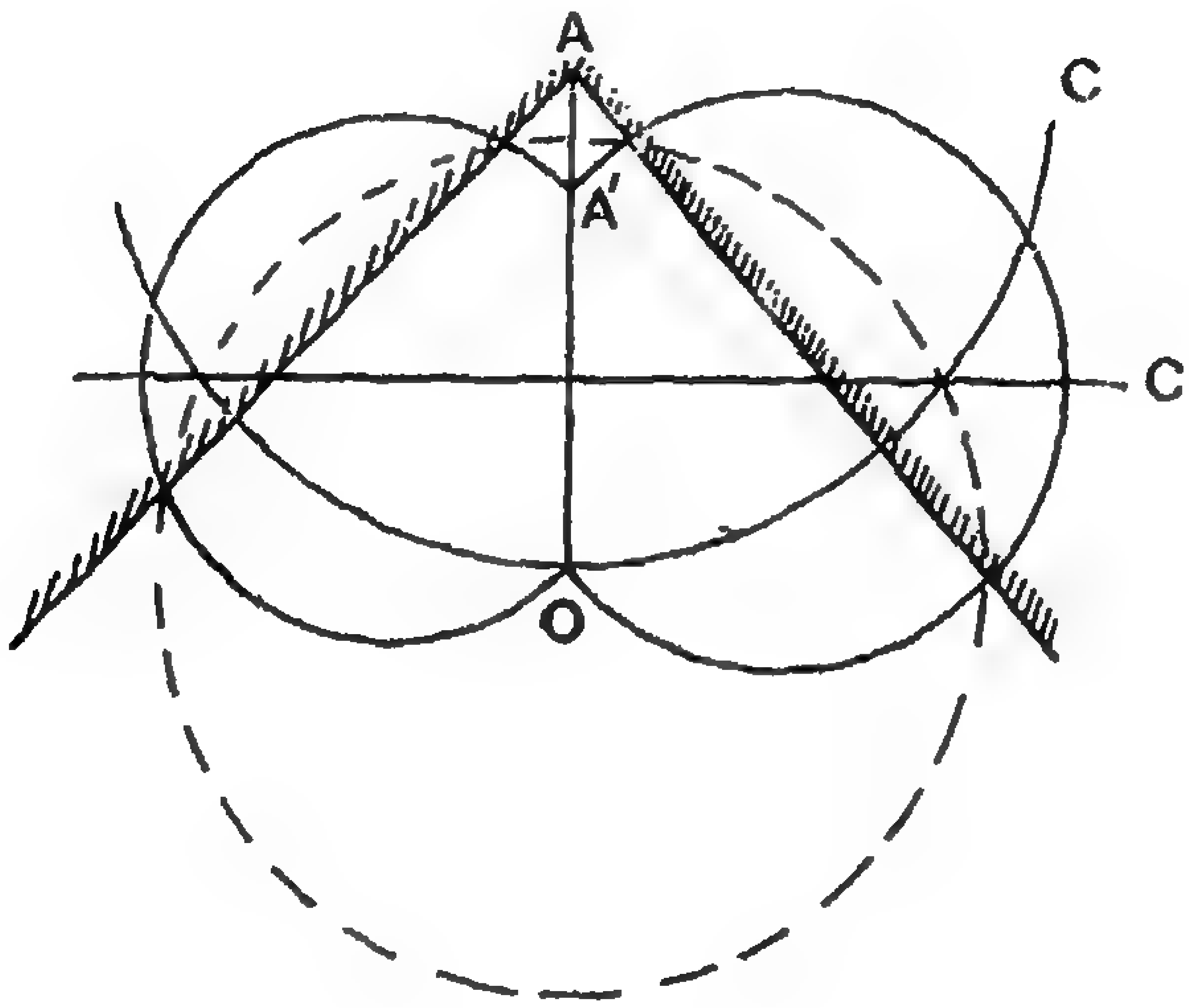


Fig. 9.69.

53. *The potential at point P (R, α, β) due to a volume distribution of charge density $\delta(r, \theta, \phi)$ at point A (r, θ, ϕ) is $V(R, \alpha, \beta)$. Prove the Thomson transformation that the potential at point P due to the inverse point of A with respect to a sphere center origin and radius K is $V' = (K/R)\, V(K^2/R, \alpha, \beta)$ This potential is the same as if we have a volume density $\delta'(r', \theta', \phi') = (K/r')^5\, \delta(K^2/r, \theta, \phi)$ at A'.*

The potential at point P due to the given distribution of charges is (Fig. 9.70),

$$V_p \quad (1/4\pi\epsilon_0) \int_v [\delta(r, \theta, \phi) / AP]\, dv \tag{1}$$

where, $AP = (R^2+r^2-2rR \cos \psi)^{1/2}$. The Thomson transformation states that the potential at P due to the inverse system is,

$$V_p' = (K/R)\ V_p\ (K^2/R,\ \alpha,\ \beta) \tag{2}$$

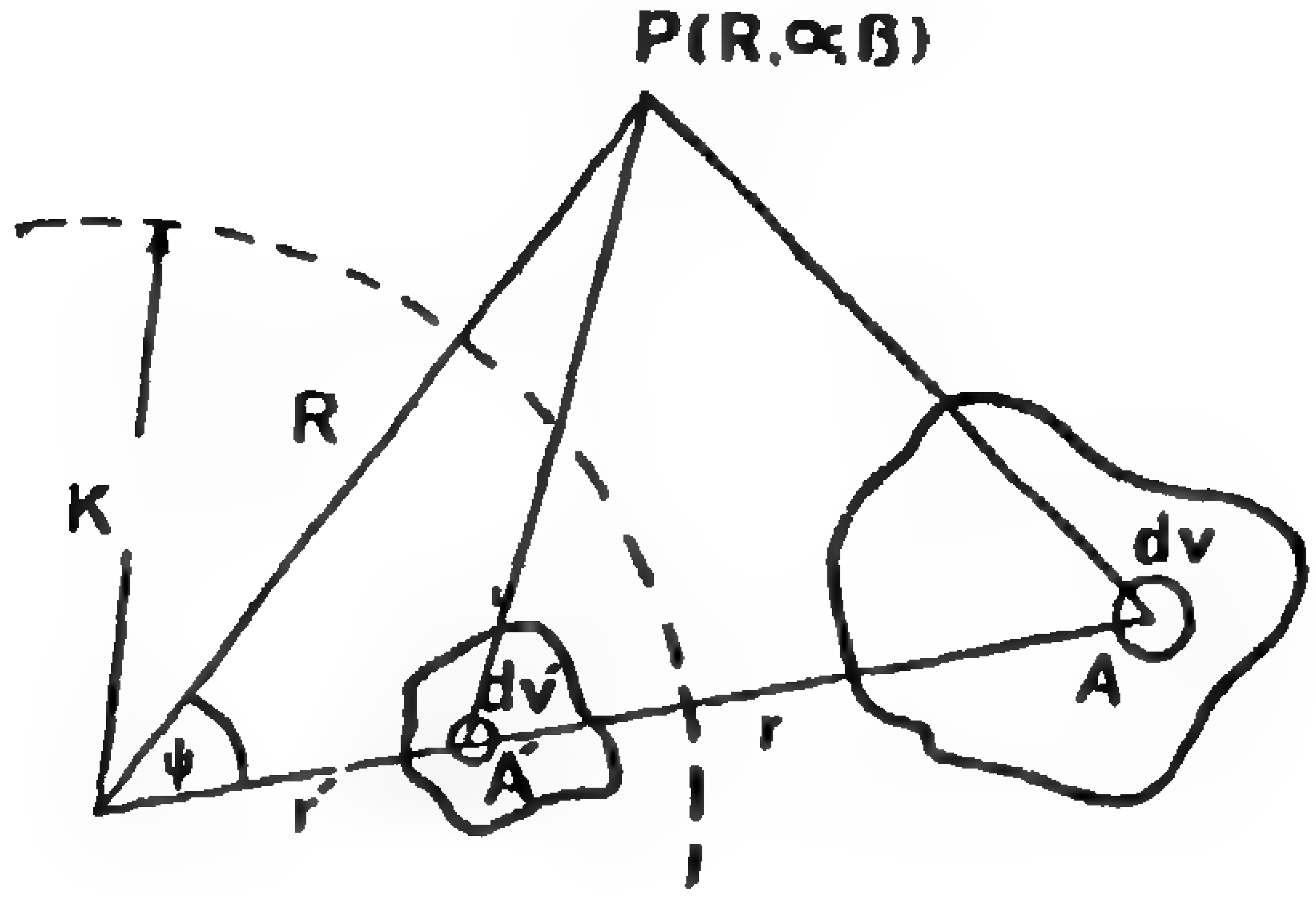


Fig. 9.70.

The potential V_p' can be obtained from (1) by transforming the integration in the right hand side from (r, θ, ϕ) to (r', θ', ϕ') where, $r' = K^2/r$, $\theta' = \theta$, $\phi' = \phi$. Using the results of problem 9.51, we have,

$$dv = (r/K)^6\ dv'$$

$$[r^2+K^4/R^2-2r\ (K^2/R)\cos\psi]^{1/2}$$

$$= (r/R)\ [R^2+K^4/r^2-2R\ (K^2/r)\cos\psi]^{1/2}$$

$$= (r/R)\ A'P$$

Thus by substituting from (1) in the right hand side of (2) and using the above relations we get,

$$V_p' = (1/4\pi\epsilon_0) \int_{v'} (r/K)^5\ \delta\ (K^2/r',\ \theta,\ \phi)\ dv'/A'P$$

with $r = K^2/r'$ the required result follows immediately.

Note : The inverse of point charge, linear and surface distributions of charges are,

$$q' = (K/'r)\ q$$

$$\lambda'(r', \theta, \phi) = (K/r')\ \lambda\ (K^2/r', \theta, \phi)$$

$$\sigma'(r', \theta, \phi) = (K/r')^3\ \sigma\ (K^2/r', \theta, \phi)$$

with $\qquad V'/V = r/K = K/r'.$

54. *A point charge Q is placed at distance b from the center of an earthed conducting sphere of radius a. An infinite conducting plate is placed in contact with the sphere perpendicular to the line joining the sphere center to the point charge. Find the charge induced on each of the plate and the sphere.*

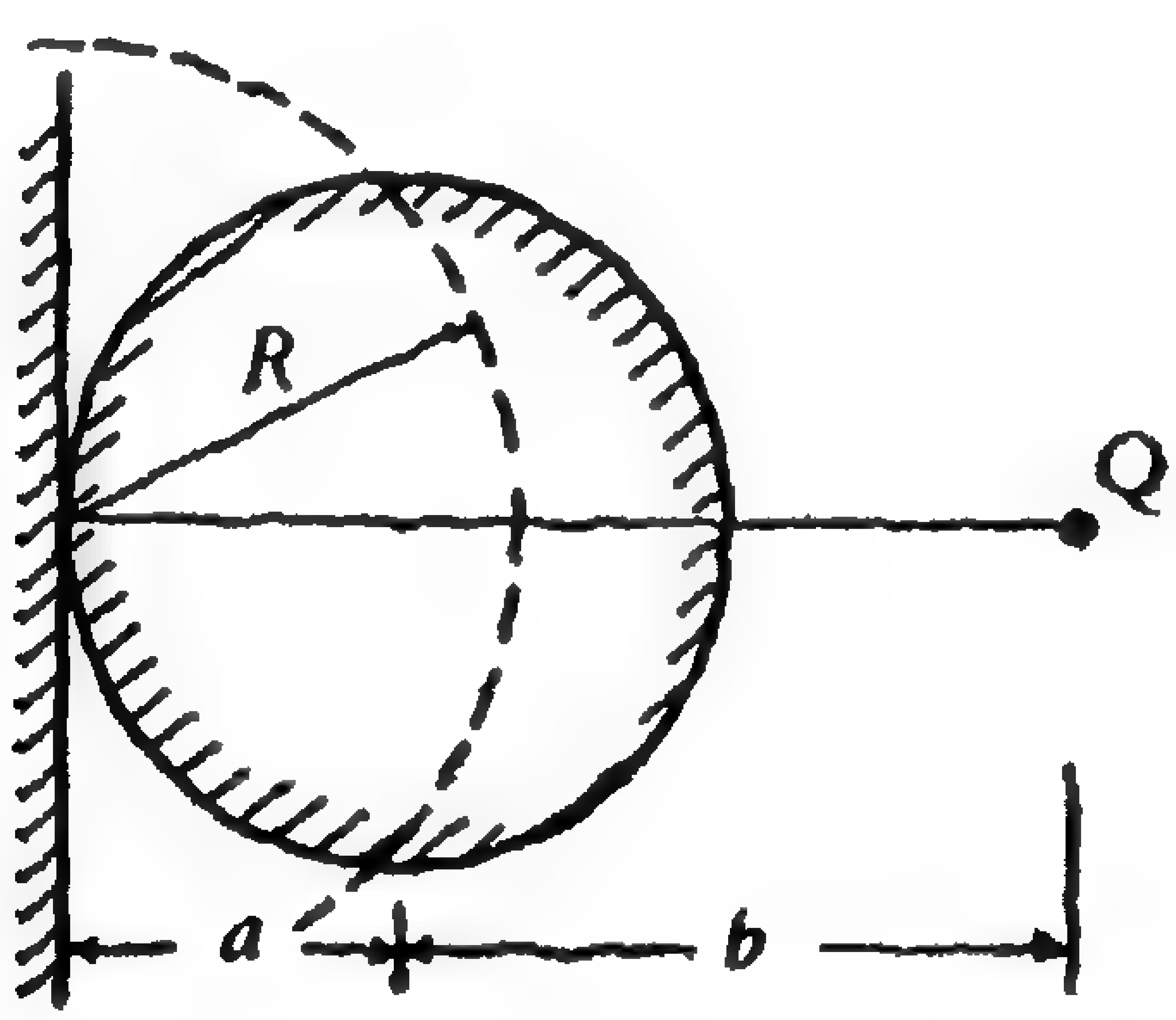


Fig. 9.71

Taking the point of contact (Fig. 9.71) as the center of inversion, the system is inverted into two parallel infinite conducting planes at ze ro potential (Fig. 9.72.) The point charge Q is inverted to q Q R/x_o

at the point $x_0 = R^2/(a+b)$, where R is the radius of inversion. The distance between the two planes is $d = R^2/2a$. The interior of the sphere is transformed into the region $x > d$. Thus the total charge on the sphere is the inverse of the total charges on the conducting plane at $x = d$. To the right of $x = d$ the positive charges are at $x = x_0 + 2nd$, $n = 1, 2, 3, \ldots$ while the negative charges are at $x = 2nd - x_0$, $n = 1, 2, 3, \ldots$ thus inside the sphere the image charges are at,

$$X_+ = R^2/(x_0+2nd), \text{ and } X_- = R^2/(2nd-x_0)$$

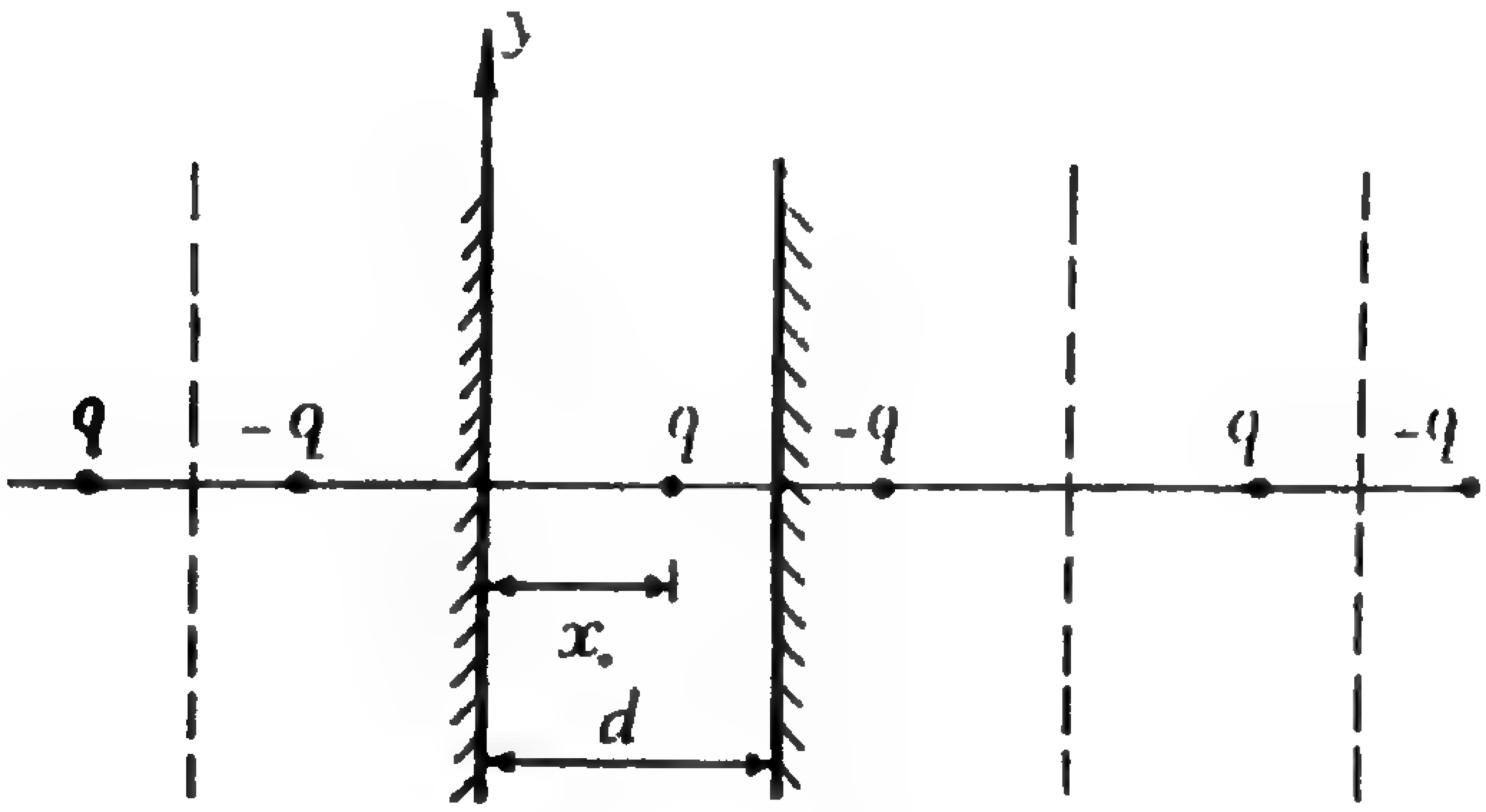


Fig. 9.72.

The total charge on the sphere is the inverse of the image charges at $x \geq d$,

$$Q_1 = \sum_1^{\infty} [qR/(x_0+2nd) - qR/(2nd-x_0)]$$

$$= \sum_1^{\infty} 2x_0Rq/(x_0^2-4n^2d^2)$$

$$= Q(a+b)\cdot\sum_1^{\infty} \theta/(\theta^2-n^2)$$

where $\theta = a/(a+b)$. From the theory of functions we have that,

$$\sum_{1}^{\infty} \theta/(\theta^2-n^2) = \tfrac{1}{2}\pi \cot(\pi\theta) - 1/2\theta$$

so that we have,

$$Q_1 = [\pi a Q/(a+b)] \cot [\pi a/(a+b)] - Q$$

All the lines of force out of Q must terminate on the conductor formed by the sphere and the plane in contact. The total charge on the plane is thus,

$$Q_2 = -Q - Q_1 = -[\pi a Q/(a+b)] \cot [\pi a/(a+b)]$$

55. *Show that if the surface density at any point on a conductor kept at potential V_0 is obtained, then the method of inversion can be used to solve the problem of a charge in front of the inverse conductor, this inverse conductor being earthed. Apply the above method to solve the problems of (i) a point charge at a distance b from the center of an earthed conducting sphere of radius a, a<b, and (ii) a point charge in front of an infinite earthed plane.*

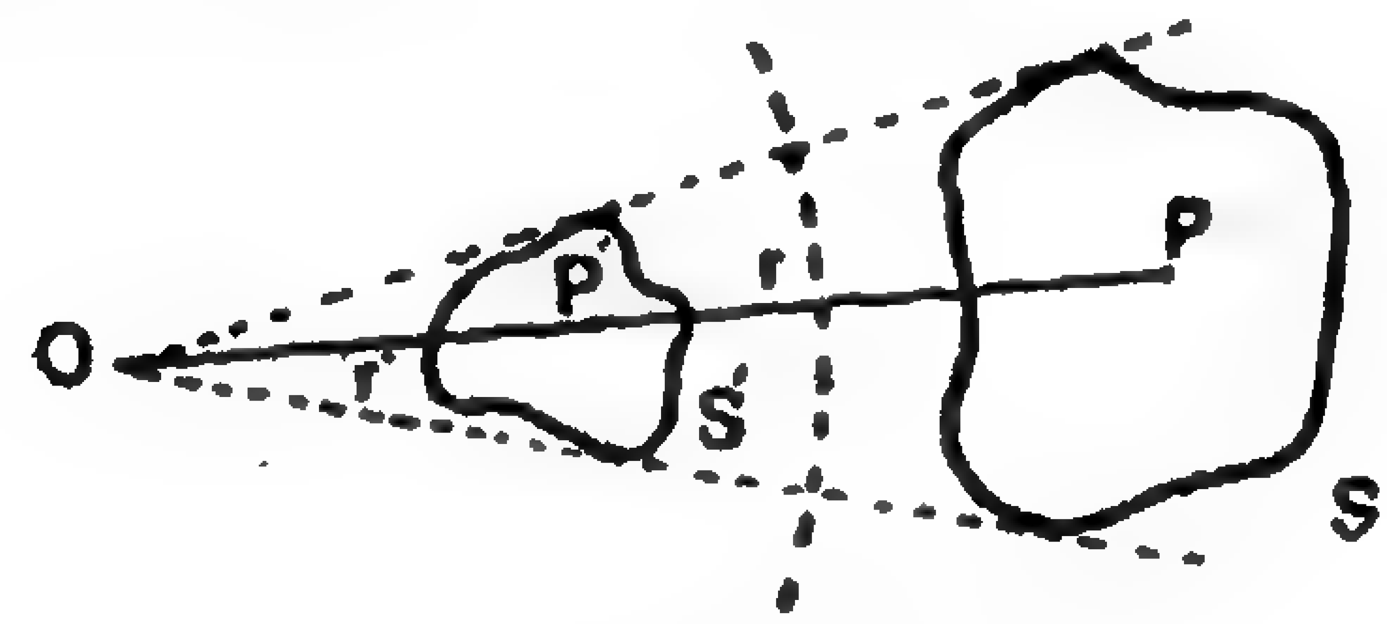


Fig. 9.73.

The point P (Fig. 9.73) lies on the surface S of the given conductor so that $V_p = V_0$. The potential of the inverse point P' is thus $V_p' = V_0\,(r/R)$. If in the inverse system a point charge $-Q$ is placed at the center of inversion, the total potential at P' becomes,

$$V_p' = V_0\,(R/r') - Q/4\pi\epsilon_0 r' \tag{1}$$

where $R=$ radius of inversion.

If we choose $Q = -4\pi \epsilon_0 V_0 R$ the potential will always be zero at any point on the inverse surface S'. Thus S' may represent the surface of an earthed conductor. Therefore the problem of a point charge in front of an earthed conductor can be solved if the inverse conductor is isolated and kept at potential V_0 and vice versa. If σ_p the surface charge density at any point on S is obtained, the surface density at the inverse point is,

$$\sigma_p' = \sigma_p (R/r')^3 \tag{2}$$

(i) Assume the conducting sphere to be raised to potential V_0 in the absence of any other charged conductors, the density of charge on the surface of the sphere is $\sigma_0 = \epsilon_0 V_0/a$. If the center of inversion is taken on the surface of the sphere and the radius of inversion $R = 2a$, the sphere is inverted into an infinite plane (Fig. 9.74). This plane can be kept at zero potential if in the inverted system a charge $Q = -4\pi\epsilon_0 V_0 R$ is placed at the center of inversion. The surface charge density at any point on the plane is thus,

$$\sigma = \sigma_0 (R/r)^3 = 2 R^2 \epsilon_0 V_0/r^3 \tag{3}$$

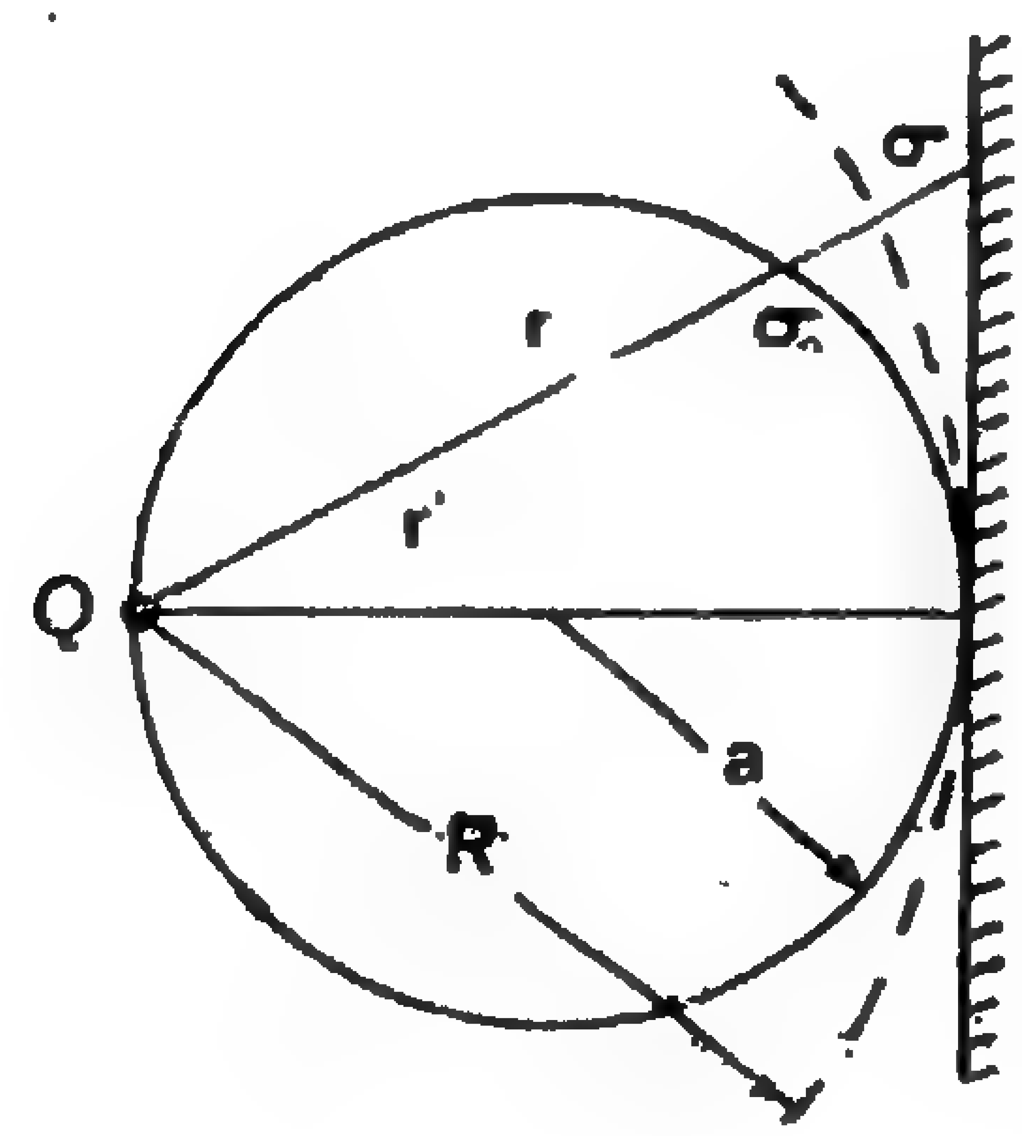


Fig. 9.74.

Substituting for $\epsilon_0 V_0 = -Q/4\pi R$ we get,

$$\sigma = -Q(2a) / 2\pi r^3 \tag{4}$$

This agrees with the solution using the method of images. (Problem 9.4).

(ii) If the center of inversion O is taken at distance b from the center of sphere, the inverted system is another sphere with center at h' from O and radius a', (Fig. 9.75). This sphere will be at zero potential if a charge $Q = -4\pi \epsilon_0 V_0 R$ is placed at O. The surface charge density at any point on the inverse sphere is,

$$\sigma' = \sigma_0 (R/r')^3 = \epsilon_0 V_0 R^3/ar'^3$$

$$= -QR^2/4\pi a r'^3 \tag{5}$$

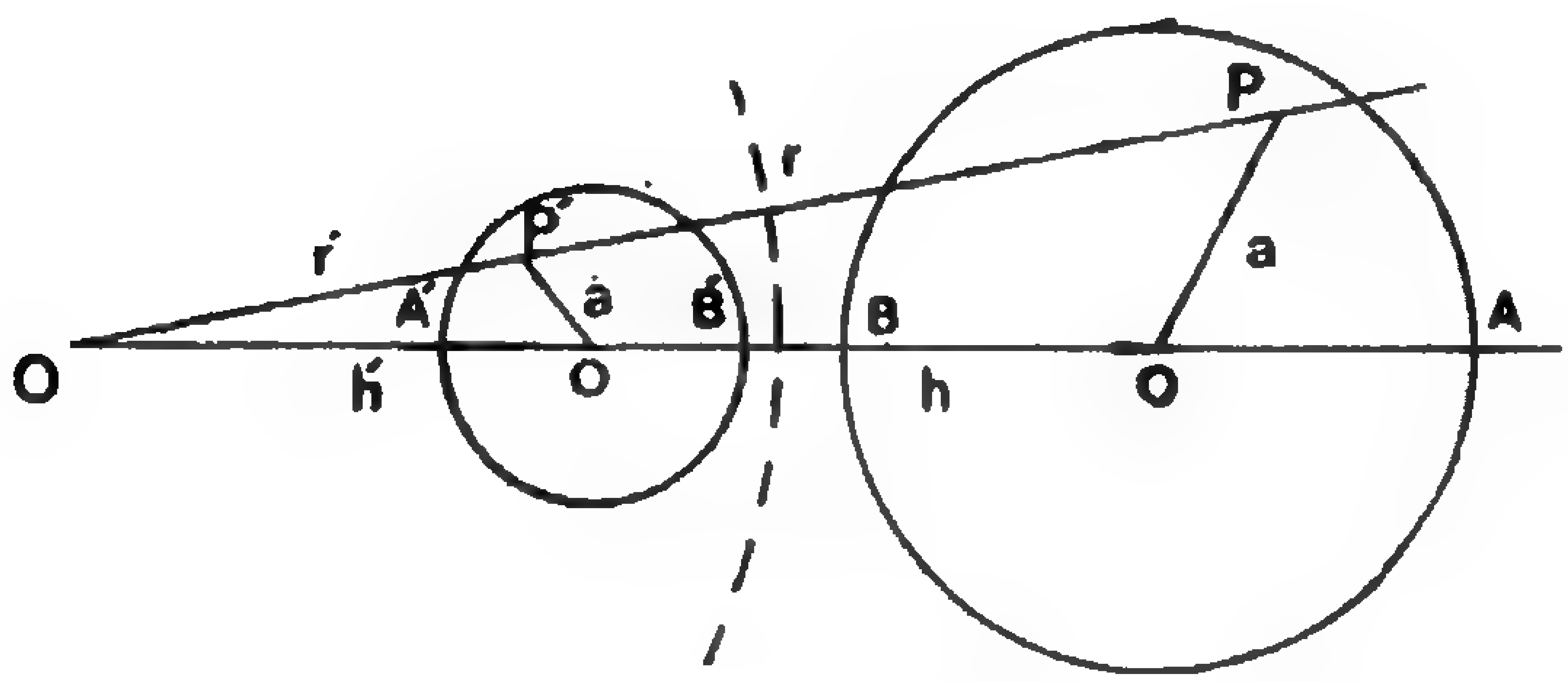


Fig. 9.75.

But we have that,

$$R^2 = (h'-a')(h+a) = (h'+a')(h-a)$$

This gives $a = a'(a+h) / (a'+h')$ and consequently $R^2/a = (h'^2-a'^2) / a'$. Substituting for R^2/a in (5) gives,

$$\sigma' = -Q(h'^2-a'^2) / 4\pi a' r'^3 \tag{6}$$

This result can be checked with that obtained before using the method of images, Problem 9.13.

The problem when the point charge lies inside a hollow conducting sphere can be treated in a similar way.

Note : As a consequence of this problem, if a conducting surface S is placed in space under the effect of a set of N point charges $q_i, i = 1, 2, 3, \ldots, N$ located at $\mathbf{r}_i$, $i = 1, 2, 3, \ldots, N$, then the potential of the inverse surface S' under the effect of the inverse charges $q'_i = R\, q_i/r_i$ is $V' = V_o\, R/r = V_o\, r'/R$. This inverse surface can be kept at zero potential if a point charge $Q = -4\,\pi\,\epsilon_o\, V_o R$ is placed at the center of inversion.

56. *A conductor is formed of two spheres of radii a and b intersecting orthogonally. If this conductor is raised to potential V_o find an an expression for the surface density at any point on it. Also find an expression for the capacitance of this conductor.*

The total charge on the conductor can be determined from the inverse system. Taking the radius of inversion R and center of inversion A (Fig. 9.76), the inverse system reduces to two perpendicular planes.

The inverse surfaces are at zero potential if a charge $Q = -4\pi R \epsilon_o V_o$ is located at A. This charge has three image charges. Thus the given conductor when charged to potential V_o the potential at any external point is that due to q_1, q_2, and q_3 the inverse charges of $-Q$ at B, Q at C, and $-Q$ at D.

$$q_1 = -Q\,(AE/R) = K\,(AE) \tag{1}$$

$$q_2 = -Q\,(AG/R) = K\,(AG) \tag{2}$$

$$q_3 = Q\,(AF)/R = -K\,(AF) \tag{3}$$

where $K = 4\pi\,\epsilon_o\, V_o$, and $AE = a$, $AG = b$, and $AF = ab/\,(a^2+b^2)^{1/2}$.

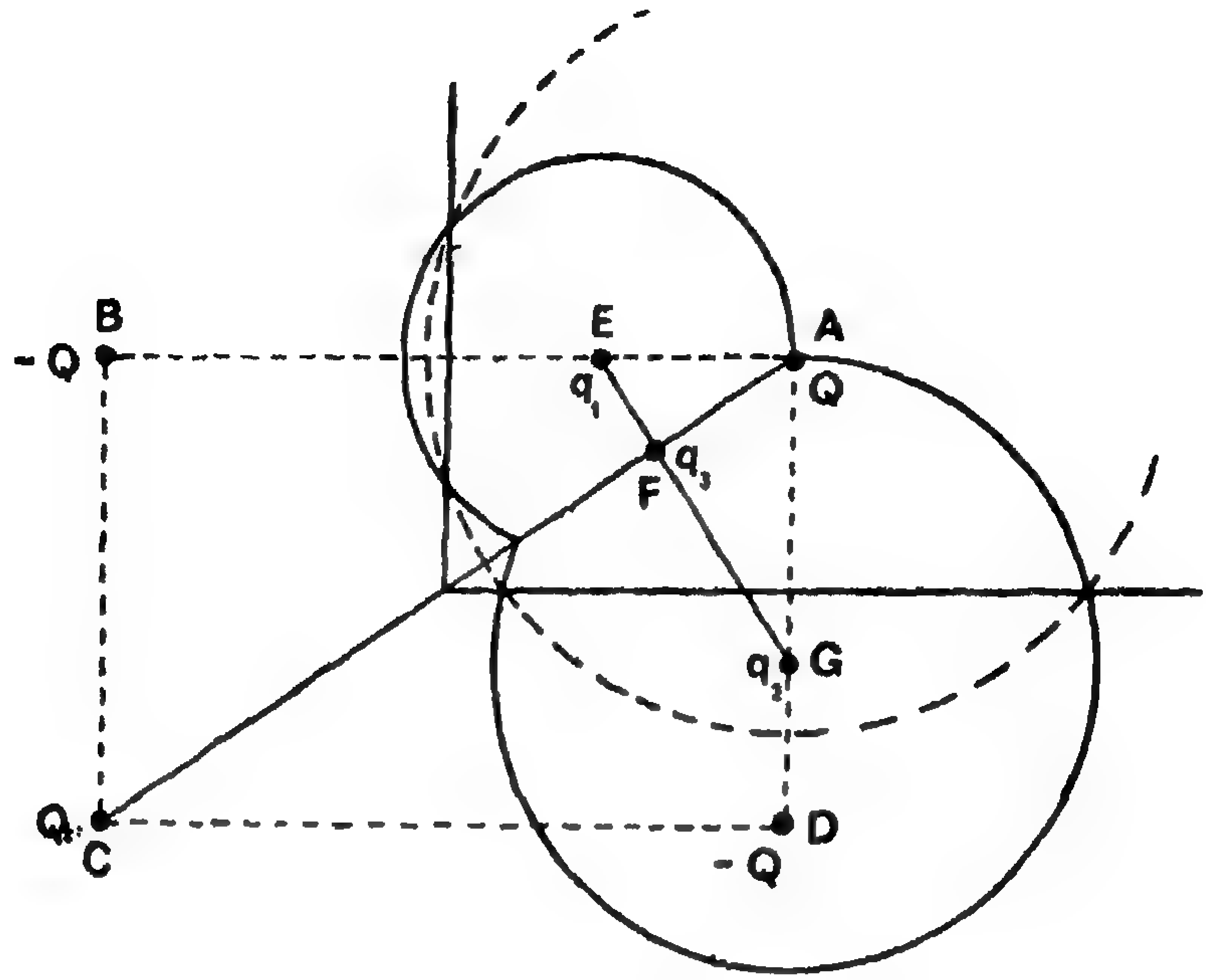


Fig. 9.76.

From (1)—(3) and the geometry of the given conductor it is clear that q is the image of q_1 with respect to sphere 2 and is the image of q_2 with respect to sphere 1. By the use of the expression of surface density of a point charge in front of a conducting sphere, the surface density at any point P on sphere 1 is (Fig. 9.77),

$$\begin{aligned} \sigma_p &= q_1/4\pi a^2 - q_2\,[(EG)^2 - a^2]\;/\;4\pi\;a\;(PG)^3 \\ &= (K/4\pi\;a)\;[1 - (b/PG)^3] \end{aligned} \tag{4}$$

Similarly the density on any point on the second sphere is,

$$\sigma \;=\; (K/4\pi\;b)\;[1 - (a/AH)^3] \tag{5}$$

On the circle of intersection of the two spheres $\sigma = 0$. The capacity of the conductor can be found from the total charge on the given conductor,

$$Q_t = q_1+q_2+q_3$$

$$= k\,[-AF + AE + AG]$$

$$= 4\pi\epsilon_0 V_o\,[-ab/(a^2+b^2)^{1/2} + a+b]$$

$$C = Q_t/V_o = 4\pi\epsilon_0\,[(a+b)\,(a^2+b^2)^{1/2} - ab]/(a^2+b^2)^{1/2} \quad (6)$$

Note that from (1)—(3) it is clear that each of the charges q_1, q_2, and q_3 is proportional to its distance from the center of inversion A.

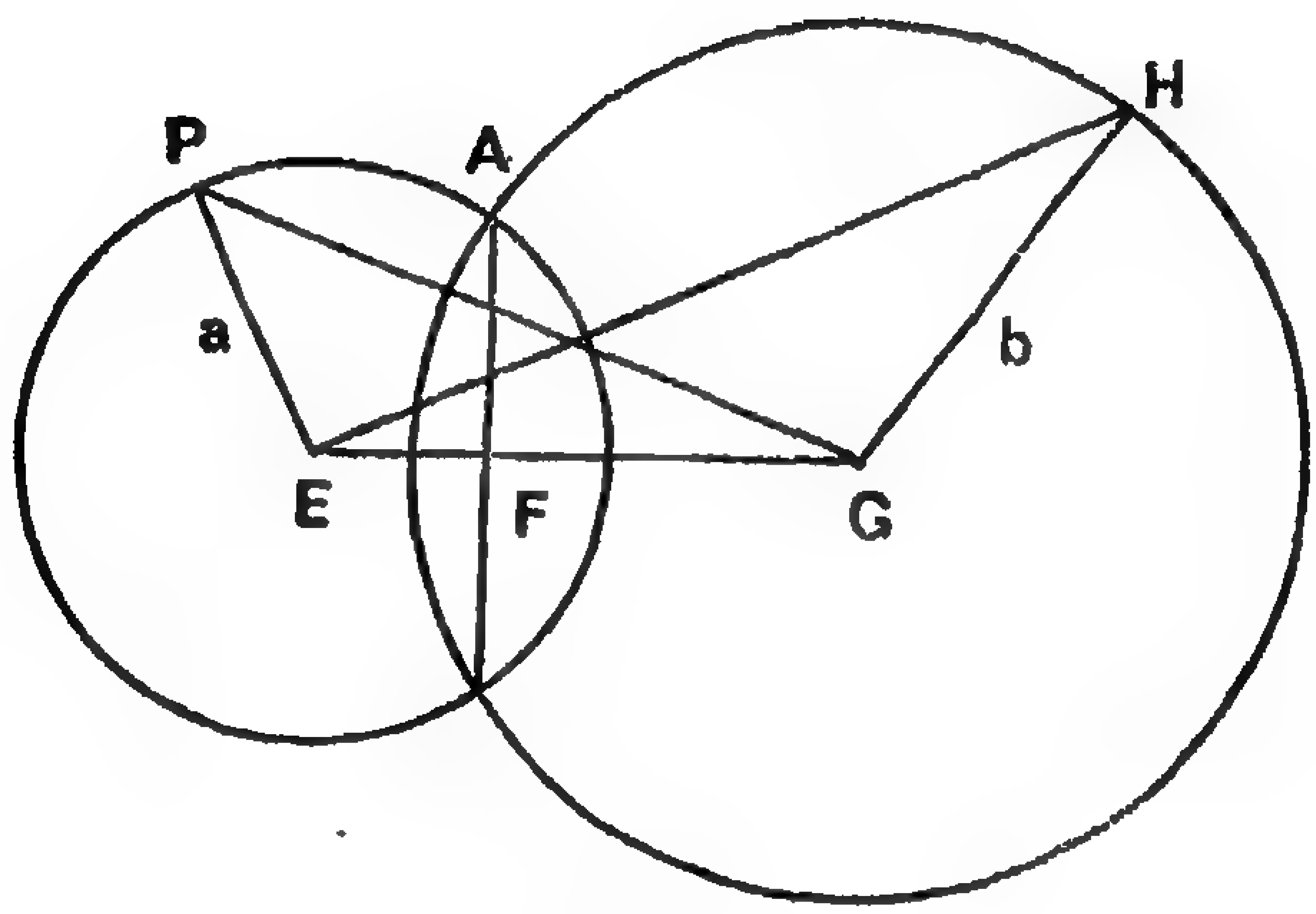


Fig. 9.77.

57. *A point charge q is placed outside the conductor of the previous problem. Determine the surface density at any point on the conducting surface. Also find an expression for the total charge on the conductor.*

With O as a center of inversion invert the given system with respect to a sphere radius unity. Aided with Fig. 9.78. let,

$$OE = \alpha\,,\ OG = \beta\,,\ GP = p\,,\ \text{and}\ PO = r$$

From the inversion properties given in problem 9 51 the inverse system remains two conducting spheres intersecting orthogonally. The in-

verse of a, b, α, β, r, and p are,

$$a' = a/(\alpha^2-a^2)\ ,\ \alpha' = \alpha/(\alpha^2-a^2)$$

$$b' = b/(\beta^2-b^2)\ ,\ \beta' = \beta/(\beta^2-b^2)$$

$$r' = 1/r\ ,\ \text{and}\ p'^2 = [b^2r^2+(\beta^2-b^2)(p^2-b^2)]\ /\ r^2(\beta^2-b^2)^2$$

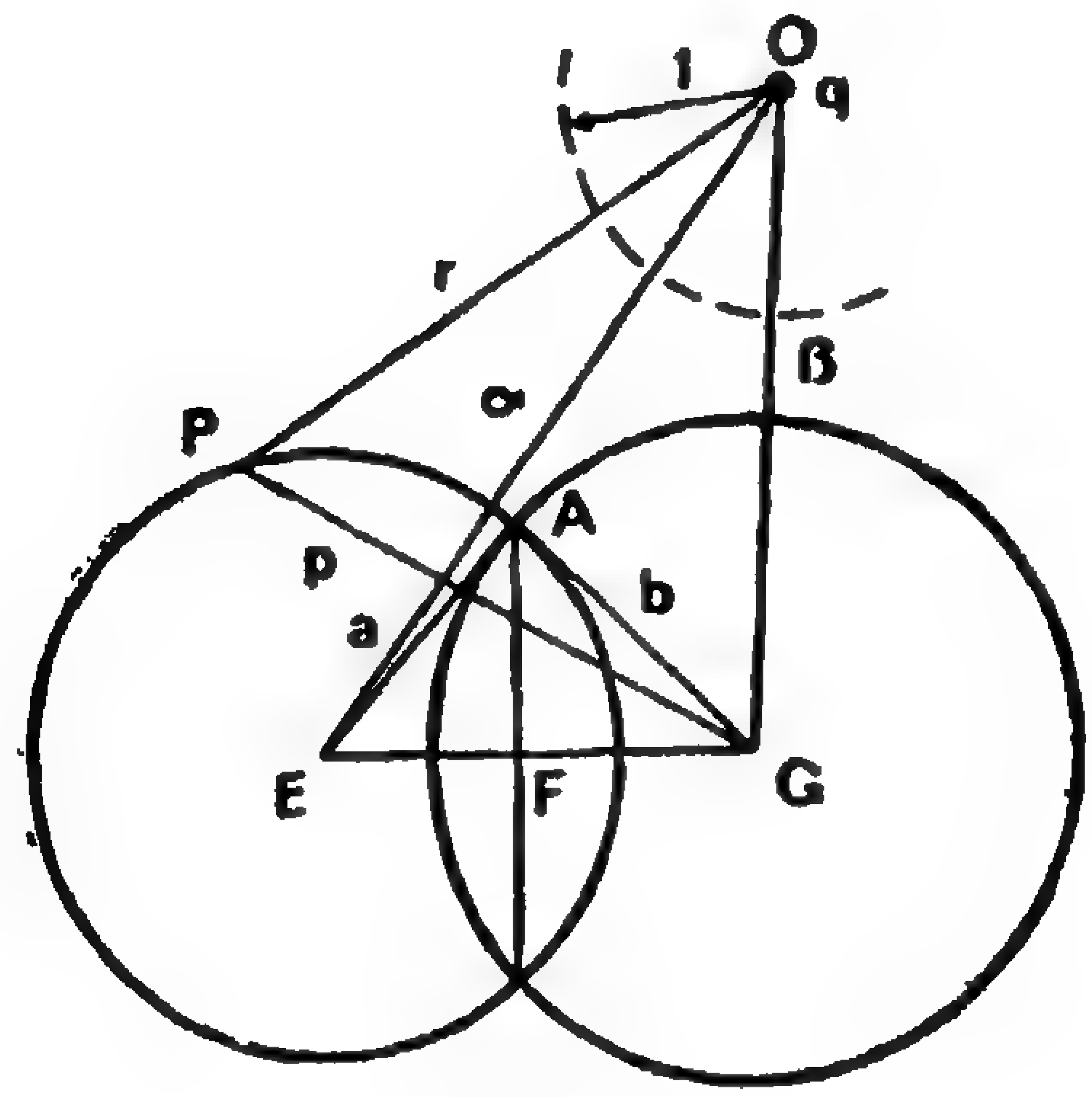


Fig. 9.78.

Now the inverted system if kept at potential V_0 the density at the point P' is (see Problem 9.56 equation (4)),

$$\sigma' = (k/4\pi a')\ [1-(b'/p')^3], \qquad k = 4\pi\epsilon_0 V_0.$$

In the original system $\sigma = \sigma'/r^3$ and $V = V_0/r$ so that if a negative charge $-4\pi\epsilon_0 V_0$ $(= -q)$ is placed at O, the potential of the original conductor is reduced to zero, while σ becomes

$$\sigma_p = \sigma'/r^3 = [k(\alpha^2-a^2)/4\pi a r^3]\ \{1-b^3r^3\ /\ [b^2r^2+(\beta^2-b^2)(p^2-b^2)]^{3/2}\}$$

where, $k = 4\pi\epsilon_0 V_0 = q$. If point P lies on the other conductor a and b. α and β are interchanged and p is the distance between G and O.

The induced charges on the given conductor can be determined by considering the charges on the inverse system. The inverse system has charges ka' at E', kb' at G' and $-ka'b'/(a'^2+b'^2)^{1/2}$ at F'. Inverting we get the charges at E, G and F.

$$ka'/\alpha' = ka/\alpha,\ kb'/\beta' = kb/\beta,\ OF' = (\alpha'^2 b'^2 + \beta'^2 a'^2 - a'^2 b'^2)/$$

$$(a'^2+b'^2) - k\,a'b'/(OF')(a'^2+b'^2)^{1/2} = -ab/(\alpha^2 b^2 + \beta^2 a^2 - a^2 b^2)^{1/2}$$

so that the total charge on the conductor is,

$$Q_t = qa/\alpha + qb/\beta - qab\,/\,[\alpha^2 b^2 + \beta^2 a^2 - a^2 b^2]^{1/2}$$

This potential can be reduced to zero by placing a point charge $Q = -4\pi\epsilon_0 V_0 R$ at the center of inversion.

58. *An infinite conducting plate with circular hole of radius h is kept at zero potential and a point charge q is placed at any point in the hole. Find an expression for the surface charge density at any point on the conducting plate.*

The problem of a charged disc at potential V_0 can be reduced to the problem of an infinite plate at zero potential under the influence of a point charge $-4\pi\epsilon_0 V_0 R$ at the center of inversion.

The surface charge density at any point on a disc radius a charged with total charge Q is, (see problem 15 Chap. 3, Vol. I),

$$\sigma = Q/4\pi a\,(a^2-r^2)^{1/2} \tag{1}$$

where r is the distance from the disc center and a its radius. Since the capacity of the disc is $8\epsilon_0 a$ we have that,

$$\sigma = 2\epsilon_0 V_0\,/\,\pi\,(a^2-r^2)^{1/2} \tag{2}$$

where V_0 is the potential of the disc. If APB is a chord through point P on the disc we have that $a^2 - r^2 = AP.BP$ (Fig. 9.79). Thus,

$$\sigma = 2\,\epsilon_0\, V_0 / \pi\,(AP.BP)^{1/2} \tag{3}$$

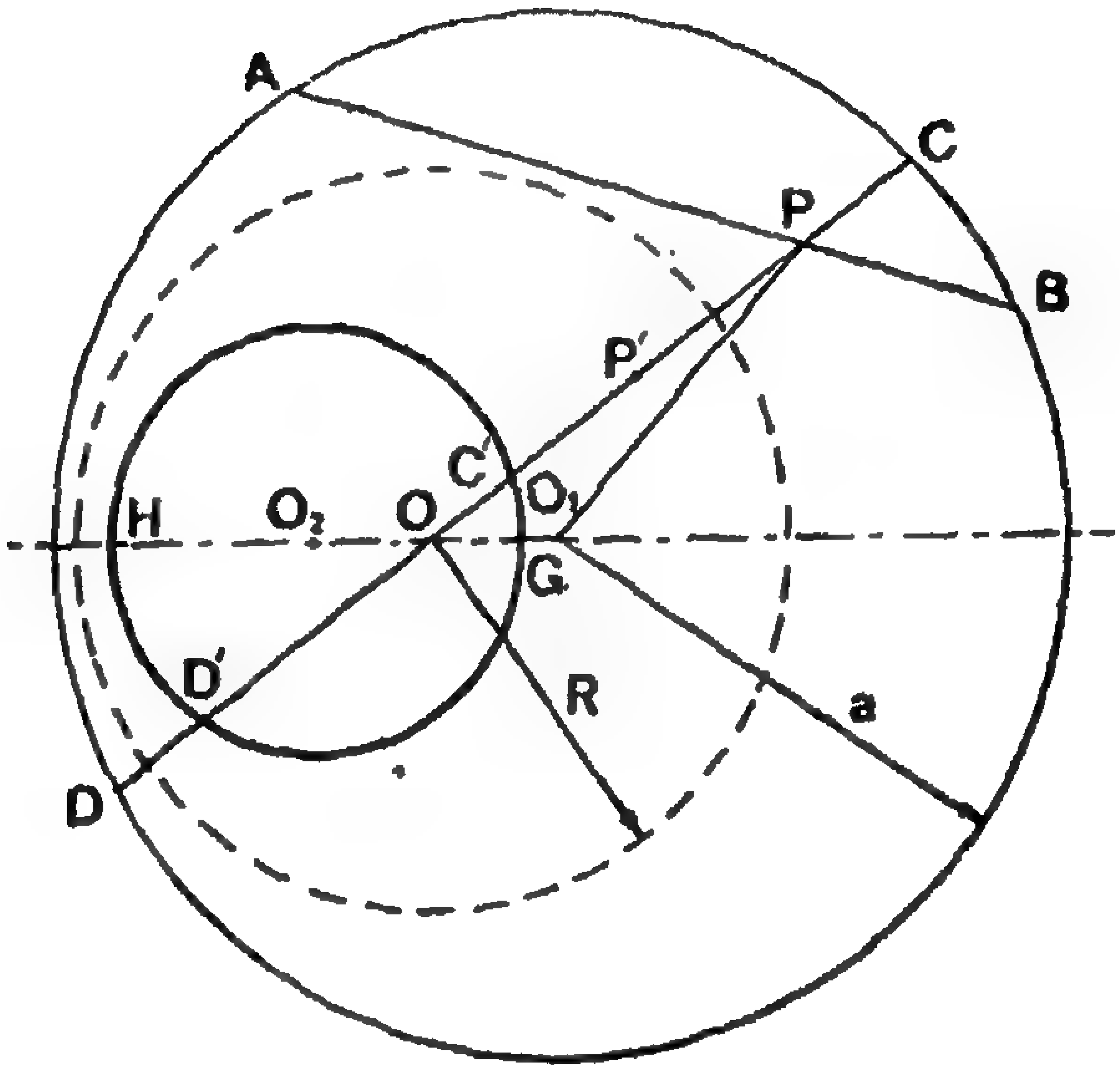


Fig. 9.79.

This disc will be inverted to become the given conducting plate with circular hole, with $q = 4\pi\,\epsilon_0\, V_0\, R$

$$\sigma = q / 2\pi^2 R\,(AP.BP)^{1/2} \tag{4}$$

The total charge at the center of inversion becomes zero, and the inverse distribution is,

$$\sigma' = \sigma R^3 / (OP')^3 = q R^2 / 2\pi^2 (OP')^3\,(AP.BP)^{1/2} \tag{5}$$

From Fig. 9.79 we have the relations,

$$AP.BP = PC.PD = OC'.OD'.PC.PD / (OC'.OD')$$

$$= (OD'.PD).(OC'.PC) / (OG.OH)$$

$$OD'.PD = OD'.(PO+OD) = R^2+(OD'.PO)$$

$$= PO.(PO'+OD') = PO'.P'D'$$

$$OC'.PC = OC'.(OC-PO) = R^2-OC'.PO$$

$$= PO.(P'O-OC') = PO.P'C'$$

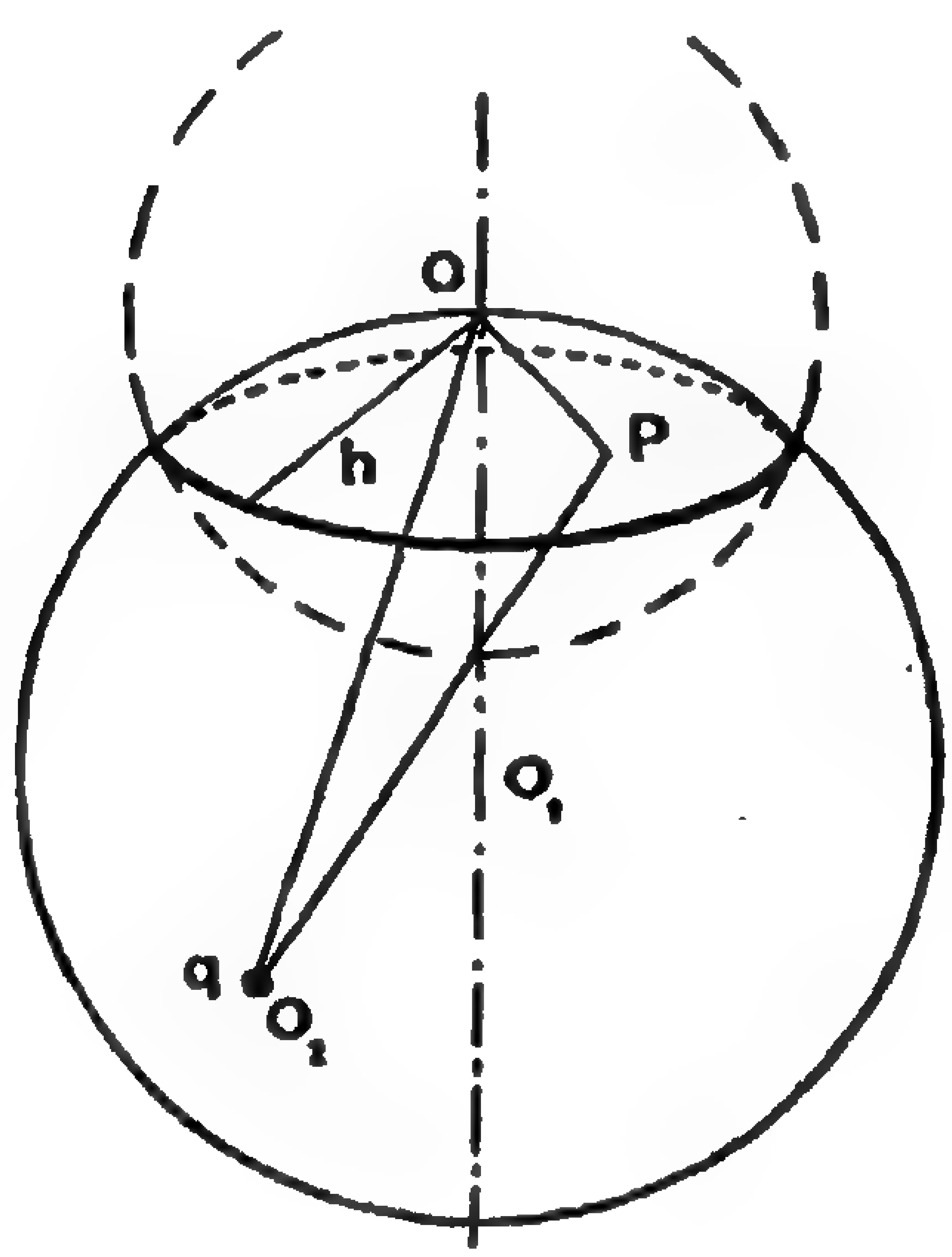


Fig. 9.80.

Thus,

$$AP.BP = (PO)^2.\ P'C'.\ P'D' / (OG.OH)$$

and (5) becomes,

$$\sigma' = [qR^2/2\pi (OP')^3] \; [OG.OH/(PO)^2 \; P'C'.P'D']^{1/2}$$

$$= [q/2\pi^2 (OP'^2)] \; [OG.OH \;/\; P'C'.P'D']^{1/2}$$

$$= [q/2\pi^2 (OP')^2] \left\{ [h^2-(OO_2)^2] / [(P'O_2)^2-h^2] \right\}^{1/2}$$

Note that if the center of inversion is taken out of the plane of the disc, the disc is inverted to a spherical surface passing through the center of inversion while the disc itself becomes a spherical surface called **bowl**. The surface distribution is,

$$\sigma = [q/2\pi^2 (O_2P)^2] \left\{ [h^2-OO_2^{\,2}] \;/\; [OP^2-h^2] \right\}^{1/2}$$

where OP', $P'O_2$, OO_2 and h are shown in Fig. 9.80. It is interesting that σ is independent of the radius of the sphere of which the bowl is a part.

10 - 9 مسائل إضافية
SUPPLEMENTARY PROBLEMS

1. An infinite conducting plate at zero potential is under the effect of a charge Q. Show that the induced charge on any area on the plane is proportional to the solid angle subtended by the charge Q.

2. An electric doublet of moment p is at perpendicular distance a from an infinite plane conductor at zero potential and its axis makes an angle θ with the normal to the plane. Find the force on the doublet.

3. A charge Q_1 lies on the axis of symmetry of a circular disc of radius a at a distance h from the plane of the disc. Find the charge Q_2 which must be placed at a point located symmetrically relative to the disc in order that the flux through the disc in the direction towards Q_2 is $1/8\ Q_1$.

4. Two equal point charges Q are at distance a apart, and each of them is $a/2$ from a grounded infinite plane. Prove that the force between the charges is $3Q^2/8\pi\,\epsilon_0\,a^2$. If the sign of one of the charges is reversed, why is the force not simply reversed also.

5. Two equal charges are placed at a distance a from an infinite conducting plane at zero potential, and at a distance $(3/2)\,a$ from each other. Show that if these charges have the same sign, the resultant mechanical force experienced by either charge makes an angle $\theta = tan^{-1}\,(224/243)$ with the normal to the plane.

6. A point charge Q is placed between two intersecting earthed conducting planes at an angle θ. Prove that there is a finite number

of image point charges if $\theta = \pi/n$, where n is a positive integer. Find an expression for the total charge induced on one of the planes.

7. A point charge Q is placed at the point (a, a, a) in the positive octant, and the positive portions of the coordinate planes are occupied by three mutually perpendicular plane conductors maintained at zero potential. Prove that the point charge is acted on by a force towards origin of amount

$$(Q^2/96\pi \epsilon_0 a^2)\,(2+6\sqrt{3}-3\sqrt{6})$$

8. If two infinite plane uninsulated conductors meet at an angle $\pi/3$, and there is a charge Q at a point equidistant from each, and distant r from the line of intersection, find the charge density at any point of the plane. Show that at a point in a principal plane through the charged point at a distance $r\sqrt{3}$ from the line of intersection, the surface density is

$$-(Q/4\pi r^2)\,(3/4+1/7\sqrt{7})$$

9. A point charge Q is placed inside a large rectangular earthed conducting box at equal distances a from three faces and near the corner formed by these faces. Neglecting effects due to the distant faces, find an image system for the field inside the box. Show that the charge density induced at any one of the three points of the box nearest to the charge is $-(Q/\pi a^2)(14/27-\sqrt{5}/25)$.

10. Two point charges A and B each of amount Q are placed at equal distance from each other and from an infinite conducting plane sheet at zero potential. Find the charges induced on the portion of the sheet cut off by the plane through A perpendicular to AB. Sketch the lines of force in the plane through AB perpendicular to the sheet.

11. An electric doublet p is held at distance a from an infinite grounded plane. Show that the image is an equal dipole. If the original dipole makes an angle θ with the normal to the plane show that the mutual potential energy is $U = -(p^2/32\pi \epsilon_0 a^3)(1+\cos^2\theta)$. Deduce that if it is free to turn about its center it will take up a position perpendicular to the plane, and the period of small oscillation about this direction is $4\pi(\sqrt{a^3 I/p})$ where I is the moment of inertia.

12. A infinite slab of homogeneous material of relative permittivity ϵ_r fills the region $z \leq 0$, being separated from free space by the plane $z = 0$. A charge Q_1 lies in air at distance a in front of the separating plane. Another charge Q_2 is placed at distance a on the other side of the separating plane in such away that the line joining Q_1 and Q_2 has the separating plane as a perpendicular bisector. Show how to choose suitable image charges and prove that the charge Q_1 is repelled from the slab by a force, $[2Q_1Q_2 - (\epsilon_r - 1)Q_1^2] / [16\pi \epsilon_0 a^2 (\epsilon_r + 1)]$.

13. Prove that if a small charged conductor be at distance r from the center in a space bounded by a spherical conducting surface of radius a at zero potential, the force repelling it from the center is $rQ^2/[4\pi \epsilon_0 (a^2 - r^2)^2]$, where Q is the charge.

14. A point charge Q is located inside a conducting sphere at zero potential, of internal radius a. Determine the surface charge density on the sphere.

15. A sphere is insulated and carries a charge Q, and another charge Q is placed at a point external to the sphere. Prove that if the ratio of the distance of the point from the center to the radius is greater than 2.62: 1.0, the surface density on the sphere will be nowhere negative.

16. Within a spherical hollow, in a conductor connected to earth, equal point charges Q are placed at equal distances h from the center on the same diameter. Show that each is acted on by a force equal to.

$$(Q^2/4\pi\epsilon_0)\,[4a^3h^3/(a^4-h^4)^2 + (1/4h^2)]$$

17. A conductor maintained at potential V contains a spherical cavity of radius a filled with a dielectric of permittivity ϵ. If a point charge Q is placed at a distance h from the center of the cavity, ($h<a$), determine the field in the cavity and the equivalent system of images.

18. An insulated conducting sphere of radius a is under the influence of a point charge Q at distance h ($>a$) from its center. What is the least positive charge that must be given to the sphere in order that the surface density may be everywhere positive.

19. A conducting sphere of radius a is placed in a dielectric of permittivity ϵ_1. The sphere contains a spherical cavity of radius b filled with a dielectric of permittivity ϵ_2. If a point charge Q is located at distance R from the center of the cavity ($R < b$), find the potential in all space.

20. The outer of two concentric spherical conductors of radii a and b, ($a < b$) is earthed and the inner is charged to potential V. A point charge Q is then placed in the space between the spheres at a distance h from the center. Prove that the charges on the spheres are as

$$(Q/4\pi\epsilon_0 h)-(Q/4\pi\epsilon_0 b)-V : (Q/4\pi\epsilon_0 a)-(Q/4\pi\epsilon_0 h)+V$$

21. A charge Q is placed midway between two equal spherical conductors which are kept at zero potential. Show that the charge induced on each is

$$-Q\,(m-m^2/2+m^3/4-3\,m^4/8)$$

neglecting higher powers of m, which is the ratio of the radius of a conductor to half the distance between their centers.

22. Show that, if the inducing charge is at a distance from the center of a sphere at zero potential equal to double the radius, the surface densities at the nearest and most remote points of the sphere are in the ratio 27 : 1.

23. A hollow conductor has the form of a quarter of a sphere bounded by two perpendicular diameteral planes. Find the images of a charge placed at any point inside it.

24. A charge Q is placed at a point P distant h from the center O of a spherical cavity of radius a, within a conductor, $h<a$. Show that a line of force from Q making initially an angle α with PO will meet the surface of the cavity at a distance $(a^2 - h^2) / (a - h \cos \alpha)$ from P.

25. A charge Q is placed inside a hemispherical hollow of radius a in an isolated conductor. The charge Q is on the axis of symmetry at a distance h from the planar boundary. Show that it is in equilibrium if h/a is equal to the root of the equation,

$$u^8 - 8u^7 - 2u^4 - 8u^3 + 1 = 0$$

Show also that this equation has one and only one root between 1 and 0.

26. A charge Q is placed at a distance h from the center of a conducting sphere of radius a. If the sphere is insulated and uncharged, find the rise in its potential due to the presence of the charge and show that there is an attractive force between the charge and sphere of magnitude, $Q^2 a^3 (2h^2 - a^2) / 4\pi \epsilon_0 h^3 (h^2 - a^2)^2$.

27. Show that the capacity of a spherical conductor of radius a is increased in the ratio $1:1+(\epsilon-\epsilon_0)\, a/2b\, (\epsilon+\epsilon_0)$ by the presence of a large mass of dielectric with a plane face, at a distance b from the center of the sphere, if a/b is considered small and only first order terms are considered.

28. A point charge Q is placed outside an insulated conducting sphere of radius a that carries a charge q. Show that, even if Q and q are of like sign, the force between the charge and the sphere is attractive provided that the distance of the point charge from the sphere lies within a certain range. Verify that when q = 7Q/18, this range extends from a to $2a$.

29. A point charge q is brought near to a spherical conductor of radius a having a charge Q. Show that the particle will be repelled by the sphere, unless its distance from the nearest point of its surface is less than $\frac{1}{2}\, a\, (q/Q)^{1/2}$ approximately, (take $q << Q$).

30. A charged sphere of radius a is at height b above the ground, $b>>a$, and raised to potential V_0. Determine approximately the potential which it will attain, when it is raised to a height c.

31. An insulated uncharged conducting sphere is placed centrally between two charges of equal magnitude. Show that if they are of like signs the repulsion between them is diminished, but if of unlike signs the attraction between them is increased.

32. Two equal point charges Q are at distance $2h$ apart. Show that the effect of placing an earthed sphere radius a with its center at the mid-point of the two equal charges, is to reduce the repulsion between the charges in the ratio,

$$(h^4-a^4)^2 - 8ah^3\,(h^4+a^4) : (h^4-a^4)$$

33. An infinite plane conductor at zero potential with a hemispherical boss of radius a is under the influence of a point charge Q at a point on the axis of symmetry distant $2a$ from the plane. Find the potential at any point and sketch the lines of force. Determine the slope at the point charge of the critical line of force separating lines of force going to the boss and to the plane.

34. A dome of conducting material is built in the form of a hemisphere on the ground; a small charged conductor is situated midway between the dome and the ground, on the vertical through the center of the dome. Obtain the system of image charges, and prove that the mechanical force on the small conductor is upwards, and equal to $(27\ Q^2/900\pi\ \epsilon_0\ a^2)$, where Q is the charge on the conductor and a is the radius of the dome.

35. A hollow conductor has the form of a quarter of a sphere bounded by two perpendicular diametral planes. Find the images of a charge placed at any point inside.

36. Two identical spherical conductors of radius a carry a charge q and are placed so that their centers are distant b apart. Show that if a/b is small, the density of charge at a point (a, θ, ϕ) on the surface of either sphere is,

$$(Q/4\pi a^2)\ [1-3\,a^2 \cos\theta/b^2+O\ (a^3/b^3)]$$

where θ is measured from the line joining the centers of the spheres.

37 If a point charge is placed at distance c from the center of an earthed conducting sphere of radius a, $(a < c)$, show that the density of charge induced on the sphere is inversely proportional to the cube of the distance from the point charge.

Show also that if the sphere is insulated and uncharged, then the part of the sphere on which the induced charge-density has the same sign as the point charge has area,

$$(\pi a/c)\,[(c+a)^2-(c^3-ca^2)^{2/3}].$$

38. An infinite conducting plate with a hemispherical boss of radius a is kept at zero potential, and a charge Q is placed on the axis of symmetry distant h from the plate. Show that the lines of force which reach the plate at its junction with the hemisphere leave the charge Q at an angle,

$$\theta = \arccos\,[1-2\,(h^2-a^2)\,/\,f\sqrt{4f\,a^2}\,]$$

with the axis of symmetry.

39. A vacuum tube has a planar anode and a parallel plane cathode, the linear dimensions of both being large compared with the separation a between them. The cathode is maintained at zero potential and the anode at V_0. An electron of charge e between the electrodes is in equilibrium at a distance d from the anode, $d<<a$. Neglecting the images in cathode, show that if $e<0$, V_0 must be negative. Show also that the value of d is $\frac{1}{2}\,[4\pi\,\epsilon_0\,ae/V_0]^{1/2}$ approximately and discuss the stability of the equilibrium.

40. An insulated conducting sphere of radius a is placed midway between two parallel infinite uninsulated planes at a great distance $2c$ apart. If only first order terms of a/c are considered, show that the capacity of the sphere is approximately given by,

$$4\pi\,\epsilon_0\,a\,[1+(a/c)\,\log 2].$$

41. A small metal ball of mass m and charge Q is suspended from a string of length L. If the point of fixation O is at distance H from a vertical grounded conducting plane, find the periodic time of oscillation of the system. First order terms of L/H will only be considered.

42. A simple pendulum consists of a string of length L and mass m carrying a charge Q. The pendulum is placed in an electric field E_0 making an angle α with the vertical. Find an expression for the periodic time of oscillation.

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